Internat. J. Math. & Math. Sci.
Vol. 4 No. 1 (1981) 155-164

ON THE CONSISTENCY OF LIMITATION METHODS FOR (N,p_n) SUMMABLE SEQUENCES

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(Received November 8, 1978 and in revised form March 12, 1979)

ABSTRACT. Two limitation methods, A and B, are said to be consistent for a class b of sequences, iff, every sequence belonging to b is limitable both by A and B and that the A-limit equals the B-limit. Any two regular limitation methods are consistent for the class-c of convergent sequences. However, this is not true in general and in fact, corresponding to every bounded non-convergent sequence it is possible to determine two T-matrices such that they limit the sequence to two different values. In this paper, we establish the necessary and sufficient conditions for the consistency of two limitation methods, for (N,p_n) summable sequences.

KEY WORDS AND PHRASES. *Summable sequence, Limitation methods, Infinite matrices.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES: 40A05.

1. INTRODUCTION.

Let $\{s_n\}$ be a sequence of real or complex terms. Let $A \equiv (a_{m\,n})$, $\infty \times \infty$, be an infinite matrix over a real or complex field. Then the transform, given by

$$T_m = \sum_{n=1}^{\infty} a_{m\,n} s_n, \tag{1.1}$$

if it exists for every m, is called the A-transform of the sequence $\{s_n\}$. If $\lim_{m \to \infty} T_m = s$, $\{s_n\}$ is said to be A-limitable to s. Moreover if $\lim_{n \to \infty} s_n = s$ implies $\lim_{m \to \infty} T_m = s$, the matrix A is said to be regular. In 1911, Toeplitz obtained necessary and sufficient conditions for a regular matrix as follows:

A Matrix $A \equiv (a_{m\,n})$ is regular, iff,

(i) $$\sup_m \sum_n |a_{m\,n}| \leq M, \text{ an absolute constant;} \tag{1.2}$$

(ii) $$\lim_{m \to \infty} a_{m\,n} = 0, \text{ for every fixed } n; \tag{1.3}$$

(iii) $$\sum_{n=1}^{\infty} a_{m\,n} = A_m \to 1 \text{ as } m \to \infty. \tag{1.4}$$

Let T be a class of matrices satisfying the conditions (1.2) to (1.4). Any matrix of the class T is called a Toeplitz matrix or simply a T-matrix. Thus, a matrix A is regular if it belongs to the class T.

Let $\{p_n\}$ be a sequence of constants, real or complex, such that $P_m = (p_o + p_1 + \ldots + p_m) \neq 0$, for any $m = 0, 1, 2, \ldots\ldots$ Then the limitation method for which

$$a_{m\,n} = \begin{cases} \dfrac{p_{m-n}}{P_m}, & \text{for } n \leq m \\ 0, & \text{for } n > m \end{cases} \tag{1.5}$$

is called the Nörlund method, or simply (N,p_n) method. A (N,p_n) method is regular, iff,

(i) $$\sum_{n=o}^{m} |p_n| = O(|P_n|), \text{ for all } m, \tag{1.6}$$

(ii) $$\lim_{m \to \infty} \frac{p_m}{P_m} = 0. \tag{1.7}$$

We use the following notations:

(i) $$p(x) = \Sigma p_n x^n; \tag{1.8}$$

(ii) $$\frac{1}{p(x)} = \Sigma c_n x^n; \tag{1.9}$$

(iii) $$\{p_n\} \in M, \text{ iff, } p_o = 1,\ p_n > 0 \text{ and } p_{n+1}^2 \leq p_n p_{n+2}; \tag{1.10}$$

(iv) $$t_m = \frac{1}{P_m} \sum_{n=o}^{m} p_{m-n}\, s_n,\ P_m \neq 0; \tag{1.11}$$

(v) Throughout the paper, M is taken for an absolute constant not necessarily the same at each occurence.

2. MAIN RESULTS.

Two limitation methods, A and B, are said to be consistent for a class b of sequences, iff, every sequence belonging to b is limitable both by A and B, and the A-limit is equal to the B-limit. Thus, any two matrix methods, generated by the matrices of the class T, are consistent for the class-c of convergent sequences. However, this is not true in general, and in fact, corresponding to every bounded non-convergent sequence it is always possible to determine two T-matrices such that they limit the sequence to two different values (see Cooke [1], page 97).

A limitation method Q is said to include a limitation method P if every sequence limitable by P is limitable by Q and to the same limit. Sometimes we indicate this by set theoretic inclusion as, $P \subseteq Q$, meaning thereby that space of sequences limitable by Q includes that limitable by P.

Two limitation methods, determined by the matrices $A \equiv (a_{m\,n})$ and $B \equiv (b_{m\,n})$ are said to be equivalent for a class b of sequences, iff, for every $\{s_n\} \in b$

$$\lim_{m \to \infty} (T_m - T_m^1) = 0, \tag{2.1}$$

where

$$T_m = \sum_{n=1}^{\infty} a_{m\,n}\, s_n \text{ and } T_m^1 = \sum_{n=1}^{\infty} b_{m\,n}\, s_n .$$

Hence two limitation methods A and B are consistent for a class b of sequences iff,

(i) A and B are equivalent, for class b, (2.2)

(ii) A and B limit every sequence $\in b$. (2.3)

Let b be the class of all sequences that are (N,p_n) summable. To ensure condition (2.2), we prove the following theorems in section 3.

THEOREM 1. Let (N,p_n) be a regular Nörlund method and let $\{p_n\} \in M$. The necessary and sufficient conditions that any two limitation methods A and B, determined respectively by the matrices $(a_{m\,n})$ and $(b_{m\,n})$, are equivalent for all such (N,p_n) summable sequences are

(i) $$\lim_{m \to \infty} \gamma_{m\,n} = 0, \text{ for all fixed } n; \tag{2.4}$$

(ii) $$\lim_{m \to \infty} \sum_n \gamma_{m\,n} = 0; \tag{2.5}$$

(iii) $$\sum_{k=0}^{\infty} P_k \left| \sum_{n=k}^{\infty} \gamma_{m\,n}\, c_{n-k} \right| \le M, \text{ for every } m, \tag{2.6}$$

where

$$(a_{m\,n}) - (b_{m\,n}) = (\gamma_{m\,n}).$$

Condition (2.3) implies that $(N,p_n) \subseteq A$ and $(N,p_n) \subseteq B$. In section 4, we prove the following theorem:

THEOREM 2. Let (N,p_n) be a regular Nörlund method and let $\{p_n\}_1 \in M$. Then the limitation method A, determined by the matrix $A \equiv (a_{m\,n})$, $\infty \times \infty$, belonging to the class T, includes (N,p_n), iff,

$$\sup_m \sum_{k=o}^{\infty} P_k \left| \sum_{n=k}^{\infty} a_{m\,n}\, c_{n-k} \right| \leq M, \tag{2.7}$$

where c_k is as defined in (1.9).

We required the following lemma of Kaluza (see Hardy [2], page 68) in proving our theorem.

LEMMA. If $p(x) = \Sigma p_n x^n$ is convergent for $|x| < 1$, and $\{p_n\}_1 \in M$, and further

$$p(x)^{-1} = 1 + c_1 x + c_2 x^2 + \ldots\ldots,$$

then

$$\Sigma |c_n| \leq 2. \quad \text{If } \Sigma p_n = \infty, \text{ then } \sum_{n=1}^{\infty} |c_n| = 2.$$

3. PROOF OF THEOREM 1.

At the outset we observe that if (N,p_n) is regular and $p_n \geq 0$, for each n, then in view of the regularity condition, $\lim_{n \to \infty} \frac{P_{n+1}}{P_n} = 1$ and the series $\Sigma\, p_n x^n$ is absolutely convergent for $|x| < 1$, as such the series

$$\sum_{n=o}^{\infty} P_n x^n (1 - x), \tag{3.1}$$

is also so, for $|x| < 1$. But then the series (3.1) equals $\Sigma p_n x^n$, and accordingly the series $\Sigma p_n x^n$ is absolutely convergent for $|x| < 1$.

Now we lay down the proof.

(If part): We have

$$t_n = \frac{1}{P_n} \sum_{k=o}^{n} p_{n-k} s_k.$$

Then, for $|x| < 1$, we have

$$\sum_{n=o}^{\infty} t_n P_n x^n = \sum_{n=o}^{\infty} \left(\sum_{k=o}^{n} p_{n-k}\, s_k\right) x^n = p(x)\, s(x).$$

Hence

$$s(x) = \frac{1}{p(x)} \sum_{n=o}^{\infty} t_n p_n x^n$$

$$= \sum_{n=o}^{\infty} \left(\sum_{k=o}^{n} t_k p_k c_{n-k}\right) x^n$$

Equating the coefficients of x^n, from both sides, we have

$$s_n = \sum_{k=o}^{n} t_k p_k c_{n-k}. \tag{3.2}$$

Now

$$\sum_{n=o}^{\infty} \gamma_{m\,n}\, s_n = \sum_{n=o}^{\infty} \gamma_{m\,n} \sum_{k=o}^{n} p_k t_k c_{n-k}$$

$$= \sum_{k=o}^{\infty} t_k p_k \sum_{n=k}^{\infty} \gamma_{m\,n}\, c_{n-k}$$

Let us put $t_k = t + \epsilon_k$, where $t = \lim_{k \to \infty} t_k$ and $\{\epsilon_k\}$ is a null sequence. Then

$$\sum_{n=o}^{\infty} \gamma_{m\,n}\, s_n = t \sum_{k=o}^{\infty} P_k \sum_{n=k}^{\infty} \gamma_{m\,n}\, c_{n-k} + \sum_{k=o}^{\infty} \left(P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k}\right) \epsilon_k$$

$$= t \sum_{k=o}^{\infty} \gamma_{m\,n} \sum_{k=o}^{n} P_k c_{n-k} + \sum_{k=o}^{\infty} \left(P_k \sum_{n=k}^{\infty} \gamma_{m\,n}\, c_{n-k}\right) \epsilon_k$$

$$= t \sum_{k=o}^{\infty} \gamma_{m\,n} + \sum_{k=o}^{\infty} \left(P_k \sum_{n=k}^{\infty} \gamma_{m\,n}\, c_{n-k}\right) \epsilon_k.$$

Taking the limit as $m \to \infty$ and making use of condition (2.5), we have

$$\lim_{m \to \infty} \sum_{n=o}^{\infty} \gamma_{m\,n}\, s_n = \lim_{m \to \infty} \sum_{k=o}^{\infty} \left(P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k}\right) \epsilon_k$$

$$= \lim_{m \to \infty} \sum_{k=o}^{\infty} d_{m\,k}\, \epsilon_k, \text{ say,}$$

where

$$d_{m\,k} = p_k \sum_{n=k}^{\infty} \gamma_{m\,n}\, c_{n-k}$$

Hence the proof is completed if we show that

$$\lim_{m \to \infty} \sum d_{mk}\, \epsilon_k = 0.$$

But since $\{\epsilon_k\} \to 0$ as $k \to \infty$, it is sufficient to show

(i) $$\sum_k |d_{m\,k}| < M, \text{ for every } m; \tag{3.3}$$

(ii) $$\lim_{m \to \infty} d_{m\,k} = 0, \text{ for every fixed } k. \tag{3.4}$$

Here (3.3) follows from (2.6). For establishing (3.4), it is sufficient to show that

$$\lim_{m \to \infty} \left(\sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k}\right) = 0, \tag{3.5}$$

for every fixed k.

Now since $\Sigma |c_n|$ is convergent, for any arbitrary small $\epsilon > 0$, we can have n_o such that

$$\sum_{n>n_o} |c_n| < \frac{\epsilon}{2M}, \tag{3.6}$$

where M is as specified in (2.6).

Further let

$$\sum_{n\le n_o} |c_n| \le A, \tag{3.7}$$

where A is a finite constant. In view of (2.4) and the fact that n_o is a finite positive integer, we can have m_o dependent on ϵ such that

$$|\gamma_{m\,n}| < \frac{\epsilon}{2A}, \tag{3.8}$$

for all $m > m_o(\epsilon)$ and all $n=k, k+1, \ldots, n_o + k$. Also

$$\begin{aligned} |\gamma_{m\,h}| &= \left|\sum_{n=h}^{\infty} \gamma_{m\,n} \sum_{k=o}^{n-h} p_k c_{n-h-k}\right| \\ &= \left|\sum_{k=o}^{\infty} p_k \sum_{n=h+k}^{\infty} \gamma_{m\,n} c_{n-h-k}\right| \\ &\le \sum_{k=o}^{\infty} P_{k+h} \left|\sum_{n=h+k}^{\infty} \gamma_{m\,n} c_{n-h-k}\right| \\ &\le \sum_{v=h}^{\infty} P_v \left|\sum_{n=v}^{\infty} \bar{\gamma}_{m\,n} c_{n-v}\right| \\ &\le M, \text{ by (2.6)}, \end{aligned} \tag{3.9}$$

for all m and h. Hence, finally

$$\begin{aligned} \left|\sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k}\right| &= \left|\sum_{n=o}^{\infty} \gamma_{m,n+k} c_n\right| \\ &\le \left|\sum_{n=o}^{n_o} \gamma_{m,n+k} c_n\right| + \left|\sum_{n>n_o} \gamma_{m,n+k} c_n\right| \\ &\le \frac{\epsilon}{2A} \sum_{n=o}^{n_o} |c_n| + M \sum_{n>n_o} |c_n|, \text{ for } m > m_o, \text{ by (3.8) and (3.9)}. \\ &\le \frac{\epsilon}{2A} A + M \frac{\epsilon}{2M}, \text{ by (3.7) and (3.6)}. \\ &= \epsilon\,. \end{aligned} \tag{3.10}$$

Hence $\lim_{m\to\infty} d_{m\,k} = 0$, for every fixed k, and the "if part" of the theorem is proved.

(Only if part):

Let the limitation method of A and B be equivalent for all (N,p_n) summable sequences. Then

$$\lim_{m\to\infty} \sum_n \gamma_{m\,n}\, s_n = 0,$$

where $(a_{m\,n}) - (b_{m\,n}) = (\gamma_{m\,n})$, and $\{s_n\}$ is a (N,p_n) summable sequence.

We have

$$\sum_{n=o}^{\infty} \gamma_{m\,n} s_n = \sum_{k=o}^{\infty} (P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k})\, t_k.$$

As $\{s_n\}$ is a (N,p_n) summable sequence,

(i) take $\{s_n\} = \{\delta_n^h\}$, so that $t_k = \frac{p_{k-h}}{P_k}$, for every $k > h$.

Then

$$\begin{aligned}\sum_n \gamma_{m\,n} s_n &= \sum_{k=o}^{\infty} p_{k-h} \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k}\\ &= \sum_{n=o}^{\infty} \gamma_{m\,n} \sum_{k=o}^{n} p_{k-h} c_{n-k}\\ &= \sum_{n=o}^{\infty} \gamma_{m\,n} \sum_{k=o}^{n-h} p_k\, c_{n-k}\\ &= \lambda_{m\,n}\end{aligned}$$

Hence

$$\lim_{m\to\infty} \sum_n \gamma_{m\,n} s_n = \lim_{m\to\infty} \gamma_{m\,n} = 0, \text{ for every fixed } h. \qquad (3.11)$$

Thus hypothesis (2.4) is necessary.

(ii) Take $s_n = 1$ in $\{s_n\}$, so that $t_k = 1$, for every k. Then

$$\begin{aligned}\lim_{m\to\infty} \sum_n \gamma_{m\,n} s_n &= \lim_{m\to\infty} \sum_{k=o}^{\infty} P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k}\\ &= \lim_{m\to\infty} \sum_{n=o}^{\infty} \gamma_{m\,n} \sum_{k=o}^{n} p_k c_{n-k}\\ &= \lim_{m\to\infty} \sum_{n=o} \gamma_{m\,n} = 0, \qquad (3.12)\end{aligned}$$

which is hypothesis (2.5).

(iii) Take $t_k = t + \epsilon_k$, where $\{\epsilon_k\}$ is a null sequence. Then

$$\begin{aligned}\lim_{m\to\infty} \sum_n \gamma_{m\,n} s_n &= \lim_{m\to\infty} t \sum_{k=o}^{\infty} P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k} + \lim_{m\to\infty} \sum_{k=o}^{\infty} (P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k})\, \epsilon_k\\ &= \lim_{m\to\infty} \sum_{k=o}^{\infty} (P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k})\, \epsilon_k, \text{ by (3.12).}\end{aligned}$$

But $\lim_{m \to \infty} \sum_n \gamma_{m\,n} s_n = 0$, for all sequences $\{s_n\}$ that are (N,p_n) summable, as such

$$\lim_{m \to \infty} \sum_{k=o}^{\infty} (P_k \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k}) \epsilon_k = 0,$$

for all null sequences $\{\epsilon_k\}$.

Hence

$$\sup_m \sum_k P_k \left| \sum_{n=k}^{\infty} \gamma_{m\,n} c_{n-k} \right| \leq M$$

Thus Theorem 1 is established.

4. PROOF OF THEOREM 2.

(If part): We have

$$T_m = \sum_{n=o}^{\infty} a_{m\,n} s_n$$

$$= \sum_{n=o}^{\infty} a_{m\,n} \sum_{k=o}^{n} t_k P_\kappa c_{n-k}, \text{ by (3.2)},$$

$$= \sum_{k=o}^{\infty} t_k P_k \sum_{n=k}^{\infty} a_{m\,n} c_{n-k} = \sum_{k=o}^{\infty} b_{m\,k} t_k, \text{ say},$$

where

$$b_{m\,k} = P_k \sum_{n=k}^{\infty} a_{m\,n} c_{n-k}. \tag{4.1}$$

In order to establish this part, it is sufficient to show that $(b_{m\,k})$ is regular, that is, it belongs to T.

Clearly

$$\sum_k |b_{m\,k}| = \sum_k P_k \left|\sum_{n=k}^{\infty} a_{m\,n} c_{n-k}\right| \leq M, \text{ by (2.7)}.$$

Since $(a_{m\,n}) \in T$, for every fixed positive integer n and $\epsilon > 0$,

$$|a_{m\,n}| < \epsilon, \text{ for all } m > m_o (n, \epsilon), \tag{4.2}$$

and also

$$\sup_{m,n} |a_{m,n}| \leq M. \tag{4.3}$$

Now, making use of (4.2) and (4.3), and proceeding along the lines of (3.10), we can easily establish that

$$\lim_{m \to \infty} b_{m\,k} = \lim_{m \to \infty} P_k \sum_{n=k}^{\infty} a_{m\,n} c_{n-k} = 0, \text{ for every fixed k}.$$

Finally, since

$$\sum_{k=o}^{n} P_k c_{n-k} = 1, \text{ for every n},$$

$$\sum_{k=o}^{\infty} b_{m\,k} = \sum_{k=o}^{\infty} P_k \sum_{n=k}^{\infty} a_{m\,n} c_{n-k} = \sum_{n=o}^{\infty} a_{m\,n} \sum_{k=o}^{n} P_k c_{n-k}$$

$$= \sum_{n=o}^{\infty} a_{m\,n} = A_m \to 1, \text{ as } m \to \infty.$$

Hence $(b_{m\,k})$ satisfies all the regularity conditions. This proves the "if part" of Theorem 2.

<u>(Only if part)</u>: We have

$$T_m = \sum_{n=o}^{\infty} a_{m\,n} s_n$$

$$= \sum_{n=o}^{\infty} a_{m\,n} \sum_{k=o}^{\infty} t_k P_k c_{n-k}$$

$$= \sum_{k=o}^{\infty} t_k P_k \sum_{n=k}^{\infty} a_{m\,n} c_{n-k}$$

$$= \sum_{k=o}^{\infty} b_{m\,k} t_k, \text{ say.}$$

Since A includes (N,p_n), $(b_{m\,k})$ is regular. Thus

$$\sum_{k=o}^{\infty} \left| b_{m\,k} \right| = \sum_{k=o}^{\infty} P_k \left| \sum_{n=k}^{\infty} a_{m\,n} c_{n-k} \right| \le M,$$

which is the required condition.

This completes the proof of Theorem 2.

5. CLOSING REMARKS.

Theorem 1 generalizes the result of Zaman [3]. It assumes a much simplified form if A and B belong to the class T and we have:

THEOREM 3. Let (N,p_n) be a regular Nörlund method and let $\{p_n\} \in M$. Then a necessary and sufficient condition that any two limitation methods, determined by the matrices $A \equiv (a_{m\,n})$ and $B \equiv (b_{m\,n})$, belonging to the class T, are equivalent for all such (N,p_n) summable sequences, is that (2.6) hold.

Theorem 2 and 3 together lead to the following Theorem of consistency of matrix limitation methods for (N,p_n) summable sequences.

THEOREM 4. Let (N,p_n) be a regular Nörlund method and let $\{p_n\} \in M$. Then the necessary and sufficient conditions that any two limitation methods, determined by the matrices of the class T, are consistent for all (N,p_n) summable sequences are that they include (N,p_n).

<u>ACKNOWLEDGEMENT</u>. Authors are indebted to the referee for his valuable criticism and useful suggestions on the original manuscript.

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