

C_0 -CONTINUITY OF THE FRÖBENIUS-PERRON SEMIGROUP

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We consider the Fröbenius-Perron semigroup of linear operators associated to a semidynamical system defined in a topological space X endowed with a finite or a σ -finite regular measure. We prove that if there exists a *faithful invariant measure* for the semidynamical system, then the Fröbenius-Perron semigroup of linear operators is C_0 -continuous in the space $L^1_\mu(X)$. We also give a geometrical condition which ensures C_0 -continuity of the Fröbenius-Perron semigroup of linear operators in the space $L^p_\mu(X)$ for $1 \leq p < \infty$, as well as in the space L^1_{loc} .

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1. Introduction. An important problem in the study of the dynamics of nonsingular transformations is to know if they admit an absolutely continuous invariant measure (acim). For interval maps, for example, we have a well-known theorem of Lasota and Yorke [5], which roughly states that if the map is smooth by parts (C^r , with $r \geq 2$) and expanding, then it admits an acim, and with some additional conditions it is exact with respect to this acim (for more details, see [5]); extensions of this result have been obtained for the n -dimensional case (see [3]). When we deal with a continuous semidynamical or a dynamical system (i.e., with a semi-flow or a flow) the problem is more complicated.

A useful technical tool for studying the problem of the existence of an acim is the Fröbenius-Perron operator (see [3, 4] for more details). Let X be a topological space and let μ be a regular measure defined on the Borel σ -algebra of X (see Section 2.1); if $\tau : X \rightarrow X$ is a nonsingular transformation, then the Fröbenius-Perron operator associated to τ , denoted by P_τ (in fact P_τ depend also on μ , and sometimes we use the notation $P_{\tau,\mu}$ in order to indicate such dependence on the measure), is a linear operator, naturally defined in the space $L^1_\mu(X)$. The central point here is that an invariant density, that is, a nonnegative measurable function of unit norm and fixed for the Fröbenius-Perron operator corresponds to a density of an acim for the transformation τ (see Section 2.4).

Let $\tau_t : X \rightarrow X$ be a semidynamical system. Denote by P_t the Fröbenius-Perron operator associated to the transformation τ_t . The family $\{P_t\}_{t \geq 0} = \{P_{t,\mu}\}_{t \geq 0}$ satisfies

$$P_0 = \text{Id}, \quad P_{t+s} = P_t \circ P_s, \quad \forall t, s \geq 0, \quad (1.1)$$

that is, $\{P_t\}_{t \geq 0}$ is a semigroup of linear operators on $L^1_\mu(X)$.

For a semigroup of continuous linear operators defined in the space $L^1_\mu(X)$, a central problem is to know if the semigroup is C_0 -continuous, that is, if the following relation

holds:

$$\lim_{t \rightarrow 0} P_t(f) = f, \quad \forall f \in L^1_\mu(X). \tag{1.2}$$

If this is the case, we may consider the infinitesimal generator of the semigroup which is defined by

$$A(f) = \lim_{t \rightarrow 0} \frac{P_t(f) - f}{t}, \tag{1.3}$$

for elements $f \in L^1_\mu(X)$ for which the above limit exists (see [2, 6]). It is known that a function $f : X \rightarrow \mathbb{R}$ satisfies $P_t(f) = f$, for all $t \geq 0$, if and only if A is defined for f and the differential equation $A(f) = 0$ is satisfied. In this way, the problem of finding or proving the existence of an acim for the semidynamical system is equivalent to the problem of finding or proving the existence of a nontrivial zero for the infinitesimal generator of the Fröbenius-Perron semigroup of linear operators associated, provided that this semigroup is strongly continuous.

Let V be a smooth vector field defined in a smooth manifold, and let $\{\tau_t\}_{t \in \mathbb{R}}$ be its flow. In this case, the Fröbenius-Perron operator, for f of class C^1 is given by the equation $A(f) = \nabla(fV)$ (where ∇ denote the divergence operator). We recover in this way a well-known theorem of Liouville which states that a flow preserves the canonical measure in the manifold if and only if the vector field has divergence equal to zero. The operator A defined in (1.3) can be viewed as a generalization of the divergence operator for continuous semi-flows for which the associated Fröbenius-Perron semigroup of linear operators is C_0 -continuous.

In this paper, we study general conditions that ensure this C_0 -continuity.

In Section 2, we establish the notation and recall some basic results from the semigroup theory, and the definition of the Fröbenius-Perron operator and some of its properties.

In Section 3, we consider the case in which we know a priori that there exists a faithful acim for the system, that is, an acim with a positive density. In that case, we prove Theorem 3.3 which implies that the problem of finding a faithful acim is equivalent to the problem of finding a zero for the infinitesimal generator.

Since the problem to deal with is exactly the problem of the existence of the acim, we have to search an intrinsic property of the flow that ensures the strong continuity of the semigroup. The condition is: *there exists $T > 0$ such that*

$$\frac{\mu(\tau_t^{-1}(A))}{\mu(A)} \leq M, \quad \forall t \leq T, \forall A \in \mathcal{A}. \tag{1.4}$$

To understand condition (1.4), we may consider the case of a dynamical system. In that case, each transformation $\tau_t : X \rightarrow X$ has an inverse, and the associated Fröbenius-Perron operator is given by

$$P_t(f) = (f \circ \tau_{-t}) \cdot J(\tau_{-t}), \tag{1.5}$$

where $J(\tau_{-t})$ is the density of the measure $(\tau_t)_*(\mu)$, where $(\tau_t)_*\mu(B) = \mu(\tau_t^{-1}(B))$ for B measurable, that is,

$$\mu(\tau_t^{-1}(A)) = \int_A J(\tau_{-t}) d\mu \tag{1.6}$$

for more details see [4].

Thus, if we have an upper bound and good behaviour for $J(\tau_{-t})$ near zero, then we use the dominated convergence theorem to prove that condition (1.2) holds for continuous functions. The extension of the result for arbitrary integrable functions is obtained by using the fact that the set of continuous functions is dense in the space of integrable functions.

For a semidynamical system, we do not have an explicit expression like (1.5) for the associated Fröbenius-Perron operator. However, a bound of $J(\tau_{-t})$ can be interpreted as an estimate of the type that appears in condition (1.4).

In Section 4, we prove that, under general hypotheses, condition (1.4) holds if and only if the Fröbenius-Perron semigroup of linear operators can be defined in the space $L^p_\mu(X)$, and it is C_0 -continuous in that space (see Theorems 4.1 and 4.2).

In Section 5, we prove that condition (1.4) also ensures strong continuity in the space $L^1_\mu(X)$ when (X, \mathcal{A}, μ) is a probability space (Theorem 5.1). The precise statement of the main result is the following (for the concepts involved, see Section 2).

THEOREM 1.1. *Let X be a topological space endowed with a regular probability measure μ . Let $\{\tau_t\}_{t \geq 0}$ be a continuous proper semidynamical system defined on X . If the semidynamical system satisfies (1.4), then the associated Fröbenius-Perron semigroup of linear operators is C_0 -continuous in $L^1_\mu(X)$.*

Finally, to make this work more complete, we deal with the $L^1_{\mu,loc}(X)$ case in Section 6. The presentation is quite informal since there are some technical difficulties derived from the fact that $L^1_{\mu,loc}(X)$ is only a locally convex space and not a Banach space (we must add some hypotheses in this case in order to make the semigroup approach to the acim problem available).

2. Basic results. In this section, we give a survey of definitions, results and notations that are necessary for the rest of the paper.

2.1. Measure theory. Let X be a topological space and let \mathcal{A} be its Borel σ -algebra. Let μ be a measure defined over \mathcal{A} . We say that μ is *regular* if, for all $A \in \mathcal{A}$, we have

$$\mu(A) = \sup \{ \mu(K) : K \subset A, K \text{ compact} \} = \inf \{ \mu(C) : A \subset C, C \text{ open} \}. \tag{2.1}$$

We note that if X is a metric space, then a probability measure defined on the Borel σ -algebra is regular. In general, if a measure μ is regular, then the set of continuous functions with compact support is dense in the space $L^p_\mu(X)$, for all $1 \leq p < \infty$.

2.2. Semidynamical systems. Let X be a topological space. A family $\{\tau_t\}_{t \geq 0}$ of continuous transformations $\tau_t : X \rightarrow X$ is a *semidynamical system* if the following conditions are satisfied:

- (i) $\tau_0 = \text{Id}$;
- (ii) $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, s \geq 0$;
- (iii) the map $[0, \infty[\times X \rightarrow X$ given by $(t, x) \rightarrow \tau_t(x)$ is continuous.

If each transformation τ_t has a continuous inverse τ_{-t} , then the family $\{\tau_t\}_{t \in \mathbb{R}}$ is a continuous flow. However, for general semidynamical systems, the maps τ_t may not necessarily have an inverse.

We say that a semidynamical system $\{\tau_t\}_{t \geq 0}$ is *proper* if, for each compact set $K \subset X$ and for each $t > 0$, the set $\cup_{s \leq t} \tau_s^{-1}(K)$ is compact.

2.3. Semigroups in Banach spaces. Let L be a Banach space with respect to a norm $\|\cdot\|$. A family $\{T_t\}_{t \geq 0}$ of continuous linear operators $T_t : L \rightarrow L$ ($t \geq 0$) is called a *semigroup of linear operators* if the following conditions are satisfied:

$$T_0 = \text{Id}, \quad T_{t+s} = T_t \circ T_s, \quad \forall t, s \geq 0. \tag{2.2}$$

For more details see [6] or [2]. We say that a semigroup $\{T_t\}_{t \geq 0}$ of linear operators is C_0 -continuous, if

$$\lim_{t \rightarrow 0} \|T_t(f) - f\| = 0, \quad \forall f \in L. \tag{2.3}$$

When a semigroup $\{T_t\}_{t \geq 0}$ is C_0 -continuous, there exist constants $M \geq 1$ and $w \geq 0$ such that, for all $f \in L$ we have

$$\|T_t(f)\| \leq M e^{wt} \|f\|. \tag{2.4}$$

If $\{T_t\}_{t \geq 0}$ is a semigroup defined on L , then the adjoint family $\{T_t^*\}_{t \geq 0}$ is a semigroup defined on the dual space L^* . By a duality theorem, we have that if $\{T_t\}_{t \geq 0}$ is C_0 -continuous and L is reflexive, then $\{T_t^*\}_{t \geq 0}$ is C_0 -continuous in L^* (see [6, Corollary 10.6, page 41]).

2.4. Fröbenius-Perron operator. Let (X, μ, \mathcal{A}) be a measure space. We say that a transformation $\tau : X \rightarrow X$ is *nonsingular* if for all $A \in \mathcal{A}$ such that $\mu(A) = 0$, we have $\mu(\tau^{-1}(A)) = 0$. If a transformation $\tau : X \rightarrow X$ is nonsingular, then associated to it there exists a linear operator $P_\tau = P_{\tau, \mu} : L_\mu^1(X) \rightarrow L_\mu^1(X)$, called *Fröbenius-Perron operator* which is characterized by the relation

$$\int_A P_\tau(f) d\mu = \int_{\tau^{-1}(A)} f d\mu, \tag{2.5}$$

for all $f \in L_\mu^1(X)$ and all $A \in \mathcal{A}$.

It is well known (see [1, 4]) that a probability measure μ on X is τ -invariant (i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$) if and only if $P_\tau(1) = 1$ (this is also true for σ -finite measure spaces, but in that case there is a problem with the space where the Fröbenius-Perron operator is defined, as we will see). In general, τ preserves a measure $d\nu = f d\mu$, with $f \in L_\mu^1(X)$ if and only if $P_\tau(f) = f$. It is also well known that the Fröbenius-Perron operator is a linear contraction in $L_\mu^1(X)$ endowed with the L_μ^1 -norm, that is, $\|P_\tau\|_{L_\mu^1} \leq 1$ (see [3, 4]). Moreover, for $f \in L_\mu^1(X)$ and a.e. $x \in X$, we have $|P(f)(x)| \leq P(|f|)(x)$. On the other hand, if we change the measure μ by an absolutely continuous one given by $d\nu = g d\mu$, then the change in the Fröbenius-Perron operator is given by

$$P_{\tau, \nu}(f) = \frac{P_{\tau, \mu}(f \cdot g)}{g}. \tag{2.6}$$

Another important property of the Fröbenius-Perron operator is given by the equality

$$\int_X P_\tau(f) \cdot g d\mu = \int_X f \cdot (g \circ \tau) d\mu, \tag{2.7}$$

valid for all $f \in L^1_\mu(X)$ and all $g \in L^\infty_\mu(X)$. Equation (2.7) permits us define a linear operator $K_\tau : L^\infty_\mu(X) \rightarrow L^\infty_\mu(X)$ given by $K_\tau(g) = g \circ \tau$. The operator K_τ is well defined if τ is a nonsingular transformation. This operator is called the *Koopman operator*. For more details about these concepts see [1] or [4].

If we have a semidynamical system $\{\tau_t\}_{t \geq 0}$ such that each transformation τ_t is nonsingular, then we denote the family of Fröbenius-Perron operators associated by $P_t = P_{\tau_t}$. This is a semigroup of continuous linear operators in the space $L^1_\mu(X)$ (see [4]). We will also use the notation K_t for the Koopman operator K_{τ_t} .

We note that the Fröbenius-Perron operator may also be defined and is bounded in other spaces of functions if the transformation τ has a good behaviour, for example, $L^p_\mu(X)$ spaces or $BV(X)$, the space of functions of bounded variation. For example, in Section 4, we consider this operator in $L^p_\mu(X)$ spaces, and we prove that the geometrical condition (1.5) ensures continuity of each operator P_t (and also the C_0 -continuity of the semigroup $\{P_t\}_{t \geq 0}$ in the space $L^p_\mu(X)$). We note that if P_t is continuous in $L^p_\mu(X)$, then the duality equation (2.7) is valid for all $f \in L^p_\mu(X)$ and for all $g \in L^q_\mu(X)$, with $1/p + 1/q = 1$, that is, K_t is the adjoint operator of P_t .

2.5. Conditional expectation and Fröbenius-Perron operator. Let (X, \mathcal{A}, μ) be a probability space. Suppose $\tau : X \rightarrow X$ preserves μ , then we have another way of introducing the Fröbenius-Perron operator associated to τ . This is given for $f \in L^1_\mu(X)$ by the equality

$$P_\tau(f) \circ \tau = E(f, \tau^{-1}(\mathcal{A})), \tag{2.8}$$

where the expression on the right-hand side denotes the *conditional expectation* of f with respect to the σ -algebra $\tau^{-1}(\mathcal{A})$. In Section 3, we use this approach and the following result of convergence that arises in Martingale theory (see [1, page 81]).

THEOREM 2.1. *Let $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ be a collection of σ -algebras such that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, for all n . Then $E(f, \mathcal{A}_n)$ converges to $E(f, \mathcal{A}_\infty)$, in the L^1 sense, where \mathcal{A}_∞ denotes the σ -algebra generated by all the \mathcal{A}_n .*

3. Strong continuity with an absolutely continuous invariant measure. Let X be a topological space and let μ be a regular probability measure defined on the Borel σ -algebra on X . In this section, we consider a nonsingular semidynamical system $\{\tau_t\}_{t \geq 0}$ defined over X . We prove that if there exists a *faithful acim* (an acim with positive density), then the associated Fröbenius-Perron semigroup is C_0 -continuous in $L^1_\mu(X)$. We first prove the following two lemmas.

LEMMA 3.1. *Suppose that $\{\tau_t\}_{t \geq 0}$ has an acim ν such that $d\nu = g d\mu$, with $g > 0$ (a.e.). Then,*

$$\lim_{t \rightarrow 0} (P_{t,\nu}(f) \circ \tau_t) = f, \tag{3.1}$$

for each $f \in L^1_\nu(X)$ in the $L^1_\nu(X)$ sense.

PROOF. The sequence of σ -algebras $\{\mathcal{A}_t\}_{t \geq 0}$, where $\mathcal{A}_t = \tau_t^{-1}(\mathcal{A})$ for all $t \geq 0$, is increasing as t goes to zero. By the martingale convergence theorem, we have, for f in $L^1_\nu(X)$,

$$\lim_{t \rightarrow 0} (P_{t,\nu}(f) \circ \tau_t) = E(f, \mathcal{A}_\infty), \tag{3.2}$$

where \mathcal{A}_∞ is the σ -algebra generated by all the σ -algebras \mathcal{A}_t for $t > 0$. Thus, we must prove that \mathcal{A}_∞ is equal to \mathcal{A} . For this, let A be an open set. By continuity of the semidynamical system $\{\tau_t\}_{t \geq 0}$, the function $|\chi_A \circ \tau_t - \chi_A|$ converges pointwise to zero as t goes to zero, and by the dominated convergence theorem, we have

$$\nu(\tau_t^{-1}(A) \Delta(A)) = \int_X |\mathcal{X}_A \circ \tau_t - \mathcal{X}_A| d\nu \tag{3.3}$$

converges to zero as t goes to zero. Thus, for each $n \in \mathbb{N}$, we may consider a sequence $t_n > 0$ such that $\nu(\tau_{t_n}^{-1}(A) \Delta(A)) < 1/2^n$. From this, it is easy to see that $\nu(\cup_{n \geq m}^\infty \tau_{t_n}^{-1}(A) \Delta(A)) < 1/2^{m-1}$. Hence,

$$\nu((\cap_{m=1}^\infty \cup_{n \geq m}^\infty \tau_{t_n}^{-1}(A)) \Delta(A)) \leq \nu(\cap_{m=1}^\infty (\cup_{n \geq m}^\infty \tau_{t_n}^{-1}(A) \Delta(A))) = 0. \tag{3.4}$$

This implies that $A \in \mathcal{A}_\infty$, and since A is an arbitrary open set, this implies that \mathcal{A}_∞ is equal to \mathcal{A} , which completes the proof of [Lemma 3.1](#). \square

LEMMA 3.2. *Suppose that $\{\tau_t\}_{t \geq 0}$ has a faithful acim ν given by $d\nu = g d\mu$, with $g \in L^1_\mu(X)$ and $g > 0$ (a.e.). Then, for each $f \in L^1_\nu(X)$,*

$$\lim_{t \rightarrow 0} P_{t,\nu}(f) = f \tag{3.5}$$

in the $L^1_\nu(X)$ sense.

PROOF. Let ε be an arbitrary positive number. We take a sequence $\{f_n\}_{n \in \mathbb{N}}$ of bounded continuous functions that converges to a function $f \in L^1_\nu(X)$ and choose $n_0 \in \mathbb{N}$ such that $\|f_{n_0} - f\| \leq \varepsilon/4$. By the invariance of ν and since the operator $P_{t,\nu}$ is a linear contraction, we have

$$\begin{aligned} \|P_{t,\nu}(f) - f\|_{L^1_\nu} &\leq \|P_{t,\nu}(f) - P_{t,\nu}(f_{n_0})\|_{L^1_\nu} + \|P_{t,\nu}(f_{n_0}) - f_{n_0}\|_{L^1_\nu} + \|f_{n_0} - f\|_{L^1_\nu} \\ &\leq 2\|f - f_{n_0}\|_{L^1_\nu} + \|P_{t,\nu}(f_{n_0}) \circ \tau_t - f_{n_0} \circ \tau_t\|_{L^1_\nu} \\ &\leq \frac{\varepsilon}{2} + \|P_{t,\nu}(f_{n_0}) \circ \tau_t - f_{n_0} \circ \tau_t\|_{L^1_\nu} + \|f_{n_0} - f_{n_0} \circ \tau_t\|_{L^1_\nu}. \end{aligned} \tag{3.6}$$

For small t , by [Lemma 3.1](#) we have $\|P_{t,\nu}(f_{n_0}) \circ \tau_t - f_{n_0} \circ \tau_t\|_{L^1_\nu} \leq \varepsilon/4$. Now, by continuity of the semidynamical system and by the dominated convergence theorem, we have $\lim_{t \rightarrow 0} \|f_{n_0} - f_{n_0} \circ \tau_t\|_{L^1_\nu} = 0$. Thus, for t small we have $\|f_{n_0} - f_{n_0} \circ \tau_t\|_{L^1_\nu} \leq \varepsilon/4$. This implies, for t small, that $\|P_{t,\nu}(f) - f\|_{L^1_\nu} \leq \varepsilon$, which completes the proof of [Lemma 3.2](#). \square

THEOREM 3.3. *Suppose that the semidynamical system $\{\tau_t\}_{t \geq 0}$ has a faithful acim. Then, the semigroup $\{P_{t,\mu}\}_{t \geq 0}$ is C_0 -continuous in $L^1_\mu(X)$.*

PROOF. Let ν the faithful acim given by $d\nu = g d\mu$, with $g \in L^1_\mu(X)$ and $g > 0$ (a.e.). Let $f \in L^1_\mu(X)$. Then, $f/g \in L^1_\nu(X)$, and we have, by equality (2.6), that

$$\begin{aligned} \|P_{t,\mu}(f) - f\|_{L^1_\mu} &= \int_X |P_{t,\mu}(f) - f| d\mu \\ &= \int_X \left| g P_{t,\nu}\left(\frac{f}{g}\right) - f \right| d\mu \\ &= \int_X \left| P_{t,\nu}\left(\frac{f}{g}\right) - \frac{f}{g} \right| d\nu \\ &= \left\| P_{t,\nu}\left(\frac{f}{g}\right) - \frac{f}{g} \right\|_1 \rightarrow 0 \text{ as } t \rightarrow 0. \end{aligned} \tag{3.7}$$

By Lemma 3.2, the last quantity converges to zero as t goes to zero, which proves the theorem. □

4. Strong continuity in L^p . In this section, we consider a semidynamical system defined over a topological space X provided of a regular measure μ defined on its Borel σ -algebra, and prove that condition (1.4) is equivalent to strong continuity of the associated Fröbenius-Perron semigroup defined over the space $L^p_\mu(X)$.

We say that a semidynamical system is *strongly nonsingular* if it satisfies condition (1.4). It is easy to see that the following conditions for being strongly nonsingular are equivalent:

- (i) for each $t > 0$, there exists M_t such that

$$\mu(\tau_s^{-1}(A)) \leq M_t \mu(A) \tag{4.1}$$

for all $s \leq t$ and all $A \in \mathcal{A}$;

- (ii) there exist $T > 0$ and $M = M_T > 0$ such that

$$\mu(\tau_s^{-1}(A)) \leq M_T \mu(A) \tag{4.2}$$

for all $A \in \mathcal{A}$ and all $s \leq T$.

In fact, condition (4.1) implies trivially (4.2), and if condition (4.2) is assumed, then condition (4.1) holds by putting $M_t = M^{t/T+1}$.

Finally, it is easy to see that every strongly nonsingular semidynamical system is nonsingular.

THEOREM 4.1. *Let $\{\tau_t\}_{t \geq 0}$ be a nonsingular semidynamical system such that its associated semigroup of Fröbenius-Perron operators $\{P_t\}_{t \geq 0}$ is a C_0 -continuous semigroup of bounded linear operators in the space $L^p_\mu(X)$ for some $1 < p < \infty$. Then $\{\tau_t\}_{t \geq 0}$ is strongly nonsingular.*

PROOF. By the hypothesis, the semigroup of linear operators $\{P_t\}_{t \geq 0}$ is a C_0 -semigroup over a reflexive space. Thus, the semigroup of Koopman operators $\{K_t\}_{t \geq 0}$ is a C_0 -semigroup in the space $L^q_\mu(X)$, where $q = p/(p-1)$ is the conjugate of p (see [6, Corollary 10.6, page 41]). Therefore, there exist constants $M \geq 1$ and $w > 0$ such that, for all $f \in L^q_\mu(X)$, we have that $\|K_t(f)\|_{L^q_\mu} \leq M e^{wt} \|f\|_{L^q_\mu}$. Now, if $\mu(A)$ is finite, then

$\chi_A \in L_\mu^q(X)$ and $\|K_t(\chi_A)\|_{L_\mu^q} \leq Me^{wt} \|\chi_A\|_{L^q}$. Thus, $\|\chi_{\tau_t^{-1}(A)}\|_{L_\mu^q} \leq Me^{wt} (\mu(A))^{1/q}$ and from this it follows that $\mu(\tau_t^{-1}(A)) \leq (Me^{wt})^q \mu(A)$, which proves our theorem. \square

If a measure μ is regular and the semidynamical system $\{\tau_t\}_{t \geq 0}$ is proper, then we have the converse of the above result.

THEOREM 4.2. *Suppose the measure μ is regular on X . If $\{\tau_t\}_{t \geq 0}$ is a proper and strongly nonsingular semidynamical system, then, for all $1 < p < \infty$, the associated Fröbenius-Perron semigroup $\{P_t\}_{t \geq 0}$ is a C_0 -semigroup of linear bounded operators in the space $L_\mu^p(X)$.*

The idea for the proof of [Theorem 4.2](#), is to prove that condition (4.1) ensures strong continuity for the dual semigroup $\{K_t\}_{t \geq 0}$ in the dual space $L_\mu^q(X)$. For this we first prove the following two lemmas.

LEMMA 4.3. *Under the hypotheses of [Theorem 4.2](#), $\|K_s(f)\|_{L_\mu^q} \leq M_t^{1/q} \|f\|_{L_\mu^q}$ for all $f \in L_\mu^q(X)$ and all $s \leq t$.*

PROOF. Let $f \in L_\mu^q(X)$ be a simple function given by $f = \sum_{i=1}^n \lambda_i \chi_{A_i}$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, then we have

$$\|f\|_{L_\mu^q} = \left(\sum_{i=1}^n |\lambda_i|^q \mu(A_i) \right)^{1/q}, \quad \|K_t(f)\|_{L_\mu^q} = \left(\sum_{i=1}^n |\lambda_i|^q \mu(\tau_s^{-1}(A_i)) \right)^{1/q}. \tag{4.3}$$

On the other hand, by condition (4.1) we have, for $s \leq t$, that

$$\left(\sum_{i=1}^n |\lambda_i|^q \mu(\tau_s^{-1}(A_i)) \right)^{1/q} \leq \left(\sum_{i=1}^n |\lambda_i|^q M_t \mu(A_i) \right)^{1/q}, \tag{4.4}$$

and this implies that $\|K_s(f)\|_{L_\mu^q} \leq M_t^{1/q} \|f\|_{L_\mu^q}$. Finally, since the set of simple functions is dense in the space $L_\mu^q(X)$, the lemma follows. \square

LEMMA 4.4. *If the hypotheses of [Theorem 4.1](#) are satisfied, then the Koopman semigroup of operators $\{K_t\}_{t \geq 0}$ is a C_0 -continuous semigroup of linear bounded operators in the space $L_\mu^q(X)$.*

PROOF. Let $f \in L_\mu^q(X)$ and let $\varepsilon > 0$. We take a sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous functions with compact support, say K , such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{L_\mu^q} = 0$ and we choose $n_0 \in \mathbb{N}$ such that

$$\|f - f_{n_0}\|_{L_\mu^q} \leq \frac{\varepsilon}{2(M^{1/q} + 1)}, \tag{4.5}$$

where $M = M_T$ is fixed. By continuity of the Koopman semigroup, the function $K_s(f_{n_0}) - f_{n_0}$ converges pointwise to zero as s goes to zero. If K is the support of f_{n_0} , then, since the semidynamical system is proper, the set $\tilde{K} = \cup_{s \leq t} \tau_s^{-1}(K)$ is compact. Now, it is clear that $\text{supp}(K_s(f_{n_0}) - f_{n_0}) \subset \tilde{K}$ for $s \leq t$, and by the dominated convergence theorem we have $\lim_{s \rightarrow 0} \|K_s(f_{n_0}) - f_{n_0}\|_{L_\mu^q} = 0$. Thus, we may take $t_0 > 0$ such that, for all $s \leq t_0$, we have

$$\|K_s(f_{n_0}) - f_{n_0}\|_{L_\mu^q} \leq \frac{\varepsilon}{2}. \tag{4.6}$$

Using inequalities (4.5) and (4.6), we have, for $s \leq \min\{T, t_0\}$, that

$$\begin{aligned} \|K_s(f) - f\|_{L_\mu^q} &\leq \|K_s(f) - K_s(f_{n_0})\|_{L_\mu^q} + \|K_s(f_{n_0}) - f_{n_0}\|_{L_\mu^q} + \|f_{n_0} - f\|_{L_\mu^q} \\ &\leq (M^{1/q} + 1) \frac{\varepsilon}{2(M^{1/q} + 1)} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned} \tag{4.7}$$

which proves the lemma. □

PROOF OF THEOREM 4.2. It follows directly by the duality theorem (see Sections 2.3 and 2.4). □

5. Strong continuity in L^1 . One of the most important applications of Fröbenius-Perron operator arise in probability spaces by considering its action over the space $L_\mu^1(X)$. For these spaces, we have the following result.

THEOREM 5.1. *Let X be a topological space endowed with a regular probability measure μ and let $\{\tau_t\}_{t \geq 0}$ be a proper semidynamical system. If the semidynamical system is strongly nonsingular, then the associated Fröbenius-Perron semigroup of operators $\{P_t\}_{t \geq 0}$ is C_0 -continuous in the space $L_\mu^1(X)$.*

PROOF. Let $f \in L_\mu^1(X)$ and $\varepsilon > 0$. We fix $p > 0$ and consider a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $L_\mu^p(X)$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{L_\mu^1} = 0$. Since $\mu(X) = 1$ we have

$$\begin{aligned} \|P_t(f) - f\|_{L_\mu^1} &\leq \|P_t(f) - P_t(f_n)\|_{L_\mu^1} + \|P_t(f_n) - f_n\|_{L_\mu^1} + \|f_n - f\|_{L_\mu^1} \\ &\leq 2\|f_n - f\|_{L_\mu^1} + \|P_t(f_n) - f_n\|_{L_\mu^1} \\ &\leq 2\|f_n - f\|_{L_\mu^1} + \|P_t(f_n) - f_n\|_{L_\mu^p}. \end{aligned} \tag{5.1}$$

We choose n_0 such that $\|f - f_{n_0}\|_{L_\mu^1} \leq \varepsilon/3$. By Theorem 4.1, there exists $t_\varepsilon > 0$ such that $\|P_t(f_{n_0}) - f_{n_0}\|_{L_\mu^p} \leq \varepsilon/3$ for all $t \leq t_\varepsilon$. Therefore, for $t \leq t_\varepsilon$ we have $\|P_t(f) - f\|_{L_\mu^1} \leq \varepsilon$. Since f and ε are arbitrary, this finishes the proof. □

Theorem 5.1 is useful for compact metric spaces of finite measure, in that case $\{\tau_t\}_{t \geq 0}$ is proper. For noncompact or unbounded metric spaces with nonfinite measure we assume that the system is strongly continuous over bounded sets, that is, we assume that for each ball

$$B = B(x_0, r) = \{x \in X : d(x, x_0) < r\}, \tag{5.2}$$

there exists a constant $M = M(x, r)$ and $T = T(x, r)$ such that

$$\mu(\tau_t^{-1}(A) \cap B) \leq M\mu(A \cap B) \tag{5.3}$$

for all $t \leq T$ and $A \in \mathcal{A}$. Now we have the following theorem.

THEOREM 5.2. *Under the hypotheses of Theorem 5.1, suppose also that condition (5.3) is true and the measure is finite over bounded sets. Then, the Fröbenius-Perron semigroup is C_0 -continuous in $L_\mu^1(X)$.*

The proof of [Theorem 5.2](#) is based in the reduction of it to [Theorem 5.1](#) in each ball $B = B(x, r)$. For that, we define a new measure $\tilde{\mu}$ in X by putting

$$\tilde{\mu}(A) = \mu(A \cap B). \tag{5.4}$$

This new measure is finite, and the semidynamical system $\{\tau_t\}_{t \geq 0}$ is strongly non-singular with respect to $\tilde{\mu}$. By [Theorem 4.2](#), we have, for each $f \in L^1_{\tilde{\mu}}(X)$, that

$$\lim_{t \rightarrow 0} \tilde{P}_t(f) = f \tag{5.5}$$

in the $L^1_{\tilde{\mu}}$ sense, where $\tilde{P}_t = P_{t, \tilde{\mu}}$. Using this, we prove the following lemma.

LEMMA 5.3. *Let $f \in L^1_{\text{loc}}(X, \mu)$ be a function of compact support K . Then,*

$$\lim_{t \rightarrow 0} \int_K |P_t(f) - f| d\mu = 0. \tag{5.6}$$

PROOF. We take a ball $B = B(x, r)$ containing K and we have

$$\begin{aligned} \int_K |P_t(f) - f| d\mu &\leq \int_B |P_t(f) - f| d\mu \\ &\leq \int_B |P_t(f) - \tilde{P}_t(f)| d\mu + \int_B |\tilde{P}_t(f) - f| d\tilde{\mu}. \end{aligned} \tag{5.7}$$

By (5.5), we have to prove that

$$\lim_{t \rightarrow 0} \int_B |\tilde{P}_t(f) - P_t(f)| d\mu = 0. \tag{5.8}$$

Let k be a positive integer and let $C_{k,t}$ be the set

$$C_{k,t} = \left\{ x \in B : |P_t(f)(x) - \tilde{P}_t(f)(x)| \geq \frac{1}{k} \right\}. \tag{5.9}$$

We claim that $\mu(C_{k,t})$ converges to zero when t goes to zero, for all k . To prove the claim, we consider the sets

$$\begin{aligned} C^1_{k,t} &= \left\{ x \in B : P_t(f)(x) - \tilde{P}_t(f)(x) \geq \frac{1}{k} \right\}, \\ C^2_{k,t} &= \left\{ x \in B : P_t(f)(x) - \tilde{P}_t(f)(x) \leq -\frac{1}{k} \right\}. \end{aligned} \tag{5.10}$$

By (2.6), we have

$$\int_{C^1_{k,t}} P_t(f) d\mu - \int_{C^1_{k,t}} \tilde{P}_t(f) d\mu = \int_{\tau_t^{-1}(C^1_{k,t})} f d\mu - \int_{\tau_t^{-1}(C^1_{k,t})} f d\tilde{\mu}, \tag{5.11}$$

and from this, we have

$$\frac{\mu(C^1_{k,t})}{k} \leq \int_{\tau_t^{-1}(C^1_{k,t}) - B} f d\mu. \tag{5.12}$$

As in the proof of [Lemma 3.1](#), and using the fact that the system is proper, it is not difficult to see that $\mu(\tau_t^{-1}(B) - B)$ converges to zero when t goes to zero, which implies

that $\mu(C_{k,t}^1)$ converges to zero. Likewise, $\mu(C_{k,t}^2)$ converges to zero with t , and this proves our claim.

To finish the proof of [Lemma 5.3](#), let $\varepsilon > 0$. We choose $k \geq 2\mu(B)/\varepsilon$ and we have

$$\begin{aligned} \int_B |P_t(f) - \tilde{P}_t(f)| d\mu &= \int_{C_{k,t}} |P_t(f) - \tilde{P}_t(f)| d\mu + \int_{B-C_{k,t}} |P_t(f) - \tilde{P}_t(f)| d\mu, \\ \int_B |P_t(f) - \tilde{P}_t(f)| d\mu &\leq \frac{\varepsilon}{2} + \int_{C_{k,t}} |P_t(f) - \tilde{P}_t(f)| d\mu. \end{aligned} \tag{5.13}$$

Finally, we have

$$\begin{aligned} \int_{C_{k,t}} |P_t(f) - \tilde{P}_t(f)| d\mu &\leq \int_{C_{k,t}} (P_t(|f|) + \tilde{P}_t(|f|)) d\mu \\ &= \int_{\tau_t^{-1}(C_{k,t})} |f| d\mu + \int_{\tilde{\tau}_t^{-1}(C_{k,t})} |f| d\tilde{\mu}. \end{aligned} \tag{5.14}$$

Since $\tilde{\mu}(\tau_t^{-1}(C_{k,t})) \leq \tilde{M}\mu(C_{k,t})$ and $\mu(\tau_t(C_{k,t}^{-1})) \leq \tilde{M}\mu(C_{k,t}) + \mu(\tau_t^{-1}(B) - B)$, these measures converge to zero (by the claim), and then the last two integrals have values less than $\varepsilon/4$ for t small. This implies that

$$\int_B |P_t(f) - \tilde{P}_t(f)| d\mu \leq \varepsilon \tag{5.15}$$

for t small, which finishes the proof. \square

PROOF OF THEOREM 5.2. Let $f \in L^1_\mu(X)$ and $\varepsilon > 0$ arbitrary. We take a sequence $\{f_n\}_{n \in \mathbb{N}}$ converging to f and such that each f_n has compact support. Then we have, as in the proof of [Theorem 4.2](#), that

$$\int_X |P_t(f) - f| d\mu \leq 2\|f - f_n\|_{L^1} + \int_X |P_t(f_n) - f_n| d\mu. \tag{5.16}$$

If we take $n_0 \in \mathbb{N}$ such that $\|f - f_{n_0}\| \leq \varepsilon/4$, then we have

$$\int_X |P_t(f) - f| d\mu \leq \frac{\varepsilon}{2} + \int_X |P_t(f_{n_0}) - f_{n_0}| d\mu. \tag{5.17}$$

If K is a compact set containing $\text{supp}(f_{n_0})$, then

$$\int_X |P_t(f) - f| d\mu \leq \frac{\varepsilon}{2} + \int_K |P_t(f_{n_0}) - f_{n_0}| d\mu + \int_{X-K} |P_t(f_{n_0})| d\mu. \tag{5.18}$$

By [Lemma 5.3](#) we have, for t small enough,

$$\int_K |P_t(f_{n_0}) - f_{n_0}| d\mu \leq \frac{\varepsilon}{4}. \tag{5.19}$$

Also,

$$\int_{X-K} |P_t(f_{n_0})| d\mu \leq \int_{X-K} P_t(|f_{n_0}|) d\mu = \int_{\tau_t^{-1}(X-K)} |f_{n_0}| d\mu. \tag{5.20}$$

Since $\int_{X-K} |f_{n_0}| d\mu = 0$ and $\mu\{\tau_t^{-1}(X-K) - (X-K)\}$ converges to zero, we have, for t small,

$$\int_{X-K} |P_t(f_{n_0})| d\mu \leq \frac{\varepsilon}{4}. \tag{5.21}$$

This implies that for small t , we have

$$\int_X |P_t(f) - f| d\mu \leq \varepsilon, \tag{5.22}$$

which finishes the proof. □

6. Strong continuity in L^1_{loc} . Since $L^1_{\text{loc}}(X)$ is not a Banach space, the approach of Section 2.3 is not advisable here. However, there is a more general setting of semigroup theory for locally convex spaces (see [7]). In this case, the hypothesis for the family $\{T_t\}_{t \geq 0}$ of continuous linear operators $T_t : L \rightarrow L$ are the following conditions:

- (i) $T_0 = \text{Id}$;
- (ii) $T_t \circ T_s = T_{t+s}$ for all $t, s \geq 0$;
- (iii) $\lim_{t \rightarrow t_0} T_t(f) = T_{t_0}(f)$ for all $t_0 \geq 0$ and $f \in L$;
- (iv) $\{T_t\}_{t \geq 0}$ is an equicontinuous family, that is, for any continuous seminorm p on L , there exists a continuous seminorm q such that $p(T_t(f)) \leq q(f)$ for all $t \geq 0$ and $f \in L$.

If these conditions are satisfied, then the equality $A(f) = 0$ is equivalent to $T_t(f) = f$ for all $t \geq 0$.

We consider the case where $L = L^1_{\mu, \text{loc}}(X)$ and $\{T_t\}_{t \geq 0}$ is the Fröbenius-Perron semigroup of linear operators associated to a semidynamical system $\{\tau_t\}_{t \geq 0}$. Since conditions (i) and (ii) hold trivially, we consider conditions (iii) and (iv). For this, we assume condition (5.3) and that the semidynamical system is strongly proper, that is, for any compact set K the set $\tilde{K} = \cup_{t \geq 0} \tau_t^{-1}(K)$ is a compact set. This hypothesis is restrictive, but it allows us to continue with our approach. In fact, condition (iv) follows from it since for all $f \geq 0$ and all $t \geq 0$, we have

$$\int_K P_t(f) d\mu = \int_{\tau_t^{-1}(K)} f d\mu \leq \int_{\tilde{K}} f d\mu. \tag{6.1}$$

Finally, using the method of the proof of Theorem 5.2, it is possible to prove that condition (iii) also holds. Then, we have extended Theorem 5.2 to $L^1_{\mu, \text{loc}}(X)$ for a large class of systems defined over X .

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