Internat. J. Math. & Math. Sci.
Vol. 3 No. 4 (1980) 731-738

SOME TAUBERIAN THEOREMS FOR EULER AND BOREL SUMMABILITY

J. A. FRIDY

Department of Mathematics
Kent State University
Kent, Ohio 44252 U.S.A.

K. L. ROBERTS

Department of Mathematics
The University of Western Ontario
London, Ontario
Canada

(Received August 6, 1979 and in revised form December 7, 1979)

ABSTRACT. The well-known summability methods of Euler and Borel are studied as mappings from ℓ^1 into ℓ^1. In this ℓ-ℓ setting, the following Tauberian results are proved: if x is a sequence that is mapped into ℓ^1 by the Euler-Knopp method E_r with $r > 0$ (or the Borel matrix method) and x satisfies $\Sigma_{n=0}^{\infty}|x_n - x_{n+1}|\sqrt{n} < \infty$, then x itself is in ℓ^1.

KEY WORDS AND PHRASES. *Tauberian condition, ℓ-ℓ method, Euler-Knopp means, Borel exponential method.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES: *Primary - 40E05, 40G05, 40G10; Secondary - 40D25.*

1. INTRODUCTION.

In [2, p. 121], G. H. Hardy described a Tauberian theorem as one which asserts that a particular summability method cannot sum a divergent series that oscillates too slowly. In this paper we shall state the results in sequence-to-sequence form, so a typical order-type Tauberian theorem for a method A would have the form, "if x is a sequence such that Ax is convergent and $\Delta x_k = x_k - x_{k+1} = o(d_k)$, then x itself is convergent." Our present task is not to give more theorems in the setting of ordinary convergence, but rather, we shall develop analogous results for methods that map ℓ^1 into ℓ^1. Such a transformation is called an ℓ-ℓ method, and we shall henceforth write ℓ for ℓ^1. In [5] Knopp and Lorentz proved that the matrix A determines an ℓ-ℓ method if and only if $\sup_k \Sigma_{n=0}^{\infty}|a_{nk}| < \infty$.

In order to prove Tauberian theorems in an ℓ-ℓ setting, it is necessary to formulate an ℓ-ℓ analogue of the above Tauberian condition $\Delta x_k = o(d_k)$. Since this condition means that $\Delta x/d$ is in c_0, an ℓ-ℓ analogue would be "$\Delta x/d$ is in ℓ," which we shall write in series form as $\Sigma_{k=0}^{\infty}|\Delta x_k|/d_k < \infty$.

2. EULER-KNOPP AND BOREL ℓ-ℓ METHODS.

The Euler-Knopp means [6, pp. 56-60] are given by the matrix

$$E_r[n,k] = \begin{cases} \binom{n}{k} r^k (1-r)^{n-k}, & \text{if } k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

In [1, Theorem 4] it is shown that E_r determines an ℓ-ℓ method if and only if $0 < r \le 1$. Moreover, for such r, $E_r^{-1}[\ell] \ne \ell$.

The customary form of Borel exponential summability is the sequence-to-function transformation ([2, p. 182], [6, p. 54]) given by

$$\text{if } \lim_{t\to\infty}\{e^{-t}\Sigma_{k=0}^{\infty} x_k t^k/k!\} = L, \text{ then x is Borel summable to L.}$$

In order to consider this method in an ℓ-ℓ setting, we must modify it into a sequence-to-sequence transformation. This can be achieved by letting t tend to ∞ through integer values and considering the resulting sequence Bx. Then B is the Borel matrix method [6, p. 56], which is given by the matrix

$$b_{nk} = e^{-n}n^k/k!.$$

By a direct application of the Knopp-Lorentz Theorem [5], one can show that B is an ℓ-ℓ matrix. We shall not use this direct approach, however, because the assertion will follow from our first theorem, which is an inclusion theorem between B and E_r.

THEOREM 1. If $r > 0$ and x is a sequence such that $E_r x$ is in ℓ, then Bx is in ℓ.

PROOF. We use the familiar technique of showing that BE_r^{-1} is an ℓ-ℓ matrix. Since $Bx = (BE_r^{-1})E_r x$, this will ensure that Bx is in ℓ whenever $E_r x$ is in ℓ. Since $E_r^{-1} = E_{1/r}$, we replace $1/r$ by s and show that BE_s is an ℓ-ℓ matrix for all positive s. The n,k-th term of BE_s is given by

$$\begin{aligned} BE_s[n,k] &= \Sigma_{j=k}^{\infty} \frac{e^{-n}n^j}{j!}\binom{j}{k}(1 - s)^{j-k}s^k \\ &= \frac{e^{-n}n^k s^k}{k!}\Sigma_{j=k}^{\infty} \frac{n^{j-k}}{(j-k)!}(1 - s)^{j-k} \\ &= \frac{(ns)^k e^{-ns}}{k!}. \end{aligned}$$

Summing the k-th column of BE_s, we get

$$\begin{aligned} \Sigma_{n=0}^{\infty} |BE_s[n,k]| &= \frac{1}{k!}\Sigma_{n=0}^{\infty}(ns)^k e^{-ns} \\ &= 0\left(\frac{1}{k!}\int_0^{\infty}(ts)^k e^{-ts}dt\right) \\ &= 0(1/s). \end{aligned}$$

Hence, $\sup_k \Sigma_{n=0}^{\infty} |BE_s[n,k]| < \infty$, so BE_s is an ℓ-ℓ matrix.

Combining Theorem 1 with the knowledge that E_r is an ℓ-ℓ matrix, we get the following result as an immediate corollary.

THEOREM 2. The Borel matrix B determines an ℓ-ℓ method.

In addition to the inclusion relation given in Theorem 1, we can show that the ℓ-ℓ method B is <u>strictly</u> stronger than all E_r methods by the following example.

EXAMPLE. Suppose $r > 0$ and $x_k = (-s)^k$, where $s \geq -1 + 2/r$; then Bx is in ℓ but $E_r x$ is not in ℓ. For,

$$(Bx)_n = \Sigma_{k=0}^{\infty} e^{-n} \frac{n^k}{k!}(-s)^k = e^{-n}e^{-sn} = e^{-n(s+1)},$$

and

$$(E_r x)_n = \Sigma_{k=0}^{n} \binom{n}{k}(1 - r)^{n-k}(-rs)^k = (1 - r - rs)^n.$$

By solving $-1 < 1 - r' - rs < 1$, we see that $E_r x$ is in ℓ if and only if $-1 < s < -1 + 2/r$.

3. <u>TAUBERIAN THEOREMS</u>.

We are now ready to prove the principal results which show that B and E_r can not map a sequence from $\sim\ell$ into ℓ if the sequence oscillates too slowly.

THEOREM 3. If x is a sequence such that Bx is in ℓ and

$$(*) \qquad \Sigma_{r=0}^{\infty} |\Delta x_r|\sqrt{r} < \infty,$$

then x is in ℓ.

PROOF. It suffices to show that $Bx - x$ is in ℓ; that is, $\Sigma_{n=0}^{\infty}|\Sigma_{k=0}^{\infty} b_{nk}x_k - x_n| < \infty$. Since $\Sigma_{k=0}^{\infty} b_{nk} = 1$ for each n, this sum can be written as $\Sigma_{n=0}^{\infty}|\Sigma_{k=0}^{\infty} b_{nk}(x_k - x_n)|$, and so it suffices to show that $A = \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{\infty} b_{nk}|x_k - x_n| < \infty$.

We can write $A = C + D$, where

$$C = \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{\infty} b_{nk}|x_k - x_n|$$

and

$$D = \Sigma_{n=0}^{\infty}\Sigma_{k=n+1}^{\infty} b_{nk}|x_k - x_n|.$$

Then

$$C \leq \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{n-1} b_{nk}\Sigma_{r=k}^{n-1} |\Delta x_r|$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|\Sigma_{n=r+1}^{\infty}\Sigma_{k=0}^{r} b_{nk}$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|C_r, \text{ say.}$$

Also,

$$D \leq \Sigma_{n=0}^{\infty}\Sigma_{k=n+1}^{\infty} b_{nk}\Sigma_{r=n}^{k-1} |\Delta x_r|$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|\Sigma_{n=0}^{r}\Sigma_{k=r+1}^{\infty} b_{nk}$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|D_r, \text{ say.}$$

By the Lemma following, $C_r = O(\sqrt{r})$ and $D_r = O(\sqrt{r})$, so

$$C + D \leq H\, \Sigma_{r=0}^{\infty} |\Delta x_r|\sqrt{r} < \infty,$$

which proves the theorem.

LEMMA. If $b_{nk} = e^{-n}n^k/k!$ and r is a positive integer, then

(i) $$\Sigma_{n=r+1}^{\infty}\Sigma_{k=0}^{r} b_{nk} = O(\sqrt{r}),$$

and

(ii) $$\Sigma_{n=0}^{r}\Sigma_{k=r+1}^{\infty} b_{nk} = O(\sqrt{r}).$$

PROOF. Let $p = [\sqrt{r}]$, and write the sum in (i) as

$$\Sigma_{n=r+1}^{\infty}\Sigma_{k=0}^{r-p} b_{nk} + \Sigma_{n=r+1}^{\infty}\Sigma_{k=r-p+1}^{r} b_{nk} = F_r + G_r, \text{ say.}$$

If $s < n$, then (cf. [2, p. 202])

$$\Sigma_{k=0}^{s} \frac{n^k}{k!} = \frac{n^s}{s!}(1 + \frac{s}{n} + \frac{s}{n}\frac{s-1}{n} + \cdots)$$

$$\leq \frac{n^s}{s!}(1 + \frac{s}{n} + (\frac{s}{n})^2 + \cdots)$$

$$= \frac{n^s}{s!}(\frac{n}{n-s}).$$

In F_r we have $s = r - p$ and

$$\max_{n\geq r+1} \frac{n}{n-r+p} = \frac{r+1}{p+1} \leq \sqrt{r} + 1,$$

so

$$F_r < (\sqrt{r} + 1) \frac{1}{(r-p)!} \Sigma_{n=r+1}^{\infty} e^{-n}n^{r-p} \leq \sqrt{r} + 1.$$

In G_r we have

$$\Sigma_{k=r-p+1}^{r} b_{nk} \leq \sqrt{r} \max_{k\leq r} b_{nk} = \sqrt{r}\ e^{-n} \frac{n^r}{r!},$$

so

$$G_r \leq \sqrt{r} \frac{1}{r!} \Sigma_{n=r+1}^{\infty} e^{-n}n^r \leq \sqrt{r}.$$

Hence, (i) is proved.

Next write the sum in (ii) as

$$\Sigma_{n=0}^{r}\Sigma_{k=r+1}^{r+p-1} b_{nk} + \Sigma_{n=0}^{r}\Sigma_{k=r+p}^{r} b_{nk} = H_r + I_r, \text{ say.}$$

(Assume that $H_r = 0$ if $p = 1$.) Then

$$H_r \leq (p - 1) \Sigma_{n=0}^{r} e^{-n} \max_{k>r} \frac{n^k}{k!}$$

$$\leq (\sqrt{r} - 1) \frac{1}{(r+1)!} \Sigma_{n=0}^{r} e^{-n}n^{r+1}$$

$$\leq \sqrt{r} - 1.$$

If $s \geq n$, then

$$\Sigma_{k=s}^{\infty} \frac{n^k}{k!} = \frac{n^s}{s!}(1 + \frac{n}{s+1} + \frac{n}{s+1}\frac{n}{s+2} + \cdots)$$

$$\leq \frac{n^s}{s!}(1 + \frac{n}{s+1} + (\frac{n}{s+1})^2 + \cdots)$$

$$= \frac{n^s}{s!}\left(\frac{s+1}{s+1-n}\right).$$

Taking $s = r + p$, we have

$$I_r \le \frac{1}{(r+p)!} \Sigma_{n=0}^{r} e^{-n} n^{r+p} \left(\frac{r+p+1}{r+p+1-n}\right)$$

$$\le \left(\frac{r+p+1}{p+1}\right) \frac{1}{(r+p)!} \Sigma_{n=0}^{r} e^{-n} n^{r+p}$$

$$\le \sqrt{r} + 1.$$

This completes the proof of the Lemma.

By combining Theorem 3 with Theorem 1, we get an ℓ-ℓ Tauberian theorem for the Euler-Knopp means.

THEOREM 4. If $r > 0$ and x is a sequence satisfying (*) such that $E_r x$ is in ℓ, then x is in ℓ.

Next we give an application of these Tauberian theorems.

EXAMPLE. The following sequence is not mapped into ℓ by B -- or, a fortiori, by E_r, with $r > 0$. Define x by

$$x_0 = \pi^2/6 \quad \text{and} \quad \Delta x_j = 1/(j+1)^2.$$

Then x satisfies (*), but x is not in ℓ because if $k \geq 1$,

$$x_k = x_0 - \Sigma_{j=0}^{k-1} \Delta x_j = \frac{\pi^2}{6} - \Sigma_{m=1}^{k} m^{-2}$$

$$= \Sigma_{m\geq k+1} m^{-2} \sim 1/k.$$

Hence, by Theorem 3, Bx is not in ℓ.

It is possible -- but much more tedious -- to construct a real number sequence x such that $\Delta x_j = \pm(j+1)^{-2}$ and x changes sign infinitely many times, yet x is not in ℓ. For such an x, Theorem 3 implies that Bx cannot be in ℓ.

REFERENCES

1. Fridy, J. A., Absolute summability matrices that are stronger than the identity mapping, Proc. Amer. Math. Soc. 47(1975) 112-118.

2. Hardy, G. H., Divergent Series, Clarendon Press, Oxford, 1949.

3. Hardy, G. H. and J. E. Littlewood, Theorems concerning the summability of series by Borel's exponential method, Rendiconti Palermo, 41(1916) 36-53.

4. Hardy, G. H. and J. E. Littlewood, On the Tauberian theorem for Borel summability, J. London Math. Soc. 18(1943) 194-200.

5. Knopp, K. and G. G. Lorentz, Beiträge zur absoluten Limitierung, Arch. Math. 2(1949) 10-16.

6. Powell, R. E. and S. M. Shah, Summability Theory and its Applications, Van Nostrand Reinhold, London, 1972.