Internat. J. Math. & Math. Sci.
Vol. 4 No. 4 (1981) 655-660

ON UNIFORM CONVERGENCE FOR (μ,ν)-TYPE RATIONAL APPROXIMANTS IN $\mathbb{C}^n$ - II

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(Received June 3, 1980)

ABSTRACT. This paper shows that if f(z) is analytic in some neighborhood of the origin, but meromorphic in $\mathbb{C}^n$ otherwise, with a denumerable non-accumulating pole sections in $\mathbb{C}^n$, and if for each fixed ν, the pole set of each (μ,ν) - unisolvent rational approximant $\pi_{\mu\nu}(z)$ tends to infinity as $\mu' = \min_{i\leq n}(\mu_i) \to \infty$, then f(z) must be entire in $\mathbb{C}^n$. This paper also shows a monotonicity property for the "error sequence" $e_{\mu\nu} = ||f(z) - \pi_{\mu\nu}(z)||_K$ on compact subsets K of $\mathbb{C}^n$.

KEY WORDS AND PHRASES. *Uniform convergence, entire functions, approximations and expansions.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. *41.*

1. INTRODUCTION.

Two earlier papers by Lutterodt [1,2] gave results on uniform convergence under restricted assumptions made about the (μ,ν) - rational approximants. In [1], the B^1-type (μ,ν)-rational approximants, were assumed to be uniformly bounded on a polydisk; whereas, in [2], the $(\mu,1)$-rational approximants were under the assumption that the coefficients of the denominator polynomial of degree $\nu = (1,1,\ldots,1) = \underline{1}$ vanished as $\mu \to (\infty,\ldots,\infty)$ except for $b^{(\mu)}_{0\ldots0} \neq 0$. In fact, $b^{(\mu)}_{0\ldots0}$ is normalized to unity.

In this paper, we attempt to provide a general result about uniform convergence of (μ,ν)-rational approximants to entire functions in $\mathbb{C}^n$.

The main results of this paper are Theorems 1 and 2. Theorem 1 establishes uniform convergence for (μ,ν) unisolvent rational approximants with infinite pole sections that tend to infinity as $\mu \to (\infty,\ldots,\infty)$ on compact subsets of $\mathbb{C}^n$; Theorem 2 introduces an "error sequence"

$$e_{\mu\nu} = ||f(z) - \pi_{\mu\nu}(z)||_K$$

on any compact-subset of $\mathbb{C}^n$ and shows that $e_{\mu\nu}$ is monotonic in ν for sufficiently large values of μ.

2. NOTATION AND DEFINITIONS.

Let $z: = (z_1,\ldots z_n)$ be an n-tuple point in $\mathbb{C}^n$; let $\mu: = (\mu_1,\ldots,\mu_n)$ and $\nu: = (\nu_1,\ldots,\nu_n)$ be n-tuples of non-negative integers in $\mathbb{N}^n$.

Let $\mathcal{R}_{\mu\nu}$ be the class of all rational functions of the form

$$R_{\mu\nu}(z) = P_\mu(z)/Q_\nu(z), \quad Q_\nu(0) \neq 0$$

where $P_\mu(z)$ and $Q_\nu(z)$ are polynomials of multiple degree of at most μ and ν, respectively, with $(P_\mu(z), Q_\nu(z)) = 1$ in some neighborhood of the origin.

DEFINITION 1. Suppose $f(z)$ is analytic at the origin and $f(0) \neq 0$. An $R_{\mu\nu}(z) \in \mathcal{R}_{\mu\nu}$ is said to be a (μ,ν)-type rational approximant to $f(z)$ at $z = 0$ if

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} (Q_\nu(z)f(z) - P_\mu(z))\Big|_{z=0} = 0 \tag{2.1}$$

for $\lambda \in E^{\mu\nu} \subset \mathbb{N}^n$, a lattice interpolation set with the following properties:

(i) $0 \in E^{\mu\nu}$

(ii) $\lambda \in E^{\mu\nu} \Rightarrow \gamma \in E^{\mu\nu}$, $\gamma_i \le \lambda_i \quad i = 1,\ldots,n$

(iii) $E_\mu :\doteq \{\lambda \in \mathbb{N}^n: 0 \le \lambda_i \le \mu_i, \quad i = 1,\ldots,n\} \subset E^{\mu\nu}$

(iv) $|E^{\mu\nu}| \le \prod_{i=1}^{n} (\mu_i + 1) + \prod_{i=1}^{n} (\nu_i + 1) - 1$

(v) Each projected variable has the Padé index set

(vi) Each $\nu_i \le \mu_i \quad i = 1,\ldots,n$.

Here $|E^{\mu\nu}|$ is the cardinality of $E^{\mu\nu}$ and

$$\frac{\partial^{|\lambda|}}{\partial z^\lambda} \equiv \frac{\partial^{\lambda_1 + \ldots + \lambda_n}}{\partial z_1^{\lambda_1} \ldots z_n^{\lambda_n}}$$

DEFINITION 2. An $R_{\mu\nu}(z) \in \mathcal{R}_{\mu\nu}$ is said to have multiple degree $\mu^* = (\mu_1^*,\ldots,\mu_n^*)$ if, in the z_j-variable, $R_{\mu\nu}(z)$ expressed as a quotient of two pseudo-polynomials in z_j, has degree given by $\mu_j^* = \max(\mu_j,\nu_j)$, $1 \le j \le n$.

It follows from property (vi) of $E^{\mu\nu}$, that the multiple degree of a (μ,ν)-type rational approximant is always μ.

We shall refer the reader to the definition of a unisolvent (μ,ν)-type rational approximant to $f(z)$ in Lutterodt [3]. We shall denote this by

$$\pi_{\mu\nu}(z) = P_{\mu\nu}(z)/Q_{\mu\nu}(z).$$

We then normalize the denominator polynomial $Q_{\mu\nu}(z)$, dividing numerator and denominator by the modulus of largest coefficient of the denominator polynomial. Thus, we get

$$\pi_{\mu\nu}(z) = P^*_{\mu\nu}(z)/Q^*_{\mu\nu}(z)$$

where $Q^*_{\mu\nu}(z)$ is a normalized polynomial.

3. CONVERGENCE.

The uniform convergence for the (μ,ν)-rational approximants to $f(z)$ entire in $\mathbb{C}^n$ rests on the assumptions made about $f(z)$ and the hypothesis that, for each fixed multiple denominator degree ν of $\pi_{\mu\nu}(z)$, the pole set tends to infinity as $\mu \to (\infty,\ldots,\infty)$. In Theorem 1 below, we assume that $f(z)$ is possibly meromorphic, not with a finite pole set as in Theorem 2 of [3], but with a pole set having infinite sections such that only a finite number of such pole sections overlap with any given polydisk. Thus, Theorem 1 of this paper extends the result in [3].

THEOREM 1: Suppose $f(z)$ is analytic at the origin and is possibly meromorphic with an infinite pole set in $\mathbb{C}^n$ without accumulation of pole sections such that given $\rho > 1$, the polydisk

$$\Delta^n_\rho: =' \{z \in \mathbb{C}^n: \ |z_j| < \rho, \quad j = 1,\ldots,n, \quad \rho > 1\}$$

overlaps with only a finite number of these pole sections.

Suppose $\pi_{\mu\nu}(z)$ is a unisolvent (μ,ν)-rational approximant to $f(z)$ such that for each fixed ν, the pole set of $\pi_{\mu\nu}(z)$ tends to infinity as $\mu \to (\infty,\ldots,\infty)$. Then

(i) $f(z)$ must be entire in $\mathbb{C}^n$

(ii) $\pi_{\mu\nu}(z) \to f(z)$ uniformly on every compact subset of $\mathbb{C}^n$.

THEOREM 2: Suppose the conditions of Theorem 1 are satisfied. Let K be any compact subset of $\mathbb{C}^n$. Let

$$e_{\mu\nu} = ||f(z) - \pi_{\mu\nu}(z)||_K = \sup_{z \in K} |f(z) - \pi_{\mu\nu}(z)| \tag{3.1}$$

for each fixed ν.

Then for sufficiently large ν, $e_{\mu\nu}$ is monotonic in ν and satisfies

$$e_{\mu,\nu+1} \le e_{\mu\nu} \quad \text{with} \quad \nu_j \le \nu_j + 1, \quad 1 \le j \le n .$$

LEMMA 1. Let ν be fixed and let $Q^*_{\mu\nu}(z)$ be a normalized denominator polynomial of $\pi_{\mu\nu}(z)$. The zero set of $Q^*_{\mu\nu}(z)$ tends to infinity as $\mu \to (\infty,\ldots,\infty) = Q^*_{\mu\nu}(z)$ tends to a constant.

PROOF. Suppose the result is false; i.e., for fixed $\nu, Q^{-1}{}_{\mu\nu}(0)$ tends to infinity, but $Q^*_{\mu\nu}(z)$ does not tend to a constant.

By Lemma 1 in [3], given $\rho > 1$ and a polydisk Δ^n_ρ, and μ sufficiently large,

$$Q^{-1}{}_{\mu\nu}(0) \cap \Delta^n_\rho = \emptyset \tag{3.2}$$

Suppose that $Q^*_{\mu\nu}(z) \to Q^*{}_m(z)$ is not constant as $\mu \to (\infty,\ldots,\infty)$ where $m = (m_1,\ldots,m_n)$ $m_i \le \nu_i$, $1 \le i \le n$ and that $Q^*{}_m(z)$ is a polynomial of multiple degree in less than ν in a partial ordered sense. Then since $Q^*{}_m(z)$ is non-constant, it has a set of non zero coefficients. Thus, $Q^{-1}{}_m(0)$, the zero set of $Q^*{}_m(z)$ cannot be empty. Now, taking $\rho_o > 1$, we find that

$$Q^{-1}{}_m(0) \cap \Delta^n_{\rho_o} \neq \emptyset \tag{3.3}$$

a contradiction. Hence the above supposition must be false and the Lemma holds.

PROOF OF THEOREM 1. f(z) is analytic at z = 0 and is possibly meromorphic with an infinite pole set

$$G = \bigcup_{k=1}^{\infty} G_{\sigma_k}$$

where

$$G_{\sigma_k} \subset G_{\sigma_{k+1}}$$

and

$$G_{\sigma_k} := \{z \in \mathbb{C}^n : q_{\sigma_k}(z) = 0\} .$$

$q_{\sigma_k}(z)$ is a polynomial of at most multiple degree,

$$\sigma_k = (\sigma_{k1}, \ldots, \sigma_{kn}) .$$

Given any real number $\rho > 1$, and a polydisk Δ^n_ρ, then $\exists\ k_o = k_o(\rho)$ such that the zero set $G_{\sigma_{k_o}}$ overlaps the polydisk Δ^n_ρ. Now, by Theorem 1 of [3], if we choose $\nu = \sigma_{k_o}$, then we must have on Δ^n_ρ as $\mu \to (\infty, \ldots, \infty)$

$$\Delta^n_\rho \cap Q^{-1}{}_{\mu\nu}(0) \to \Delta^n_\rho \cap G_{\sigma_{k_o}} \tag{3.4}$$

But by hypothesis, the pole set of $\pi_{\mu\nu}(z)$ tends to infinity as $\mu \to (\infty, \ldots, \infty)$ for each fixed ν. Therefore, for the given $\rho > 1$ above as $\mu \to (\infty, \ldots, \infty)$, we must have

$$\Delta^n_\rho \cap Q^{-1}{}_{\mu\nu}(0) = \emptyset \tag{3.5}$$

Thus by (3.4) and (3.5) we must have

$$\Delta^n_\rho \cap G_{\sigma_{k_o}} = \emptyset .$$

Since $k_o = k_o(\rho)$ and ρ is arbitrary, it follows that $G_{\sigma_{k_o}}$ must tend to infinity as $k_o \to \infty$. Hence, all the poles of $f(z)$ must tend to infinity and $f(z)$ must therefore be entire. This completes (i).

To prove (ii), we note that the result follows immediately from Theorem 1 of [3] and the (i) part just proved above.

PROOF OF THEOREM 2. Let K be any compact subset of $\mathbb{C}^n$. Then we can find $\rho > 1$ and a polydisk $\mathbb{C}^n$ such that $K \subset \Delta^n_\rho$. Then, for μ sufficiently large and $z \in K$, we find by the hypothesis of Theorem 1, that for each fixed ν,

$$Q^*_{\mu\nu}(z) \neq 0 \quad \text{i.e.} \qquad \delta > 0$$

such that

$$|Q^*_{\mu\nu}(z)| > \delta .$$

Hence, under these conditions, we get

$$||\pi_{\mu,\nu+1}(z) - \pi_{\mu\nu}(z)||_K \leq \frac{2||P^*_{\mu\nu}(z)||_K}{\delta^2} \; ||Q^*_{\mu,\nu+1}(z) - Q^*_{\mu\nu}(z)||_K$$

By Lemma 1, we know that $Q^*_{\mu\nu}(z)$ tends to a constant as $\mu \to (\infty,\ldots,\infty)$ for any fixed ν. Hence, given $\varepsilon > 0$, $\mu_0 = (\mu_{10}, \ldots, \mu_{n0})$ such that for $\mu_{i0} < \mu_i$, $1 \le i \le n$

$$||Q^*_{\mu,\nu+1}(z) - Q^*_{\mu\nu}(z)||_K < \varepsilon \frac{\delta^2}{2M_\rho} . \tag{3.7}$$

$M_\rho = ||P^*_{\mu\nu}(z)||_{\Delta^n_\rho} \ge ||P^*_{\mu\nu}(z)||_K$ by the maximum modulus principle, and M_ρ is dependent on ρ but independent of μ. Hence, by combining (3.6), (3.7) and (3.8) for each fixed ν and $\mu_{i0} < \mu_i$, $1 \le i \le n$, we obtain

$$||\pi_{\mu,\nu+1}(z) - \pi_{\mu\nu}(z)||_K < \varepsilon . \tag{3.8}$$

To get the desired inequality, we note by triangular for sup-norms on K that

$$e_{\mu,\nu+1} \le e_{\mu\nu} + ||\pi_{\mu,\nu+1}(z) - \pi_{\mu\nu}(z)||_K , \tag{3.9}$$

where we have used the definition of $e_{\mu\nu}$ as in (3.1).

For $\mu_{i0} < \mu_i$, $1 \le i \le n$, and for each fixed ν,

$$e_{\mu,\nu+1} < e_{\mu\nu} + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, the results follows.

ACKNOWLEDGEMENT. This paper was written while I was at the Mathematics Department, University of South Florida, Tampa, Florida.

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