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NOTES ON WHITEHEAD SPACE OF AN ALGEBRA

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Let R be a ring, and denote by $[R,R]$ the group generated additively by the additive commutators of R. When $R_n = M_n(R)$ (the ring of $n\times n$ matrices over R), it is shown that $[R_n,R_n]$ is the kernel of the regular trace function modulo $[R,R]$. Then considering R as a simple left Artinian F-central algebra which is algebraic over F with $\operatorname{Char} F = 0$, it is shown that R can decompose over $[R,R]$, as $R = Fx+[R,R]$, for a fixed element $x \in R$. The space $R/[R,R]$ over F is known as the Whitehead space of R. When R is a semisimple central F-algebra, the dimension of its Whitehead space reveals the number of simple components of R. More precisely, we show that when R is algebraic over F and $\operatorname{Char} F = 0$, then the number of simple components of R is greater than or equal to $\dim_F R/[R,R]$, and when R is finite dimensional over F or is locally finite over F in the case of $\operatorname{Char} F = 0$, then the number of simple components of R is equal to $\dim_F R/[R,R]$.

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1. Introduction. Additive commutator elements of a ring R and the groups and structures they make have a great role in the general specification of a ring, and their study is one of the approaches to recognize rings in noncommutative ring theory [2, 3, 4, 5]. The reason is clear, they have covered the secrets of noncommutative behaviour of the structure. In recent years, these elements are returned once again under a full consideration, and a lot of wonderful works has been done on them [1, 10, 11, 12, 13]. Our study here is also among these studies, and it reveals some of bilateral relations between substructure given by additive commutators (the additive commutator group $[R,R]$, the additive Whitehead group, and the space $R/[R,R]$) and some characteristics of the ring. In what follows let R be a ring. By $[R,R]$ we denote the group generated additively by the additive commutators of R. Following [2], the additive group $R/[R,R]$ is called the additive Whitehead group of R. This group is an F-vector space when R is a central F-algebra, and is called the Whitehead space of R.

2. Results. Our first result is about the additive commutator subgroup of a matrix ring over a given ring.

Proposition 2.1. *Let R be a unitary ring and let $R_n = M_n(R)$ be the ring of $n\times n$ matrices over R. Consider the regular trace function on R_n, as $\operatorname{tr}: R_n \to R$, then*

$$[R_n,R_n] = \{A \in R_n \mid \operatorname{tr}(A) \in [R,R]\}. \tag{2.1}$$

Proof. The inclusion "$\subseteq$" follows by the fact that $\operatorname{tr}(AB-BA) \in [R,R]$. In order to show the reverse inclusion, let $\{E_{ij}\}$ be the matrix units and note that if $i \neq j$, we have $E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii} \in [R_n,R_n]$ and $E_{ii}-E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij} \in [R_n,R_n]$. For any

$A = (a_{ij}) \in R_n$, we have the following congruence:

$$A = \Sigma a_{ij}E_{ij} \equiv \Sigma a_{ii}E_{ii} \equiv \Sigma a_{ii}E_{11} (\operatorname{mod}[R_n,R_n]). \tag{2.2}$$

In particular, if $\operatorname{tr}(A) \in [R,R]$, then $A \in [R_n,R_n]$. □

Corollary 2.2. *Consider the trace function on R_n module of $[R,R]$. Clearly the group isomorphism $R_n/[R_n,R_n] \cong R/[R,R]$ can be derived.*

Theorem 2.3. *Let R be a left Artinian central simple F-algebra which is algebraic over F with $\operatorname{Char} F = 0$. Then R decomposes over $[R,R]$ as $R = Fx + [R,R]$, for a fixed $x \in R$.*

Proof. By Wedderburn-Artin theorem, $R = M_n(D)$ for a division ring D and suitable $n \in \mathbb{N}$ [6, 14]. We divide our proof into two parts.

(i) Let $n = 1$, in other words let $R = M_1(D) = D$ be a division ring. Let $a \in R$ and let $f(t) = t^r + b_1t^{r-1} + \cdots + b_r$ be the minimal polynomial of a over F, where $b_i \in F$, $i = 1,2,\ldots,r$ and $r = \dim_F F(a)$. By the Wedderburn theorem [9, page 265], $f(t)$ splits completely in $R[t]$, this means that there exists $c_i \in R^* = D - \{0\}$, $i = 1,2,\ldots,r-1$, such that $f(t) = (t-a)(t-c_1ac_1^{-1})\cdots(t-c_{r-1}ac_{r-1}^{-1})$. Then we have

$$\begin{aligned} \operatorname{Tr}_{F(a)/F}(a) &= a + c_1ac_1^{-1} + c_2ac_2^{-1} + \cdots + c_{r-1}ac_{r-1}^{-1} \\ &= ra + \left(c_1ac_1^{-1} - a\right) + \cdots + \left(c_{r-1}ac_{r-1}^{-1} - a\right) \\ &= ra + \left(c_1\left(ac_1^{-1}\right) - \left(ac_1^{-1}\right)c_1\right) + \cdots + \left(c_{r-1}\left(ac_{r-1}^{-1}\right) - \left(ac_{r-1}^{-1}\right)c_{r-1}\right) \\ &= ra + d_1 + d_2 + \cdots + d_{r-1} = ra + d, \end{aligned} \tag{2.3}$$

where $d_1,\ldots,d_{r-1}$, $d \in [R,R]$. This simply yields $a \in F + [R,R]$ which imply that $R = F + [R,R]$, $x = 1$.

(ii) Let $n \in \mathbb{N}$ be an arbitrary positive integer. We have $R = M_n(D)$, where D is a division ring. By (i), $D = F + [D,D]$, so

$$R = M_n(D) = M_n(F + [D,D]) = M_n(F) + M_n([D,D]) \subseteq M_n(F) + [R,R] \subseteq R. \tag{2.4}$$

This implies that $R = M_n(F) + [R,R]$. By this formula, given $A \in R$, there exist $B \in M_n(F)$ and $C \in [R,R]$ such that $A = B + C$, hence $A = (B - (\operatorname{tr}B/n)I) + (\operatorname{tr}B/n)I + C$, where I is the identity matrix of size n. By Proposition 2.1, $(B - (\operatorname{tr}B/n)I) \in [R,R]$, and $A = (\operatorname{tr}B/n)I + ((B - (\operatorname{tr}/n)I) + C)$, consequently

$$R = FI + [R,R], \quad x = I. \tag{2.5}$$

□

To see a different statements and initial ideas of these theorems we refer the reader to [1, 2]. Also a multiplicative version of Theorem 2.3 could be found in [11].

Now, we are going to state our main result, which is about the Whitehead space of a semisimple ring. This theorem is a generalization of a nice theorem due to R. Brauer [8, page 130].

Theorem 2.4. *Let R be a left Artinian semisimple central F-algebra and let k be the number of left simple components of R. Then,*

(i) *if R is algebraic over F and* $\operatorname{Char} F = 0$*, then* $k \geq \dim_F R/[R,R]$*;*

(ii) *if R is finite dimensional over F, or is locally finite over F, and* $\operatorname{Char} F = 0$*, then* $k = \dim_F R/[R,R]$.

Proof. Consider the following chain of functions:

$$R \xrightarrow{f_1} M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k) \xrightarrow{f_2} D_1/[D_1,D_1]\times\cdots\times D_k/[D_k,D_k], \tag{2.6}$$

where f_1 is the isomorphism given by the Wedderburn-Artin theorem for the decomposition of a semisimple left Artinian ring into a direct product of simple ring [6, 14], and f_2 is the F-algebra homomorphisms, by considering component-wise the trace function on $M_{n_i}(D_i) \bmod [D_i,D_i]$, $i = 1,\dots,k$.

By Proposition 2.1 we have, $\ker(f_2\circ f_1) = [R,R]$, noting that $[R,R] \cong [R_1,R_1]\times\cdots\times [R_k,R_k]$, where $R_{n_i} = M_{n_i}(D_i)$, $i = 1,\dots,k$. Therefore the following F-isomorphism holds:

$$R/[R,R] \cong D_1/[D_1,D_1]\times\cdots\times D_k/[D_k,D_k]. \tag{2.7}$$

It remains to compute the dimension of Whitehead space of a division ring in the two cases (i) and (ii) above.

First let D be algebraic over F and $\operatorname{Char} F = 0$. We show that any two elements $\bar a, \bar b \in D/[D,D]$ are linearly dependent. By Theorem 2.3, there exist elements $\alpha,\beta \in F$ and $d_1, d_2 \in [D,D]$, such that $a = \alpha + d_1$ and $b = \beta + d_2$. In other words, $\beta\bar a - \alpha\bar b = \bar 0$ in $D/[D,D]$. Hence in this case $\dim_F D/[D,D] \leq 1$.

Now let D be finite dimensional F-central algebra. Let $\mathrm{RT}_{D/F} : D \to F$ be the reduced trace function which is surjective by [7, page 148]. Furthermore, by a theorem of Amitsur and Rowen [5, page 171] its kernel is equal to $[D,D]$ and so it is a hyperplane over F, in this case $\dim_F D/[D,D] = 1$.

As a latter case let D be a locally finite division ring over it's center F and $\operatorname{Char} F = 0$. Now consider the function $\mathrm{TR} : D \to F$ defined by

$$\mathrm{TR}(x) = \frac{1}{\deg_F(x)}\operatorname{Tr}_{F(x)/F}(x), \tag{2.8}$$

we show that this function is an F-linear surjective map, whose kernel is $[D,D]$. The claim then is clear.

First note that in this case $1 \notin [D,D]$, for if $1 \in [D,D]$, then there exist some x_i's and y_i's in D, such that $1 = \sum(x_iy_i - y_ix_i)$. Let D_1 be the division ring generated by F together with x_i's and y_i's. Taking the reduced trace of D_1 over its centre of both sides of $1 = \sum(x_iy_i - y_ix_i)$, we get a contradicting result. Therefore $[D,D]\cap F = \{0\}$. Now, by considering the trace formula (given in the proof of Theorem 2.3) for elements a, b and $\lambda a + b$ $(\lambda \in F)$ in D, it is readily verified that

$$\frac{1}{r}\operatorname{Tr}(\lambda a + b) = \frac{\lambda}{n}\operatorname{Tr}(a) + \frac{1}{m}\operatorname{Tr}(b), \tag{2.9}$$

where r, n, and m are degrees of $\lambda a + b$, a and b. So TR is F-linear. The surjectivity is clear. In order to specify the kernel of TR, consider the trace formula for elements of $[D,D]$. Suppose that $a \in [D,D]$. Now, we have $\mathrm{Tr}_{F(a)/F}(a) = na + d \in [D,D] \cap F$, where n is the degree of a over F and $d \in [D,D]$. Therefore $\mathrm{TR}(a) = 0$. By the same argument we can see that if $\mathrm{TR}(a) = 0$, then $a \in [D,D]$. □

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