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A GENERALIZATION OF CONTRACTION PRINCIPLE

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ABSTRACT: In this paper, a generalized mean value contraction is introduced. This contraction is an extension of the contractions of earlier researchers and of the generalized mean value non-expansive mapping. Using the generalized mean value contraction, some fixed point theorems are discussed.

KEY WORDS AND PHRASES: Fixed Point, Mean Value Iteration.

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1. INTRODUCTION.

Let T be a self mapping of a Banach space E. The mapping T will be called a generalized mean value contraction mapping if for any $x,y \in E$, there exist non-negative real numbers a_i $(i = 1,2,\dots 5)$ such that

$$||TT_\lambda x-TT_\lambda y|| \le a_1||x-y||+ a_2||x-TT_\lambda x||+ a_3||y-TT_\lambda y||+ a_4||x-TT_\lambda y||+ a_5||y-TT_\lambda x|| \tag{1.1}$$

where $\sum_{i=1}^{5} a_i < 1$ and $T_\lambda x = \lambda x + (1-\lambda)\, Tx$, and $TT_\lambda x = T(\lambda x + (1-\lambda)\, Tx)$, $0<\lambda\le 1$ holds.

The contraction (1.1) is more general than the Banach contraction, contractions of

Kannan [1], Chatterjee [2], Hardy and Rogers [3]. When $\lambda=1$ all these contractions follow as a particular case of (1.1), with suitable choice of a_i's. Also, by example, we show that there exist self-mappings which satisfy (1.1), but do not satisfy the well-known contraction just mentioned.

EXAMPLE 1. Let T be a self-mapping on [0,1] defined by

$$T(0) = 1,\ T(1) = 0,\ T(x) = \frac{1}{9}\ ,\ x \in (0,1)\ .$$

EXAMPLE 2. Let T be a self-mapping on [0,1] defined by $T(x) = 1-x$, $x \in [0,1]$.

EXAMPLE 3. Let T be a self-mapping on [-1,1] defined by $Tx = -x$, $x \in [-1,1]$.

The mapping T of the above examples satisfies (1.1) for $\lambda = \frac{1}{2}$. However, for $x=0$, $y=1$, T of Example 1 or Example 2, and for $x=1$, $y=-1$, T of Example 3 do not satisfy the above well-known contractions. Next, we define generalized mean value non-expansive mapping: Let T be a self-mapping of a Banach space E. Then T will be called a generalized mean value non-expansive mapping if for any x, y in E, there exists non-negative real numbers a_i $(i = 1, 2,\ldots 5)$ such that

$$||TT_\lambda x-TT_\lambda y|| \le a_1||x-y||+ a_2||x-TT_\lambda x||+ a_3||y-TT_\lambda y||+ a_4||x-TT_\lambda y||+ a_5||y-TT_\lambda x||, \tag{1.2}$$

where $\sum_{i=1}^{5} a_i = 1$ and $T_\lambda x = \lambda x + (1-\lambda)\ Tx$, $0 < \lambda \le 1$ holds.

Now we define a new contraction which is more general than (1.1) as follows:

Let X be subset of a normed linear space E. A mapping $T\colon\ X \to X$ is called an iteratively mean value contraction mapping if for every $x \in X$ there exist non-negative real numbers a, such that

$$||TT_\lambda\ (TT_\lambda x)\ -\ TT_\lambda x|| \le a||TT_\lambda x\ -\ x||, \tag{1.3}$$

where $0 < \lambda \le 1$ and $T_\lambda x = \lambda x + (1-\lambda)\ T_\lambda x$ and $TT_\lambda x = T\ (\lambda x + (1-\lambda)T\ x)$ holds.

The above definition is given because there are self-mappings of a subset of a normed linear space, which do not satisfy (1.1), but satisfies (1.3). An example of self-mapping for which (1.3) holds but (1.1) does not hold, is given below:

EXAMPLE 4. Let T be a self-mapping on [-1,7] defined by

$$Tx = -x,\ x\ \ [-1,1],\ Tx = \frac{6}{7} -x,\ x \in [1,7]\ .$$

2. MAIN THEOREMS.

THEOREM 1. Let T be a self-mapping of a normed linear space E. If

(i) T satisfies (1.1),

(ii) $\{x_n\}$ converges to $u \in E$ where $x_n = TT_\lambda x_{n-1}$ $(n=1,2,\ldots.)$ for any $x_0 \in E$,

(iii) $T(\lambda u + (1-\lambda)\ Tu) = \lambda Tu + (1-\lambda)\ T^2u$, only for u;

then T has a unique fixed point in E.

PROOF: Let x_0 be any point in E. Define, $x_n = TT_\lambda x_{n-1}$ $(n = 1,2,\ldots)$. Put $x_o = x$ and $x_1 = y$ in (1.1), then we have

$$\|x_1 - x_2\| \leq a_1\| x_0 - x_1\| + a_2\| x_0 - x_1\| + a_3\| x_1 - x_2\| + a_4\| x_0 - x_2\| , \quad (2.1)$$

Again, put $x_1 = x$ and $y = x_0$ in (1.1). Then

$$\|x_2 - x_1\| \leq a_1\| x_1 - x_0 \|+ a_2\| x_1 - x_2\| + a_3\| x_0 - x_1\| + a_5\| x_0 - x_2\| . \quad (2.2)$$

Adding (2.1) and (2.2), we obtain $\|x_2 - x_1\| \leq r \|x_1 - x_0\|$,

where $r = \dfrac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5}$ and $r < 1$, since $\sum_{i=1}^{5} a_i < 1$.

By induction it may be proved that $\|x_n - x_{n+1}\| \leq r^n\| x_1 - x_0\|$

It may be shown by routine calculation that $\{x_n\}$ is a Cauchy sequence. Hence $\{x_n\}$ is convergent. So, by (ii), $x_n \to u \in E$, as $n \to \infty$.

Now, $\|u - TT_\lambda u\| \leq \|u - x_{n+1}\| + \|TT_\lambda x_n - TT_\lambda u\|$

$\leq \|u-x_{n+1}\| +a_1\| x_n-u\| +a_2\| x_n-x_{n+1}\| +a_3\| u-TT_\lambda u\| +a_4\| x-TT_\lambda u\| +a_5\| u-x_{n+1}\|$

$\leq (a_3 + a_4) \|u - TT_\lambda u\|$, as $n \to \infty$.

Therefore, $(1 - a_3 - a_4) \|u - TT_\lambda u\| \leq 0$, which implies that $u = TT_\lambda u$, since $\sum_{i=1}^{5} a_i < 1$. Now, $Tu = T(TT_\lambda u) = T(T(\lambda u + (1-\lambda)\ Tu) = T(\lambda\ Tu + (1-\lambda)\ Tu^2)$, by

Therefore ,

$\|u - Tu\| = \|T(\lambda u + (1-\lambda)\ Tu) - T(\lambda Tu + (1-\lambda)\ T^2u)\| \leq r\| u - Tu\|$, by (i).

Since $r < 1$, $(1-r)\ \| u - Tu\| \leq 0$ implies $Tu = u$ i. e. u is a fixed point of T. Uniqueness of the fixed point follows easily.

THEOREM 2. Let T be a self-mapping of a bounded convex subset M of a normed linear space E. If for any $x \in M$,

(i) T satisfies (1.3)

(ii) $\{x_n\}$ converges to $u \in M$, whenever $\{x_n\}$ is convergent, where $x_n = TT_\lambda x_{n-1}$, $(n = 1,2,3,\ldots)$ for any $x_0 \in M$.

(iii) $\lim_{n\to\infty} T(\lambda x_n + (1-\lambda)\, T x_n) = T(\lambda \lim_{n\to\infty} x_n + (1-\lambda)\, T \lim_{n\to\infty} x_n)$

(iv) $T(\lambda u + (1-\lambda)\, Tu) = \lambda Tu + (1-\lambda)\, T^2 u$, for all u;

then T has a fixed point.

PROOF: Proof is exactly similar to that of Theorem 1, so we omit it.

THEOREM 3. Let E be a rotund Banach space, M be a compact convex subset of E and T be a self-mapping of M. If T is continuous and T satisfies (1.2) and $TT_\lambda x = T_\lambda Tx$ for any $x \in M$, then T has a fixed point in M.

PROOF: Let x be any point in M. Define $f(x) = \|x - Tx\|$. Since T and $\|.\|$ are continuous functions, therefore, f(x) is also continuous. So f(x) attains its minimum for some x(say $x = z \in M$).

First suppose $\|Tz - z\| = 0$, then z is a fixed point of T. Now let $\|Tz - z\| \neq 0$. Hence

$$f(TT_\lambda z) = \|TT_\lambda z - T(TT_\lambda z)\| = \|TT_\lambda z - TT_\lambda(Tz)\|$$
$$\leq \|z - Tz\| < \|z - Tz\|, \text{ since E is rotund.}$$
$$= f(z), \text{ which contradicts the minimality of } f(z).$$

Therefore $\|T(z) - z\| = 0$ i.e. $Tz = z$ is a fixed point of T.

THEOREM 4. Let E be a Banach space, M be a compact convex subset of E, and T be a continuous self-mapping of M. If for any x,y $(x \neq y) \in M$, T satisfies (1.1) (where $\leq$ is replaced by <) and $\sum_{i=1}^{5} a_i = 1$ and $TT_\lambda x = T_\lambda Tx$, then T has a unique fixed point in M.

PROOF: Proof is similar to that of Theorem 3.

3. CONCLUDING REMARKS.

(i) That the condition (iii) of Theorem 1 is necessary for existence of fixed point of T as illustrated by the following example.

EXAMPLE 4. Let T be a self-mapping on [0,1] defined by $Tx = 1 - x$, $x \in [0,1]$, $T(1) = 0$. Here T satisfies conditions (i) and (ii) of Theorem] for $\lambda < 1$, but it does not satisfy (iii) and T has no fixed point in [0,1].

(ii) The self-mapping T of Example 1 and Example 2 are non-expansive ($\|Tx - Ty\| \leq \|x - y\|$). Kirk [4] has proved the following fixed point theorem on non-expansive mapping:

"If K be a nonempty closed convex bounded subset of a reflexive Banach space X and if K possisses normal structure, then every non-expansive mapping from K into itself has a fixed point."

The same result is also established independently by Browder [5] in a uniformly convex Banach space. There is a close connection between the theorems of Kirk and Browder. This was first noted by Goebel [6] that if X be a uniformly convex Banach space, then any closed convex bounded subset K of X, must have normal structure.

We observe that for the existence of a fixed point of any non-expansive mapping in a Banach space, the Banach space must have a property either "uniform convexity" or "reflexivity with normal structure". Though self-mapping T in Example 1 and Example 2 are non-expansive, they are contractions in the sense (1.1). These mappings satisfy all the conditions of Theorem 1. Theorem 1 explains the existence of the fixed point of the above mappings without assuming "uniform convexity" or "reflexivity with normal structure".

These examples also suggest that non-expansive mappings may be converted into contraction mappings (general process of conversion is not known). Since the study of contraction mappings is easier than non-expansive mapping, so this type conversion has some importance in fixed point theory.

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