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A NOTE ON REGULAR RINGS WITH STABLE RANGE ONE

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It is known that a regular ring has stable range one if and only if it is unit regular. The purpose of this note is to give an independent and more elementary proof of this result.

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1. Introduction. All rings considered in this note are associative with identity. A ring R is said to be (*von Neumann*) *regular* if, given any $x \in R$, there exists $y \in R$ such that $xyx = x$. If, given any $x \in R$, there exists an invertible element $u \in R$ such that $xux = x$, then R is said to be *unit regular*. A ring R is said to have *stable range one* if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. By Vaserstein [4, Theorem 1], this definition is left-right symmetric.

It has been shown independently in [1, 3] that a regular ring has stable range one if and only if it is unit regular (see also [2]). The aim of this note is to provide a rather straightforward and more elementary proof of this result.

We need the following proposition.

Proposition 1.1. *A ring R has stable range one if and only if for any $a, x, b \in R$ satisfying $ax + b = 1$, there exists $y \in R$ such that $a + by$ is invertible.*

Proof. Assume that R has stable range one and let $a, x, b \in R$ satisfy $ax + b = 1$. Then $aR + bR = R$ and by definition, there exists $y \in R$ such that $a + by$ is right invertible. By [5, Theorem 2.6], it follows that $a + by$ is left invertible. The converse is obvious. □

We also need the following known result (see, e.g., [6]).

Proposition 1.2. *Let R be a ring. Then R is unit regular if and only if every element of R is the product of an idempotent and an invertible element (which do not necessarily commute).*

2. A different proof. We are now ready to give a different proof of the following result.

Theorem 2.1. *A regular ring R has stable range one if and only if it is unit regular.*

Proof. First, assume that R has stable range one and let $a \in R$. Since R is regular, there exists $x \in R$ such that $axa = a$. Clearly, $ax + (1 - ax) = 1$. By the assumption on R and Proposition 1.1, there exists $y \in R$ such that $u = a + (1 - ax)y$ is invertible. Therefore, $axu = ax[a + (1 - ax)y] = axa = a$. It follows that $ax = au^{-1}$ from which we have $au^{-1}a = axa = a$.

Conversely, assume that R is unit regular and suppose that $ax+b=1$ for some $a,x,b\in R$. By Proposition 1.2, we may write $a=eu$, $b=gv$ for some idempotents $e,g\in R$ and some invertible elements $u,v\in R$. It follows that

$$e(ux+b)+(1-e)gv = eux+eb+(1-e)b = ax+b = 1. \tag{2.1}$$

Since R is regular, there exists $c\in R$ such that $(1-e)g=(1-e)gc(1-e)g$. Let $f=(1-e)gc(1-e)$. We then have, by (2.1), that

$$\begin{aligned} e(ux+b)+fb &= e(ux+b)+(1-e)gc(1-e)gv \\ &= 1-(1-e)gv+(1-e)gv = 1. \end{aligned} \tag{2.2}$$

Note that $0=feux=fax=f(1-b)$, that is, $fb=f$. We also have $e=e1=e(ax+b)=e(ux+b)$. Thus

$$e+f = e(ux+b)+fb = 1. \tag{2.3}$$

It is clear that $1+ebv^{-1}c(1-e)$ is invertible with inverse $1-ebv^{-1}c(1-e)$. Since $e+f=1$, we have that $e+(1-e)gc(1-e)=1$, that is, $e+(1-e)gvv^{-1}c(1-e)=1$. But since $b=gv$, it follows that $e+(1-e)bv^{-1}c(1-e)=1$ and therefore

$$e+bv^{-1}c(1-e) = 1+ebv^{-1}c(1-e). \tag{2.4}$$

Since $(1-e)e=0$, we can write

$$e+bv^{-1}c(1-e)[1+ebv^{-1}c(1-e)] = 1+ebv^{-1}c(1-e). \tag{2.5}$$

Multiplying on the right by u and noting that $eu=a$, we then obtain

$$a+bv^{-1}c(1-e)[1+ebv^{-1}(1-e)]u = [1+ebv^{-1}c(1-e)]u, \tag{2.6}$$

which is invertible. It then follows from Proposition 1.1 that R has stable range one. □

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