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THE UNION PROBLEM ON COMPLEX MANIFOLDS

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Let Ω be a relatively compact subdomain of a complex manifold, exhaustable by Stein open sets. We give a necessary and sufficient condition for Ω to be Stein, in terms of L^2-estimates for the $\bar{\partial}$-operator, equivalent to the condition of Markoe (1977) and Silva (1978).

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1. Introduction. As indicated in [7], from the beginning of the theory of Stein spaces, the following question has held great interest: is a complex space, which is exhaustable by a sequence $X_1 \Subset X_2 \Subset \cdots$ of Stein subspaces, itself Stein?

In [1], the following is proved: every domain in $\mathbb{C}^m$ which is exhaustable by a sequence of Stein domains $B_1 \Subset B_2 \Subset \cdots$ is itself Stein, and this is shown to hold more generally for unramified Riemann domain $\mathcal{B}$ over $\mathbb{C}^m$ in [6]. In [11], the following is proved: let X be a reduced complex space and $X_1 \Subset X_2 \Subset \cdots$ be an exhaustion of X by Stein domains, if every pair (X_j, X_{j+1}) is Runge then $X = UX_j$ is Stein. Recently, Markoe [9] and Silva [10] proved the following: let X be reduced and $X_1 \Subset X_2 \Subset \cdots$ be an exhaustion of X by Stein domains. Then X is Stein if and only if $H^1(X,\mathbb{O}) = 0$ ($\mathbb{O}$ being the structure sheaf of X).

More recently the following has been proved in [12]: let $\Omega_1 \subset \Omega_2 \subset \cdots$ be a sequence of open Stein subsets of a Stein space X, $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, and $\dim H^1(\Omega,\mathbb{O}) < \infty$. Then Ω is Stein.

Fornæss [4] produced an example to show that if $X_1 \Subset X_2 \Subset \cdots$ is a sequence of Stein manifolds, the limit manifold $X = \bigcup X_j$, in which each X_j is an open submanifold, need not be Stein. But it is known that if the limit manifold is itself an open submanifold of a Stein manifold then the limit manifold is necessarily Stein.

This led Fornæss and Narasimhan to pose the following problem [5]: let X be a Stein space and $\Omega_1 \Subset \Omega_2 \Subset \cdots$ an increasing sequence of Stein open sets in X. Is $\bigcup \Omega_j$ Stein? As indicated above this is the case when X is a Stein manifold, but this question remains open in the general case.

In this paper, we consider the case where X is a general complex manifold and $\Omega_1 \Subset \Omega_2 \Subset \cdots$ an increasing sequence of open Stein manifolds in X such that $\Omega = \bigcup \Omega_j$ is relatively compact in X. We give a condition for Ω to be Stein, equivalent to Markoe's and Silva's condition and involving L^2-estimates for the $\bar{\partial}$ operator.

2. Preliminaries. Let X be an n-dimensional complex manifold with a C^∞ Hermitian metric. The space $L^2_{(p,q)}(X)$ of square integrable differential forms of type (p,q) on X

is a Hilbert space under the scalar product,

$$(f,g) = \int_X f_\wedge * \bar{g}, \tag{2.1}$$

where $*$ is the Hodge $*$-operator associated with the metric and orientation of X.

Let $\Omega_1 \Subset \Omega_2 \Subset \cdots$ be an increasing sequence of Stein open sets in X such that their union $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ is relatively compact in X.

The following theorem is our main result.

Theorem 2.1. *The union Ω is Stein if and only if given an $f \in L^2_{(p,q)}(\Omega)$, which is $\bar\partial$-closed in the sense of distributions, there is a $u \in L^2_{(p,q-1)}(\Omega)$ such that $\bar\partial u = f$ in the sense of distributions and*

$$\|u\|_{L^2_{(p,q-1)(\Omega)}} \le K\|f\|_{L^2_{(p,q)(\Omega)}}, \quad q > 0, \tag{2.2}$$

where K depends on Ω.

Let U be a bounded open set in $\mathbb{C}^n$, and $\mathbb{O}$ the structure sheaf of $\mathbb{C}^n$. A section $f = (f_1,\dots,f_p) \in \Gamma(U,\mathbb{O}^p)$, where $p > 0$ is an integer, is L^2-bounded if

$$\|f\|_{L^2(U)} = \|f_1\|_{L^2(U)} + \cdots + b\|f_p\|_{L^2(U)} < \infty. \tag{2.3}$$

We then denote all sections of $\mathbb{O}^p$ over U that are L^2-bounded by $\Gamma_2(U,\mathbb{O}^p)$.

For the definition of L^2-bounded sections of coherent analytic sheaves, we require the coherent analytic sheaf $\mathcal{F}$ to be defined on a simply connected polycylinder neighborhood V of the closure of U. Then by [8, Theorem 5, Section F, Chapter VI], there is an $\mathbb{O}$-homographic in another simply connected polycylinder neighborhood V' of the closure of U,

$$\mathbb{O}^p \xrightarrow{\lambda} \mathcal{F} \longrightarrow 0, \tag{2.4}$$

where $p > 0$ is some integer; and $f \in \Gamma(U,\mathcal{F})$ is L^2-bounded if $f \in \Gamma_2(U,\mathcal{F}) := \lambda(\Gamma_2(U,\mathbb{O}^p))$. It can be shown that $\Gamma_2(U,\mathcal{F})$ is independent of λ and p, so that $\Gamma_2(U,\mathcal{F})$ is well defined.

Now let Ω be a relatively compact subdomain of an n-dimensional complex manifold X. An open subset Y of Ω is said to be admissible for the coherent analytic sheaf $\mathcal{F}$ defined in the neighborhood of the closure of Ω in X, if Y is Stein. There is a coordinate neighborhood V in X of the closure, $\bar{Y}$ of Y such that V is biholomorphic to a simply connected polycylinder V' in $\mathbb{C}^n$, and $\bar{Y}$ is contained in the neighborhood of $\bar\Omega$ where $\mathcal{F}$ is defined as $f \in \Gamma(Y,\mathcal{F})$ which is L^2-bounded if

$$f \in \Gamma_2(Y,\mathcal{F}) := \{g \in \Gamma(Y,\mathcal{F}) : \eta_*(g) \in \Gamma_2(\eta(Y),\eta_*(\mathcal{F}))\}, \tag{2.5}$$

where η is the restriction of the biholomorphic map $V \to V^1$ to Y, and $\eta_*(\mathcal{F})$ is the zero direct image of $\mathcal{F}$ on Y.

Let Ω be as in Theorem 2.1 (then clearly Ω is locally Stein). Let $\mathcal{F}$ be a coherent analytic sheaf in a neighborhood of the closure of Ω. Then it is clear that Ω is a finite union, $\Omega = \bigcup_{j=1}^{m} U_j$, where each U_j is admissible for $\mathcal{F}$. If $\mathcal{V} = \{U_j\}_{j\in I}$, $I = \{1,\dots,m\}$, where the U_j's are as above, we say that $\mathcal{V}$ is a finite admissible cover of Ω for $\mathcal{F}$ and we define the L^2 (alternate) q-cochains of $\mathcal{V}$ with values in $\mathcal{F}$ as those cochains,

$$c = (c_\alpha) \in C^q(\mathcal{V},\mathcal{F}) = \prod_{\alpha \in I^{q+1}} \Gamma(U_\alpha,\mathcal{F}),$$
$$U_\alpha = U_{i_0} \cap \cdots \cap U_{i_q}, \quad \alpha = (i_0,\ldots,i_q), \tag{2.6}$$

which are alternate and satisfy $c_\alpha \in \Gamma_2(U_\alpha,\mathcal{F})$ for all $\alpha \in I^{q+1}$. We denote by $C_2^q(\mathcal{V},\mathcal{F})$ the space of L^2-bounded cochains.

The coboundary operator,

$$\delta : C^q(\mathcal{V},\mathcal{F}) \longrightarrow C^{q+1}(\mathcal{V},\mathcal{F}), \tag{2.7}$$

maps $C_2^q(\mathcal{V},\mathcal{F})$ into $C_2^{q+1}(\mathcal{V},\mathcal{F})$. If $Z_2^q(\mathcal{V},\mathcal{F}) = \{c \in C_2^q(\mathcal{V},\mathcal{F}) : \delta c = 0\}$ and $B_2^q(\mathcal{V},\mathcal{F}) = \delta C_2^{q-1}(\mathcal{V},\mathcal{F})$, then as usual $B_2^q(\mathcal{V},\mathcal{F}) \subseteq Z_2^q(\mathcal{V},\mathcal{F})$ and we define $H_2^q(\mathcal{V},\mathcal{F}) := Z_2^q(\mathcal{V},\mathcal{F})/B_2^q(\mathcal{V},\mathcal{F})$ and call it the L^2-bounded cohomology of $\mathcal{V}$ with values in $\mathcal{F}$. We then have the following theorem.

Theorem 2.2. *For any $q > 0$, the natural map*

$$H_2^q(\mathcal{V},\mathcal{F}) \longrightarrow H^q(\Omega,\mathcal{F}) \tag{2.8}$$

is an isomorphism.

We use Theorem 2.2 as a pivot to prove Theorem 2.1, but the proof of Theorem 2.2 is not given here, since it is similar to that of [2, Theorem].

3. A triangle of isomorphisms. Let Ω be as in Theorem 2.1. By the end of the section Theorem 2.1 will be proved. If $U \neq \varnothing$ is an open set in $\bar{\Omega}$, then $\mathcal{B}_\Omega^p(U)$ is the Hilbert space of holomorphic p-forms h on $\Omega \cap U$ such that

$$\|h\|_{L^2_{(p,0)}(\Omega\cap U)} < \infty. \tag{3.1}$$

If V is open in $\bar{\Omega}$ with $\varnothing \neq V \subset U$, the restriction map $\gamma_V^U \colon \mathcal{B}_\Omega^p(U) \to \mathcal{B}_\Omega^p(V)$ is defined. Then $\mathcal{B}_0^p = \{\mathcal{B}_\Omega^p(U), \gamma_V^U\}$ is the canonical presheaf of L^2-holomorphic p-forms on $\bar{\Omega}$. The associated sheaf $\mathcal{B}_2^p$ is the sheaf of germs of L^2-holomorphic p-forms on $\bar{\Omega}$. We then have the following lemma.

Lemma 3.1. *Let $\mathcal{O}^p$ be the sheaf of germs of holomorphic p-forms on X, and $\mathcal{V}$ a finite admissible cover of Ω for $\mathcal{O}^p$. Then the following diagram is an isomorphism triangle of cohomology groups:*

$$\begin{array}{ccc} H_2^q(\mathcal{V},\mathcal{O}^p) & \xrightarrow{\ \sim\ } & H^q(\Omega,\mathcal{O}^p) \\ & \searrow \quad \nearrow & \\ & H^q(\bar{\Omega},\mathcal{B}_2^p) & \end{array} \tag{3.2}$$

for $q \geq 1$ and $p \geq 0$.

Proof. From Theorem 2.2 and the fact that any finite cover of $\bar{\Omega}$ has a refinement $\mathcal{U} = \{V_j\}_{j\in J}$ such that $\mathcal{U}_\Omega = \{V_j \cap \Omega\}_{j\in J}$ is a finite admissible cover of Ω for $\mathcal{D}^p$, the lemma follows. □

Now, using Hörmander's L^2-estimates locally we get the following lemma.

Lemma 3.2. *The cohomology group $H^q(\bar{\Omega}, \mathcal{B}_2^p)$ is isomorphic to the quotient space*

$$\{g : g \in L^2_{(p,q)}(\Omega) \text{ and } \bar{\partial} g = 0\} / \{\bar{\partial} h : h \in L^2_{p,q-1}(\Omega) \text{ and } \bar{\partial} h \in L^2_{(p,q)}(\Omega)\}, \tag{3.3}$$

where Ω is as in Theorem 2.1.

Also the following lemma is proved in [3].

Lemma 3.3. *If $\Omega \Subset X$ is Stein, where X is a complex manifold, then given $f \in L^2_{(p,q)}(\Omega)$ with $\bar{\partial} f = 0$, there is $u \in L^2_{(p,q-1)}(\Omega)$ such that*

$$\bar{\partial} u = f, \qquad \|u\|_{L^2{}_{(p,q-1)}(\Omega)} \le K \|f\|_{L^2(p,q)(\Omega)}, \tag{3.4}$$

where K depends on Ω.

To finish with the proof of Theorem 2.1 we remark that $\mathcal{D}^0 = \mathbb{O}$ is the structure sheaf of X (X, Ω as in Theorem 2.1), therefore Theorem 2.1 follows from Lemmas 3.1, 3.2, and 3.3, and from Markoe's and Silva's condition.

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