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ON SEPARABLE ABELIAN EXTENSIONS OF RINGS

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ABSTRACT. Let R be a ring with 1, G ($= \langle\rho_1\rangle\times\ldots\times\langle\rho_m\rangle$) a finite abelian automorphism group of R of order n where $\langle\rho_i\rangle$ is cyclic of order n_i for some integers n, n_i, and m, and C the center of R whose automorphism group induced by G is isomorphic with G. Then an abelian extension $R[x_1,\ldots,x_m]$ is defined as a generalization of cyclic extensions of rings, and $R[x_1,\ldots,x_m]$ is an Azumaya algebra over K ($= C^G = \{c$ in C / $(c)\rho_i = c$ for each ρ_i in $G\}$) such that $R[x_1,\ldots,x_m] \cong R^G \otimes_K C[x_1,\ldots,x_m]$ if and only if C is Galois over K with Galois group G (the Kanzaki hypothesis).

KEY WORDS AND PHRASES. Abelian ring extensions, separable algebras, Azumaya algebras, Galois extensions.

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1. INTRODUCTION.

Cyclic extensions of rings have been intensively investigated by Nagahara and Kishimoto [1], Parimula and Sridharan [2], the present author [3,4,5], and others. In [3], a separable cyclic extension R[x] with respect to a cyclic automorphism group $\langle\rho\rangle$ of R of order n for some integer n over a noncommutative ring R was studied. It was shown ([3], Theorem 3.3) that if R is Galois over $R^{\langle\rho\rangle}$ ($= \{r$ in R / $(r)\rho = r\}$) with Galois group $\langle\rho\rangle$ and if $R^{\langle\rho\rangle}$ is contained in the center **C** of R, then R[x] is an Azumaya algebra over $R^{\langle\rho\rangle}$, where x^n (= b for some b in R) and n are units in $R^{\langle\rho\rangle}$. Let G be an abelian automorphism group of R of order n such that $G = \langle\rho_1\rangle\times\ldots\times\langle\rho_m\rangle$

where $\langle\rho_i\rangle$ is a cyclic subgroup of order n_i for some integers n, m, and n_i. Noting that $(C)\rho_i = C$ for each ρ_i, we shall study an abelian extension $R[x_1,\ldots,x_m]$ with respect to G, where $rx_i = x_i(r\rho_i)$ for each r in R, $x_i^{n_i} = b_i$ which is a unit in C^G, $x_ix_j = x_jx_i$ for all i and j, and the set $\{x_1^{k_1}\ldots x_m^{k_m} / 0 \leq k_i < n_i\}$ is a basis over R. A ring R is called to satisfy the Kanzaki hypothesis ([6], P. 110) if R is Azumaya over C with a finite automorphism group G and C is Galois over K ($= C^G$) with Galois group induced by and isomorphic with G. DeMeyer [7] has shown that $R \cong R^G \otimes_K C$ under the Kanzaki hypothesis for R. The present paper will generalize the Parimula-Sridharan theorem from cyclic extensions ([2], Proposition 1.1, [3], Theorem 3.3) to abelian extensions $R[x_1,\ldots,x_m]$ with respect to an abelian automorphism group G ($= \langle\rho_1\rangle \times \ldots \times \langle\rho_m\rangle$) of R. Let G restricted to C be isomorphic with G. Then we shall show that C is Galois over K ($= C^G$) if and only if $R[x_1,\ldots,x_m]$ is an Azumaya algebra over K such that $R[x_1,\ldots,x_m] \cong R^G \otimes_K C[x_1,\ldots,x_m]$ where R^G is an Azumaya K-algebra. Thus, a structure of $R[x_1,\ldots,x_m]$ is obtained. Moreover, a structure of $C[x_1,\ldots,x_m]$ is also obtained when each direct summand of G is a G-subgroup (see definition below).

2. PRELIMINARIES.

Throughout, let R be a ring with 1, C the center of R, G ($= \langle\rho_1\rangle \times \ldots \times \langle\rho_m\rangle$) an abelian automorphism group of R of order n where ρ_i is cyclic of order n_i for some integers n, n_i, and m. Then $R[x_1,\ldots,x_m]$ is the abelian extension of R with respect to G as defined in Section 1. We denote C^G by K, and assume that the automorphism group of C is isomorphic with G. The Azumaya algebra R is called to satisfy the Kanzaki hypothesis ([6], P. 110) if C is Galois over K with Galois group induced by and isomorphic with G. For separable extensions, Azumaya algebras, and Galóis extensions, see [3], [4], and [5].

3. ABELIAN EXTENSIONS.

Keeping the notations of Sections 1 and 2, we shall show the Parimula-Sridharan theorem ([2], Proposition 1.1, [3], Theorem 3.3) and two structural theorems for abelian extensions $R[x_1,\ldots,x_m]$. We begin with a proposition on separable abelian extensions.

PROPOSITION 3.1. Let G ($= \langle \rho_1 \rangle \times \cdots \times \langle \rho_m \rangle$) be an abelian automorphism group of R of order n. If n and $x_i^{n_i}$ ($= b_i$) are units in C^G for each i, then $R[x_1,\ldots,x_m]$ is a separable extension of R.

PROOF. Since n_i divides n, n_i is a unit in C^G. Hence the cyclic extension $R[x_1]$ with respect to $\langle \rho_1 \rangle$ is a separable extension over R ([3], Lemma 3.1). Now $\langle \rho_2 \rangle$ is extended to an automorphism group of $R[x_1]$ by $(x_1)\rho_2 = x_1$, so $(R[x_1])[x_2]$ is a separable extension over $R[x_1]$ by a similar reason. Thus $R[x_1,x_2]$ ($= (R[x_1])[x_2]$) is a separable extension over R by the transitivity of separable extensions. By repeating the above argument $(m-2)$ times, $R[x_1,\ldots,x_m]$ is a separable extension over R.

We now show the Parimula-Sridharan theorem for $R[x_1,\ldots,x_m]$.

THEOREM 3.2. By keeping the notations of Proposition 3.1, if R satisfies the Kanzaki hypothesis, then $R[x_1,\ldots,x_m]$ is an Azumaya K-algebra.

PROOF. By Proposition 3.1, $R[x_1,\ldots,x_m]$ is a separable extension over R. By the Kanzaki hypothesis for R, R is separable over C and C is Galois over K, so $R[x_1,\ldots,x_m]$ is a separable extension over K by the transitivity of separable extensions. So, it suffices to show that the center of $R[x_1,\ldots,x_m]$ is K. It is easy to see that K is contained in the center. Since $\{x_1^{k_1}\cdots x_m^{k_m} \,/\, 0 \leq k_i < n_i\}$ is a basis of $R[x_1,\ldots,x_m]$ over R, we can take f in the center of $R[x_1,\ldots,x_m]$ such that $f = a_o + x_1^{k_1}\cdots x_m^{k_m}\cdot a$ where a_o and a are in R, and $0 \leq k_i < n_i$. Then, $rf = fr$ for each r in R. This implies that $ra_o = a_o r$ and $ar = (r)\rho_1^{k_1}\cdots\rho_m^{k_m}\cdot a$. Hence a_o is in C, and the second equation implies that $a(r-(r)\rho_1^{k_1}\cdots\rho_m^{k_m}) = 0$ for each r in C. Thus a is in the annihilator ideal I of $\{r-(r)\rho_1^{k_1}\cdots\rho_m^{k_m} \,/\, r \text{ in } C\}$ of R. Since R is Azumaya over C, $I = I_o R$ where I_o is the annihilator ideal of $\{r-(r)\rho_1^{k_1}\cdots\rho_m^{k_m} \,/\, r \text{ in } C\}$ of C. $I_o = \{0\}$ ([7], Proposition 1.2) because C is Galois over K with Galois group induced by and isomorphic with G. Thus $I = \{0\}$, and so $a = 0$. Therefore, $f = a_o$ in C. Also, $x_i f = f x_i$ for each i, so $a_o = (a_o)\rho_i$ for each i. Thus a_o is in K. This completes the proof.

Next is a structural theorem for $R[x_1,\ldots,x_m]$ under the Kanzaki hypothesis.

THEOREM 3.3. If R satisfies the Kanzaki hypothesis, then $R[x_1,\dots,x_m] \cong R^G \otimes_K C[x_1,\dots,x_m]$ as Azumaya K-algebras.

PROOF. By Proposition 3.1, $C[x_1,\dots,x_m]$ is an Azumaya algebra over K. Then, similar to the arguments used in the proof of Theorem 3.2, we shall show that the commutant of $C[x_1,\dots,x_m]$ in $R[x_1,\dots,x_m]$ is R^G. Clearly, R^G is contained in the commutant. Now, let $f = a_0 + x_1^{k_1}\cdots x_m^{k_m}\cdot a$ be an element in the commutant for some a_0 and a in R and $0 \leq k_i < n_i$. Then cf = fc for each c in C. This implies that a = 0. Also, $x_i f = f x_i$ for each i, so a_0 is in R^G. Thus f (= a_0) is in R^G. Noting that $C[x_1,\dots,x_m]$ and $R[x_1,\dots,x_m]$ are Azumaya algebras over K, we have that $R[x_1,\dots,x_m] \cong R^G \otimes_K C[x_1,\dots,x_m]$ by the well known commutant theorem for Azumaya algebras ([7], Theorem 4.3, P. 57).

COROLLARY 3.4. If R satisfies the Kanzaki hypothesis, then R^G is an Azumaya algebra over K.

PROOF. This is a consequence of Theorem 3.3 and the commutant theorem for Azumaya algebras.

We are going to show a converse of Theorem 3.3.

THEOREM 3.5. If $R[x_1,\dots,x_m]$ is an Azumaya algebra over K such that $R[x_1,\dots,x_m] \cong R^G \otimes_K C[x_1,\dots,x_m]$ where R^G is an Azumaya K-algebra, then C is Galois over K with Galois group induced and isomorphic with G.

PROOF. By the commutant theorem for Azumaya algebras, since $R[x_1,\dots,x_m]$ and R^G are Azumaya K-algebras, so is $C[x_1,\dots,x_m]$. Then, we claim that C is Galois over K with Galois group G. Suppose not. There is a non-identity g in G such that $\{c-(c)g \;/\; c \text{ in } C\}$ is not C ([7], Proposition 1.2). Let $g = \rho_1^{k_1}\cdots\rho_m^{k_m}$ for some k_i, $0 \leq k_i < n_i$. Since I generated by (c-(c)g) for c in C is a G-ideal of C (that is, (I)G = I), we have an Azumaya algebra $(C/I)[x_1,\dots,x_m]$ over $K/(K \cap I)$. On the other hand, one can show that $(x_1^{k_1}\cdots x_m^{k_m})$ is in the center of $(C/I)[x_1,\dots,x_m]$. This is a contradition. Thus C is Galois over K with Galois group G.

Let S be a ring Galois extension over a subring T with a finite Galois group G. A normal subgroup H of G is called a <u>G-subgroup</u> if S is Galois over S^H with Galois group H and S^H is Galois over T with Galois group G/H. Keeping the notations of Theorem 3.5, we give a structural theorem for $C[x_1,\ldots,x_m]$.

We denote the center of $C[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m]$ by C_i' for each i. Clearly, $C_i' = C^{(G/\langle\rho_i\rangle)}$. Let each direct summand of G be a G-subgroup, we have:

THEOREM 3.6. If C is Galois over K with Galois group G, then the abelian extension $C[x_1,\ldots,x_m] \cong C_1'[x_1]\otimes_K\cdots\otimes_K C_m'[x_m]$ as Azumaya K-algebras.

PROOF. Extending ρ_i from C to $C[x_1,\ldots,x_m]$ by $(x_j)\rho_i = x_j$ for each i and j, we claim that $C[x_1,\ldots,x_m] \cong (C[x_1,\ldots,x_{m-1}])^{\rho_m}\otimes_K C_m'[x_m]$. In fact, since C is Galois over K, $C^{(G/\langle\rho_m\rangle)}$ is Galois over K with Galois group $\langle\rho_m\rangle$ (for $G/\langle\rho_m\rangle \cong \langle\rho_1\rangle\times\cdots\times\langle\rho_{m-1}\rangle$ is a G-subgroup of G by hypothesis). Now, the center of $C[x_1,\ldots,x_{m-1}]$ is $C^{(G/\langle\rho_m\rangle)}$, so $C[x_1,\ldots,x_{m-1}]$ satisfies the Kanzaki hypothesis; that is, $C[x_1,\ldots,x_{m-1}]$ has an automorphism group $\langle\rho_m\rangle$ such that its center $C^{(G/\langle\rho_m\rangle)}$ is Galois over $(C^{(G/\langle\rho_m\rangle)})^{\langle\rho_m\rangle}$ (= K) with Galois group induced by and isomorphic with $\langle\rho_m\rangle$. But $C[x_1,\ldots,x_m] \cong (C[x_1,\ldots,x_{m-1}])[x_m]$, so $C[x_1,\ldots,x_m] \cong (C[x_1,\ldots,x_{m-1}])^{\langle\rho_m\rangle}\otimes_K C_m'[x_m]$ by Theorem 3.3. Next, considering $(C[x_1,\ldots,x_{m-1}])^{\langle\rho_m\rangle}$, we have that $(C[x_1,\ldots,x_{m-1}])^{\langle\rho_m\rangle} \cong (C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}])[x_{m-1}]$ such that the center of $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}] = C_{m-1}'$ which is Galois over K with Galois group $\langle\rho_{m-1}\rangle$. Since $\langle\rho_{m-1}\rangle$ is an automorphism group of $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}]$, $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-2}]$ satisfies the Kanzaki hypothesis with a center which is Galois over K with Galois group $\langle\rho_{m-1}\rangle$. Hence $C^{\langle\rho_m\rangle}[x_1,\ldots,x_{m-1}] \cong C^{\langle\rho_m\rangle\times\langle\rho_{m-1}\rangle}[x_1,\ldots,x_{m-2}]\otimes_K C_{m-1}'[x_{m-1}]$. The above arguments can be repeated for (m-2) more times. Thus the proof is completed.

As immediate consequences of Theorem 3.5 and Theorem 3.6, we have the following:

COROLLARY 3.7. If R satisfies the Kanzaki hypothesis such that each direct summand of G is a G-subgroup, then $R[x_1,\ldots,x_m] \cong R^G\otimes_K C_1'[x_1]\otimes_K\cdots C_m'[x_m]$.

COROLLARY 3.8. If R satisfies the Kanzaki hypothesis such that the center C of R has no idempotents but 0 and 1, then $R[x_1,\ldots,x_m] \cong R^G\otimes_K C_1'[x_1]\otimes_K\cdots\otimes_K C_m'[x_m]$.

PROOF. Since C is Galois over K with no idempotents but 0 and 1, each direct summand of G is indeed a G-subgroup ([7], Theorem 1.1, P. 80, or [8]).

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