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ON THE RELATION BETWEEN INTERIOR CRITICAL POINTS OF POSITIVE SOLUTIONS AND PARAMETERS FOR A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

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We consider the boundary value problem $-u''(x) = \lambda f(u(x))$, $x \in (0,1)$; $u'(0) = 0$; $u'(1)+\alpha u(1) = 0$, where $\alpha > 0$, $\lambda > 0$ are parameters and $f \in c^2[0,\infty)$ such that $f(0) < 0$. In this paper, we study for the two cases $\rho = 0$ and $\rho = \theta$ (ρ is the value of the solution at $x = 0$ and θ is such that $F(\theta) = 0$ where $F(s) = \int_0^s f(t)dt$) the relation between λ and the number of interior critical points of the nonnegative solutions of the above system.

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1. Introduction. We consider the two point boundary value problem with Neumann-Robin boundary conditions

$$-u''(x) = \lambda f(u(x)), \quad x \in (0,1), \tag{1.1}$$

$$u'(0) = 0, \tag{1.2}$$

$$u'(1)+\alpha u(1) = 0, \tag{1.3}$$

where $\alpha > 0, \lambda > 0$ are parameters, $f \in c^2[0,\infty)$ and $f(0) < 0$, and we will assume that there exist $\beta,\theta > 0$ such that $f(s) < 0$ on $[0,\beta)$, $f(\beta) = 0$, $f'(s) \geq 0$, $f''(s) > 0$, $\lim_{s\to\infty}(f(s)/s) = \infty$, and $F(\theta) = 0$ where $F(s) = \int_0^s f(t)dt$. It is proved in [1, Theorems 3.4.1(a) and 3.4.1(b)] that for any $n = 0,1,2,\ldots$, $\alpha \in (0,\infty)$, $\rho = \theta$ ($\rho = 0$), (1.1), (1.2), and (1.3) have exactly two nonnegative solutions $u_{2n,i}(u_{2n+1,i})$, $i = 1,2$ with $2n$ (and $2n+1$) interior critical points. Also it is shown in [3, Theorem 1.4] that for the following Dirichlet boundary conditions

$$\begin{gathered} -u''(x) = \lambda f(u(x)), \quad x \in (0,1), \\ u(0) = 0 = u(1), \end{gathered} \tag{1.4}$$

where n is a positive integer, there exists $\lambda^* > 0$ such that (1.4) has a unique nonnegative solution with n interior zeros if and only if $\lambda = (n+1)^2\lambda^*$. Equation (1.1) in the cases Neumann and Dirichlet-Robin boundary conditions have been studied in [2, 4], respectively. We discuss the relation between interior critical points of nonnegative solutions and λ's for problem (1.1), (1.2), and (1.3) for the case $\rho = \theta$ in Section 2, and for the case $\rho = 0$ in Section 3. Finally, in Section 4 we compare λ's in two cases $\rho = \theta$ and $\rho = 0$ for any $n = 0,1,2,\ldots$.

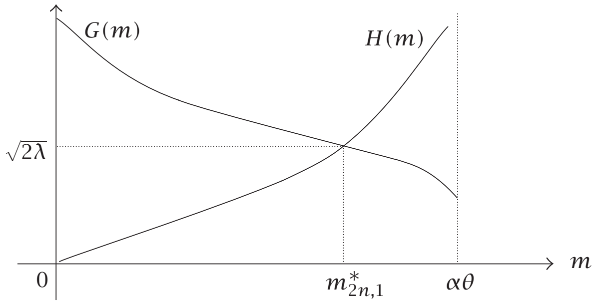

Figure 2.1

2. The case $\rho = \theta$. In [1] it has been established that for $\alpha \in (0,\infty)$, $\rho = \theta$, and $n = 0,1,2,\ldots$, there exists a unique number $m^*_{2n,1} \in (0,\alpha\theta)$ such that

$$G(m^*_{2n,1}) = H(m^*_{2n,1}), \tag{2.1}$$

where

$$\begin{aligned} G(m) &= \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}}ds + 2n\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds, \quad m \in (0,\alpha\theta), \\ H(m) &= \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m \in (0,\alpha\theta). \end{aligned} \tag{2.2}$$

So we obtain $\lambda = \lambda_{2n,1}(\theta, m^*_{2n,1})$ such that $\sqrt{2\lambda} = G(m^*_{2n,1}) = H(m^*_{2n,1})$ (see Figure 2.1), that is,

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}}\int_{m/\alpha}^{\theta}\frac{1}{\sqrt{-F(s)}}ds + \frac{2n}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds \quad m = m^*_{2n,1}. \tag{2.3}$$

Thus (1.1), (1.2), and (1.3) have exactly a nonzero solution $u_{2n,1}$ with $2n$ interior critical points where $u'_{2n,1}(1) = -m^*_{2n,1}$ and $u'_{2n,1}(0) = \theta$ at $\lambda = \lambda_{2n,1}(\theta, m^*_{2n,1})$. Also, the equation

$$\sqrt{\lambda} = \frac{2n+1}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds \tag{2.4}$$

has a unique solution $\lambda = \lambda_{2n,2}(\theta,0)$ such that for this λ problem, (1.1), (1.2), and (1.3) have exactly a nonnegative solution $u_{2n,2}$ with $2n$ interior critical points such that $u'_{2n,2}(1) = 0$ and $u_{2n,2}(0) = \theta$.

In [1], it is proved that

$$\lambda_{2n,1}(\theta, m^*_{2n,1}) < \lambda_{2n,2}(\theta,0) < \lambda_{2(n+1),1}(\theta, m^*_{2(n+1),1}). \tag{2.5}$$

Now we are ready to prove the main theorem of this section.

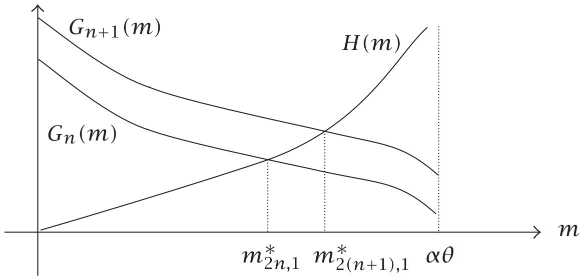

FIGURE 2.2

THEOREM 2.1. *Let $n = 0, 1, 2, \ldots$, then*

$$\lambda_{2n,2} - \lambda_{2n,1} < \lambda_{2(n+1),2} - \lambda_{2(n+1),1}, \tag{2.6}$$

that is, $n \mapsto \lambda_{2n,2} - \lambda_{2n,1}$ is a strictly increasing function.

PROOF. Since $G(m)$ is dependent on n, so we write it by $G_n(m)$, that is,

$$G_n(m) = \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + 2n \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m \in (0, \alpha\theta). \tag{2.7}$$

So it is easy to see that $\{G_n(m)\}_{n=0}^{\infty}$ is a strictly increasing sequence of n for every $m \in (0, \alpha\theta)$, that is,

$$G_n(m) < G_{n+1}(m), \quad m \in (0, \alpha\theta) \tag{2.8}$$

and we can easily see that

$$m^*_{2n,1} < m^*_{2(n+1),1}, \quad n = 0, 1, 2, \ldots \tag{2.9}$$

(see Figure 2.2). On the other hand, from (2.3) and (2.4) we have

$$\begin{aligned} \sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2n,1}, \\ \sqrt{\lambda_{2(n+1),2}} - \sqrt{\lambda_{2(n+1),1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2(n+1),1}. \end{aligned} \tag{2.10}$$

Thus combining (2.9) and (2.10) we obtain

$$\sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2(n+1),2}} - \sqrt{\lambda_{2(n+1),1}}. \tag{2.11}$$

Also combining (2.4) and (2.9) we have

$$\sqrt{\lambda_{2n,2}}+\sqrt{\lambda_{2n,1}} = \frac{4n+1}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds+\frac{1}{\sqrt{2}}\int_{m/\alpha}^{\theta}\frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2n,1}, \tag{2.12}$$

$$\sqrt{\lambda_{2(n+1),2}}+\sqrt{\lambda_{2(n+1),1}} = \frac{4n+5}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds+\frac{1}{\sqrt{2}}\int_{m/\alpha}^{\theta}\frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2(n+1),1}. \tag{2.13}$$

Since $0<m/\alpha<\theta$ for $m=m^*_{2n,1}$, so we have

$$\int_{m/\alpha}^{\theta}\frac{1}{\sqrt{-F(s)}}ds<\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2n,1}, \tag{2.14}$$

and then

$$\frac{1}{\sqrt{2}}\int_{m/\alpha}^{\theta}\frac{1}{\sqrt{-F(s)}}ds+\frac{4n+1}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds<\frac{1}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds+\frac{4n+1}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds. \tag{2.15}$$

Now from (2.12) and (2.15) we obtain

$$\sqrt{\lambda_{2n,2}}+\sqrt{\lambda_{2n,1}}<\frac{4n+2}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds. \tag{2.16}$$

On the other hand, by the positivity of

$$\frac{3}{\sqrt{2}}\int_0^{\theta}\frac{1}{\sqrt{-F(s)}}ds+\frac{1}{\sqrt{2}}\int_{m/\alpha}^{\theta}\frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2(n+1),1}, \tag{2.17}$$

and also from (2.13) and (2.16) we obtain

$$\sqrt{\lambda_{2n,2}}+\sqrt{\lambda_{2n,1}}<\sqrt{\lambda_{2(n+1),2}}+\sqrt{\lambda_{2(n+1),1}}. \tag{2.18}$$

Now combining (2.11) and (2.18) we obtain

$$\begin{aligned}\left(\sqrt{\lambda_{2n,2}}-\sqrt{\lambda_{2n,1}}\right)\left(\sqrt{\lambda_{2n,2}}+\sqrt{\lambda_{2n,1}}\right)&<\left(\sqrt{\lambda_{2n,2}}+\sqrt{\lambda_{2n,1}}\right)\left(\sqrt{\lambda_{2(n+1),2}}-\sqrt{\lambda_{2(n+1),1}}\right)\\&<\left(\sqrt{\lambda_{2(n+1),2}}-\sqrt{\lambda_{2(n+1),1}}\right)\left(\sqrt{\lambda_{2(n+1),2}}+\sqrt{\lambda_{2(n+1),1}}\right)\end{aligned} \tag{2.19}$$

and so,

$$\lambda_{2n,2}-\lambda_{2n,1}<\lambda_{2(n+1),2}-\lambda_{2(n+1),1}, \tag{2.20}$$

thus, the proof is complete. □

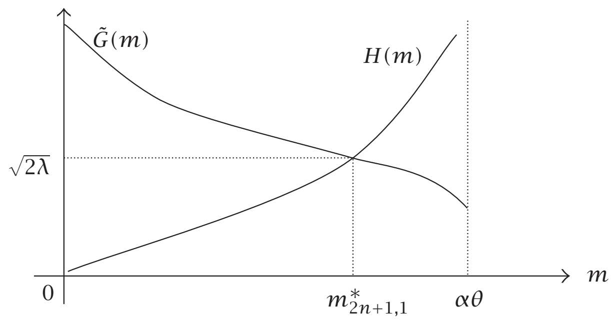

FIGURE 3.1

3. The case $\rho = 0$. Also in [1] it has been established that for $\alpha \in (0,\infty)$, $\rho = 0$, and $n = 0,1,2,\dots$, there exists a unique number $m^*_{2n+1,1} \in (0,\alpha\theta)$ such that

$$\tilde{G}(m^*_{2n+1,1}) = H(m^*_{2n+1,1}), \tag{3.1}$$

where

$$\begin{aligned} \tilde{G}(m) &= \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + (2n+1) \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m \in (0,\alpha\theta), \\ H(m) &= \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m \in (0,\alpha\theta). \end{aligned} \tag{3.2}$$

So we obtain $\lambda = \lambda_{2n+1,1}(0,m^*_{2n+1,1})$ such that $\sqrt{2\lambda} = \tilde{G}(m^*_{2n+1,1}) = H(m^*_{2n+1,1})$ (see Figure 3.1), that is,

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + \frac{2n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2n+1,1}. \tag{3.3}$$

Thus (1.1), (1.2), and (1.3) have exactly a nonzero solution $u_{2n+1,1}$ with $2n+1$ interior critical points where $u'_{2n+1,1}(1) = -m^*_{2n+1,1}$ and $u'_{2n+1,1}(0) = 0$ at $\lambda = \lambda_{2n+1,1}(0,m^*_{2n+1,1})$. Also, the equation

$$\sqrt{\lambda} = \frac{2(n+1)}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds \tag{3.4}$$

has a unique solution $\lambda = \lambda_{2n+1,2}(0,0)$ such that for this λ problem, (1.1), (1.2), and (1.3) have exactly a nonnegative solution $u_{2n+1,2}$ with $2n+1$ interior critical points such that $u'_{2n+1,2}(1) = 0$ and $u_{2n+1,2}(0) = 0$.

In [1], it is proved that

$$\lambda_{2n+1,1}(0,m^*_{2n+1,1}) < \lambda_{2n+1,2}(0,0) < \lambda_{2(n+1)+1,1}(0,m^*_{2(n+1)+1,1}). \tag{3.5}$$

Now we are ready to prove the main theorem of this section.

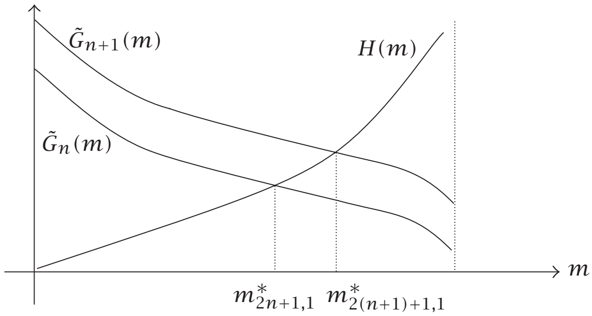

Figure 3.2

Theorem 3.1. *Let $n = 0, 1, 2, \ldots,$ then*

$$\lambda_{2n+1,2} - \lambda_{2n+1,1} < \lambda_{2(n+1)+1,2} - \lambda_{2(n+1)+1,1}, \tag{3.6}$$

that is, $n \mapsto \lambda_{2n+1,2} - \lambda_{2n+1,1}$ is a strictly increasing function.

Proof. Since $\tilde{G}(m)$ is dependent on n, so we write it by $\tilde{G}_n(m)$, that is,

$$\tilde{G}_n(m) = \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + (2n+1) \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m \in (0, \alpha\theta). \tag{3.7}$$

It is easy to see that $\{\tilde{G}_n(m)\}_{n=0}^{\infty}$ is a strictly increasing sequence of n for every $m \in (0, \alpha\theta)$, that is,

$$\tilde{G}_n(m) < \tilde{G}_{n+1}(m), \quad m \in (0, \alpha\theta) \tag{3.8}$$

(see Figure 3.2) and we can easily see that

$$m^*_{2n+1,1} < m^*_{2(n+1)+1,1}, \quad n = 0, 1, 2, \ldots \tag{3.9}$$

(see Figure 3.2). On the other hand, from (3.3) and (3.4) we have

$$\begin{aligned}
\sqrt{\lambda_{2n+1,2}} - \sqrt{\lambda_{2n+1,1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2n+1,1}, \\
\sqrt{\lambda_{2(n+1)+1,2}} - \sqrt{\lambda_{2(n+1)+1,1}} &= \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds - \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2(n+1)+1,1}.
\end{aligned} \tag{3.10}$$

Thus, combining (3.9) and (3.10) we obtain

$$\sqrt{\lambda_{2n+1,2}} - \sqrt{\lambda_{2n+1,1}} < \sqrt{\lambda_{2(n+1)+1,2}} - \sqrt{\lambda_{2(n+1)+1,1}}. \tag{3.11}$$

Also combining (3.3) and (3.4) we have

$$\begin{aligned} &\sqrt{\lambda_{2n+1,2}}+\sqrt{\lambda_{2n+1,1}} \\ &\quad = \frac{4n+3}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds+\frac{1}{\sqrt{2}}\int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2n+1,1}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\sqrt{\lambda_{2(n+1)+1,2}}+\sqrt{\lambda_{2(n+1)+1,1}} \\ &\quad = \frac{4n+7}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds+\frac{1}{\sqrt{2}}\int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2(n+1)+1,1}. \end{aligned} \tag{3.13}$$

Since $0<m/\alpha<\theta$ for $m=m^*_{2n+1,1}$, so we have

$$\int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}}ds<\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2n+1,1}, \tag{3.14}$$

and then

$$\begin{aligned} &\frac{1}{\sqrt{2}}\int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}}ds+\frac{4n+3}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds \\ &\qquad <\frac{1}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds+\frac{4n+3}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds. \end{aligned} \tag{3.15}$$

Now from (3.12) and (3.15) we obtain

$$\sqrt{\lambda_{2n+1,2}}+\sqrt{\lambda_{2n+1,1}}<\frac{4n+4}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds. \tag{3.16}$$

On the other hand, by the positivity of

$$\frac{3}{\sqrt{2}}\int_0^\theta \frac{1}{\sqrt{-F(s)}}ds+\frac{1}{\sqrt{2}}\int_{m/\alpha}^\theta \frac{1}{\sqrt{-F(s)}}ds, \quad m=m^*_{2(n+1)+1,1}, \tag{3.17}$$

and also from (3.13) and (3.16) we obtain

$$\sqrt{\lambda_{2n+1,2}}+\sqrt{\lambda_{2n+1,1}}<\sqrt{\lambda_{2(n+1)+1,2}}+\sqrt{\lambda_{2(n+1)+1,1}}. \tag{3.18}$$

Now combining (3.11) and (3.18) we have

$$\lambda_{2n+1,2}-\lambda_{2n+1,1}<\lambda_{2(n+1)+1,2}-\lambda_{2(n+1)+1,1}, \tag{3.19}$$

thus, the proof is complete. □

4. Comparing the two cases $\rho = 0$ and $\rho = \theta$. Now we compare λ's in the two cases $\rho = 0$ and $\rho = \theta$ for any $n = 0, 1, 2, \ldots$, and we are ready to prove the main theorem of this section.

Theorem 4.1. *Let $n = 0, 1, 2, \ldots$, then*

$$\lambda_{2n,2} - \lambda_{2n,1} < \lambda_{2n+1,2} - \lambda_{2n+1,1}, \tag{4.1}$$

that is, the distance between $\lambda_{2n,1}(\theta, m^*_{2n,1})$ and $\lambda_{2n,2}(\theta, 0)$ is less than the distance between $\lambda_{2n+1,1}(0, m^*_{2n+1,1})$ and $\lambda_{2n+1,2}(0,0)$.

Proof. Since $0 < m/\alpha < \theta$ for $m = m^*_{2n,1}$, we have

$$\int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds < \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2n,1}, \tag{4.2}$$

and then

$$\frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds + \frac{2n}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds < \frac{2n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2n,1}. \tag{4.3}$$

Now, from (4.3) and (2.3) we obtain

$$\sqrt{\lambda_{2n,1}} < \frac{2n+1}{\sqrt{2}} \int_0^{\theta} \frac{1}{\sqrt{-F(s)}} ds. \tag{4.4}$$

On the other hand, since $0 < m/\alpha < \theta$ for $m = m^*_{2n+1,1}$, then we have

$$0 < \frac{1}{\sqrt{2}} \int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}} ds, \quad m = m^*_{2n+1,1}, \tag{4.5}$$

and then from (3.3) we have

$$\sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2n+1,1}}. \tag{4.6}$$

Also, we know that

$$\begin{aligned} \sqrt{2\lambda_{2n,1}} &= H(m) = \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m = m^*_{2n,1}, \\ \sqrt{2\lambda_{2n+1,1}} &= H(m) = \frac{m}{\sqrt{-F(m/\alpha)}}, \quad m = m^*_{2n+1,1}. \end{aligned} \tag{4.7}$$

Now since function H is one to one on interval $(0, \alpha\theta)$ (see Figure 4.1), we have

$$m^*_{2n,1} < m^*_{2n+1,1}, \tag{4.8}$$

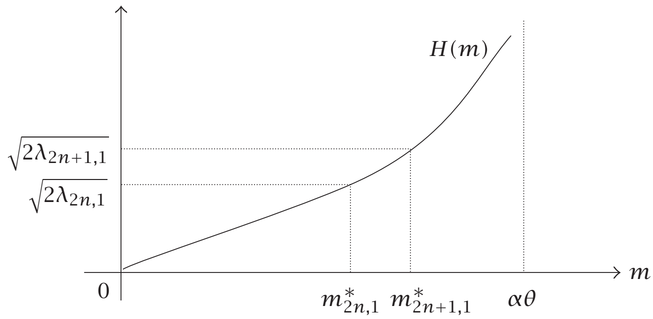

FIGURE 4.1

so we obtain

$$\int_{m_{2n+1,1}^*/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}}ds < \int_{m_{2n,1}^*/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}}ds. \tag{4.9}$$

Thus combining (2.10), (3.10), and (4.9) we obtain

$$\sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2n+1,2}} - \sqrt{\lambda_{2n+1,1}}. \tag{4.10}$$

On the other hand, since $0 < m/\alpha < \theta$ for $m = m_{2n,1}^*$, we have

$$\int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}}ds < \int_{0}^{\theta} \frac{1}{\sqrt{-F(s)}}ds, \quad m = m_{2n,1}^*, \tag{4.11}$$

and then

$$\frac{4n+1}{\sqrt{2}}\int_{0}^{\theta} \frac{1}{\sqrt{-F(s)}}ds + \frac{1}{\sqrt{2}}\int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}}ds < \frac{4n+2}{\sqrt{2}}\int_{0}^{\theta} \frac{1}{\sqrt{-F(s)}}ds, \quad m = m_{2n,1}^*, \tag{4.12}$$

and thus from (2.12) we obtain

$$\sqrt{\lambda_{2n,2}} - \sqrt{\lambda_{2n,1}} < \frac{4n+2}{\sqrt{2}}\int_{0}^{\theta} \frac{1}{\sqrt{-F(s)}}ds, \tag{4.13}$$

and also from (4.13) we have

$$\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}} < \frac{4n+3}{\sqrt{2}}\int_{0}^{\theta} \frac{1}{\sqrt{-F(s)}}ds + \frac{1}{\sqrt{2}}\int_{m/\alpha}^{\theta} \frac{1}{\sqrt{-F(s)}}ds, \quad m = m_{2n+1,1}^*. \tag{4.14}$$

So from (4.13) and (3.12), we obtain

$$\sqrt{\lambda_{2n,2}} + \sqrt{\lambda_{2n,1}} < \sqrt{\lambda_{2n+1,2}} + \sqrt{\lambda_{2n+1,1}}, \tag{4.15}$$

and also from (4.15) and (4.10), we have

$$\lambda_{2n,2} - \lambda_{2n,1} < \lambda_{2n+1,2} - \lambda_{2n+1,1} \tag{4.16}$$

thus, the proof is complete. □

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