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SMOOTHNESS PROPERTIES OF FUNCTIONS IN $R^2(X)$ AT CERTAIN BOUNDARY POINTS

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ABSTRACT. Let X be a compact subset of the complex plane $\mathbb{C}$. We denote by $R_0(X)$ the algebra consisting of the (restrictions to X of) rational functions with poles off X. Let m denote 2-dimensional Lebesgue measure. For $p \geq 1$, let $R^p(X)$ be the closure of $R_0(X)$ in $L^p(X,dm)$.

In this paper, we consider the case $p = 2$. Let $x \in \partial X$ be both a bounded point evaluation for $R^2(X)$ and the vertex of a sector contained in Int X. Let L be a line which passes through x and bisects the sector. For those $y \in L \cap X$ that are sufficiently near x we prove statements about $|f(y) - f(x)|$ for all $f \in R^2(X)$.

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1. INTRODUCTION.

Let X be a compact subset of the complex plane $\mathbb{C}$. We denote by $R_0(X)$ the algebra consisting of the (restrictions to X of) rational functions with poles off X. Let m denote 2-dimensional Lebesgue measure. For $p \geq 1$, let $L^p(X) = L^p(X,dm)$. The closure of $R_0(X)$ in $L^p(X)$ will be denoted by $R^p(X)$. Whenever p and q both appear, we will assume that $p^{-1} + q^{-1} = 1$.

In "Bounded point evaluations and smoothness properties of functions in $R^p(X)$", [6, p. 76], we proved the following:

THEOREM 1.1. Let ϕ be an admissible function and s a nonnegative integer. Suppose that $p > 2$ and that there is an $x \in X$ represented by a function $g \in L^q(X)$ such that $(z-x)^{-s}\phi(|z-x|)^{-1}g \in L^q(X)$. Then for every $\varepsilon > 0$ there is a set E in X having full area density at x such that for every $f \in R^p(X)$

(i) $f = \sum_{j=0}^{s} (D_x^j f)(z-x)^j + R$ where $R \in R^p(X)$ satisfies

(ii) $|R(y)| \leq \varepsilon|y-x|^s\phi(|y-x|)\,||f||_p$ for all $y \in E$, and

(iii) $\operatorname{app\,lim}_{y\to x} \dfrac{R(y)}{|y-x|^s\phi(|y-x|)} = 0.$

It is natural to ask whether a similar result holds for the case $p = 2$. The problem in extending the proof of Theorem 1.1 to the case $p = 2$ is that $z^{-1} \notin L^2_{loc}$. Fernström and Polking have shown at least one way in which the case $p > 2$ differs from $p = 2$ [2, pp. 5-9]. They have constructed a compact set X such that $R^2(X) \neq L^2(X)$ but no point in X is a bounded point evaluation for $R^2(X)$. In this paper we consider the case $p = 2$ when $x \in \partial X$ is a bounded point evaluation for $R^2(X)$ and is a special kind of boundary point. We will assume that $x \in \partial X$ is the vertex of a sector contained in Int X.

To prove our theorem we will need the representing functions used in [6] and a capacity defined in terms of a Bessel kernel. We will also use results of Fernström and Polking to construct a representing function for x with support outside the sector mentioned above.

2. REPRESENTING FUNCTIONS.

In this paper z will denote the identity function.

DEFINITION 2.1. A point $x \in X$ is a bounded point evaluation (BPE) for $R^2(X) \subset L^2(X)$ if there is a constant C such that

$$|f(x)| \leq C\{\int |f|^2 dm\}^{1/2} \quad \text{for all } f \in R^2(X).$$

It follows from the Riesz representation theorem that if $x \in X$ is a BPE for $R^2(X)$ then there is a function $g \in L^2(X)$ such that $f(x) = \int fg\, dm$ for all $f \in R^2(X)$. Such a g is called a representing function for x.

DEFINITION 2.2. We define the Cauchy transform of g to be

$$\hat{g}(y) = \int (z-y)^{-1} g\, dm$$

for each y such that $\int |z-y|^{-1}|g|dm < \infty$.

The following lemma was proved by Bishop for the sup norm case. The proof for our case is similar and is found in [6, p. 73].

LEMMA 2.1. Suppose that $g \in L^2(X)$ and that $\int fg\, dm = 0$ for all $f \in R^2(X)$. Suppose that $\hat{g}(y)$ is defined and $\neq 0$ and that $(z-y)^{-1}g \in L^2(X)$. Then $\hat{g}(y)^{-1}(z-y)^{-1}g$ is a representing function for y.

Let $c(y) = \int (z-x)(z-y)^{-1}g\, dm = 1 + (y-x)\hat{g}(y)$. From the above lemma there follows

COROLLARY 2.1. Let $g \in L^2(X)$ be a representing function for $x \in X$. Then $c(y)^{-1}(z-x)(z-y)^{-1}g$ is a representing function for y whenever $c(y)$ is defined and $\neq 0$, and $(z-y)^{-1}g \in L^2(X)$.

3. CAPACITY DEFINED USING A BESSEL KERNEL.

Denote the Bessel kernel of order 1 by G_1 where G_1 is defined in terms of its Fourier transform by

$$\hat{G}_1(z) = (1+|z|^2)^{-1/2}.$$

For $f \in L^2(C)$ we define the potential

$$U_1^f(z) = \int G_1(z-y)f(y)dm(y).$$

DEFINITION. $\mathcal{L}_1^2$ denotes the space of functions U_1^f, $f \in L^2$, where the norm is defined by $||U_1^f|| = ||f||_2$.

DEFINITION. L_1^2 is the Sobolev space of functions in L^2 whose distribution derivatives of order 1 are functions in L^2.

The Calderón-Zygmund theory shows that $\mathcal{L}_1^2$ equals the space of functions L_1^2 and that the norms are equivalent [4].

We recall the definition of the capacity Γ_2.

DEFINITION. Let $E \subset ¢$ be an arbitrary set. Then $\Gamma_2(E) = \inf_\omega \int |\text{grad } \omega|^2 dm$ where the infimum is taken over all $\omega \in L_1^2$ such that $\omega \geq 1$ on E. Hedberg has used this capacity to characterize BPE's for $R^2(X)$ [3]. The next theorem is proved in [6, p. 82].

THEOREM 3.1. Let $0 \in X$ be a BPE for $R^2(X)$ that is represented by a function $v \in L^2(X)$. Suppose that ϕ is an admissible function such that $\phi(|z|)^{-1}v \in L^2(X)$. Then $\sum_{n=1}^{\infty} 2^{2n}\phi(2^{-n})^{-2}\Gamma_2(A_n \backslash X) < \infty$.

REMARK. The theorem is, in fact, true if ϕ is any positive non-decreasing function defined on $(0,\infty)$.

Now we define the Bessel capacity which Fernström and Polking use to describe BPE's for $R^2(X)$.

DEFINITION. Let $E \subset ¢$ be an arbitrary set. Then $C_{1,2}(E) = \inf \int |f|^2 dm$ where the infimum is taken over all $f \in L^2(¢)$ such that $f(z) \geq 0$ and $U_1^f(z) \geq 1$ for all $z \in E$.

The equivalence of the norms on $\mathcal{L}_1^2$ and L_1^2 implies that the capacities Γ_2 and $C_{1,2}$ are equivalent.

4. A FUNDAMENTAL SOLUTION FOR $\frac{\partial}{\partial \bar{z}}$

We will use $\beta = (\beta_1, \beta_2)$ to denote a double index that may be $(0,0)$, $(0,1)$, or $(1,0)$. We set $|\beta| = \beta_1 + \beta_2$. Letting $z = x + iy$, we denote the first order partial derivatives by

$$D^\beta = \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \frac{\partial^{\beta_2}}{\partial y^{\beta_2}} .$$

The differential operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$ has the function $H(w,z) = \frac{1}{\pi}(\frac{1}{z-w})$ as a bi-regular fundamental solution. Hence $\frac{\partial}{\partial \bar{z}} H(z,w) = \delta_w$ and $\frac{{}^t\partial}{\partial \bar{w}} H(z,w) = \delta_z$ where $\frac{{}^t\partial}{\partial \bar{w}}$ is the formal adjoint of $\frac{\partial}{\partial \bar{z}}$ and δ_z is the Dirac measure supported at z. We note that for $\beta = (0,0), (0,1), (1,0)$

$$|D^\beta H(0,z)| \leq \frac{1}{\pi} |z|^{-1-|\beta|}, \quad z \neq 0.$$

The next lemma links BPE's to the function $H(w,z)$. A proof which includes this as a special case is in [2, p. 3].

LEMMA 4.1. A point $z_0 \in X$ is a BPE for $R^2(X) \subset L^2(X)$ if and only if there is a function $f \in L^2_{1,loc}(\mathbb{C})$, such that $f(z) = \frac{1}{\pi}(\frac{1}{z-z_0})$ for all $z \in \mathbb{C}\setminus X$.

The next lemma we need is proved by Fernström and Polking in [2, pp. 13-15]. It is interesting that this lemma holds for $\beta = (0,0)$ as well as $(0,1)$ and $(1,0)$. Before stating it we introduce more notation.

DEFINITION. For a compact set X, let

$$X_\varepsilon = \{z | \mathrm{Dist}(z,X) < \varepsilon\}.$$

DEFINITION. We denote $A_k(0) = \{z | 2^{-k-1} \leq |z| \leq 2^{-k+1}\}$ by A_k.

DEFINITION. Let $A_k' = \{z | 2^{-k-2} \leq |z| \leq 2^{-k+1}\}$.

LEMMA 4.2. Let $X \subset \mathbb{C}$ be compact and suppose that

$$\sum_{k=0}^{\infty} 2^{2k} C_{1,2}(A_k \setminus X) < \infty.$$

Then for each $\varepsilon > 0$ and for each $k \geq 0$ there is a function $\psi_k \in C^\infty$ such that

(i) $\psi_k(z) \equiv 1$ for z near $A'_k \setminus X_\varepsilon$, and

(ii) $$\int_{|z| \leq 2^{-k+1}} |D^\beta \psi_k(z)|^2 dm(z) \leq F2^{-2k(1-|\beta|)} C_{1,2}(A'_k \setminus X)$$

for $\beta = (0,0)$, $(0,1)$, and $(1,0)$. The constant F is independent of k.

5. THE MAIN RESULT.

It is no restriction to assume that the boundary point $x \in \partial X$ is the origin $(x = 0)$. Also, we may assume that $X \subset \{|z| < 2\}$. In taking 0 to be the vertex of a sector in $\text{Int}\, X$ we mean that there are numbers α, β, $0 \leq \alpha < \beta < 2\pi$, and a number a, $0 < a < 2$, such that if (r,θ) are polar coordinates, and $S = \{(r,\theta) | \alpha \leq \theta \leq \beta,\ 0 \leq r \leq a\}$, then $\text{Int}\, S \subset \text{Int}\, X$. Let L be the mid-line $L = \{(r,\theta) | \theta = \frac{\beta-\alpha}{2},\ 0 \leq r < a\}$. Since $y \in \text{Int}\, X$ is a BPE for $R^2(X)$, we may use $f(y)$ to represent the value of that linear functional at a given $f \in R^2(X)$. We want to study $f(y) - f(0)$ for $f \in R^2(X)$ as y approaches 0 along L.

First we will construct a function $g \in L^2(X)$ which represents 0 for $R^2(X)$ and which has support disjoint from a sector surrounding L. This second sector S' is a subset of S defined by

$$S' = \{(r,\theta) | \alpha + \frac{\beta-\alpha}{3} \leq \theta \leq \beta - \frac{\beta-\alpha}{3},\ 0 \leq r < a\}.$$

LEMMA 5.1. Suppose that 0 is a BPE for $R^2(X)$ that is the vertex of a sector S in X. Then, there is a function $g \in L^2(X)$ such that:

(i) g represents 0 for $R^2(X)$,

(ii) $m((\text{supp}\, g) \cap S') = 0$,

(iii) For all $n \geq 0$,

$$\int_{A_n \cap X} |g|^2 dm \le F \sum_{k=n-1}^{n+1} 2^{2k} C_{1,2}(A_k' \setminus X)$$

where F is a constant independent of n.

PROOF. Choose $\lambda \in C_0^\infty(R^1)$ such that

$$\lambda(t) = \begin{cases} 0 & \text{if } t \le \frac{1}{4} \text{ or } t \ge 2 \\ 1 & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

For each integer k set

$$\lambda_k(z) = \lambda(2^k|z|) / \sum_{j=-\infty}^{\infty} \lambda(2^j|z|) \quad \text{for } z \in \mathbb{C} \setminus \text{Int } S.$$

For those values of z in Int S define $\lambda_k(z)$ so that the following three conditions are satisfied:

(1) $\lambda_k(z) \in C^\infty$

(2) $\lambda_k(z) = 0$ for $z \in X \cap S'$, and

(3) There are constants F_1 and F_2 such that for all k

$$\left|\frac{\partial \lambda_k(z)}{\partial x}\right| \le F_1 2^k \quad \text{and} \quad \left|\frac{\partial \lambda_k(z)}{\partial y}\right| \le F_2 2^k.$$

The constants F_1 and F_2 are independent of k.

Given $\varepsilon > 0$ choose the functions ψ_k of Lemma 4.2. On the complement of X_ε we have $\psi_k \lambda_k \equiv \lambda_k$ since supp $\lambda_k \subset A_k'$. Thus, $\sum_0^\infty \psi_k \lambda_k \equiv 1$ on $\Delta(0,1/4) \setminus X_\varepsilon$. Choose $\chi \in C_0^\infty$ with $\chi(z) \equiv 1$ near X. Set $h(z) = \chi(z) H(0,z)$ where $H(0,z) = \frac{1}{\pi z}$.

For each double index $\beta = (0,0)$, $(0,1)$, and $(1,0)$ there is a constant F_β such that

$$|D^\beta h(z)| \le F_\beta |z|^{-1-|\beta|}.$$

These inequalities follow from those of Section 4 and the fact that χ and its derivatives are bounded. Set $f_\varepsilon = h \sum_0^\infty \psi_k \lambda_k = \sum_0^\infty \psi_k h_k$ where $h_k = \lambda_k h$. Since supp $\lambda_k \subset A_k'$, the above inequalities imply that

(*) $$|D^{\beta}h_k(z)| \le F_{\beta}2^{k(1+|\beta|)}.$$

Henceforth, we will limit the number of symbols denoting constants by letting F denote any constant. The inequalities (*) combined with Lemma 4.2 imply that

$$\begin{aligned} ||f_{\varepsilon}||^2_{L^2_1} &\le F \sum_{|\beta+\lambda|\le 1} \sum_{k=0}^{\infty} \int |D^{\beta}h_k(z)D^{\gamma}\psi_k(z)|^2 dm(z) \\ &\le F \sum_{k=0}^{\infty} \sum_{|\beta+\lambda|\le 1} 2^{2k(1+|\beta|)} \int_{|z|\le 2^{-k+1}} D^{\lambda}\psi_k(z)|^2 dm(z) \\ &\le F \sum_{k=0}^{\infty} 2^{2k} C_{1,2}(A'_k\backslash X). \end{aligned}$$

Finally, by the subadditivity of the capacity $C_{1,2}$, we have

$$||f_{\varepsilon}||^2_{L^2_1} \le F \sum_{k=0}^{\infty} 2^{2k} C_{1,2}(A_k\backslash X).$$

The net $\{f_{\varepsilon}\}$ is bounded in L^2_1. We can choose a subsequence $\{f_{\varepsilon_j}\}$ that converges weakly in L^2_1. Let $f(z) = \lim_{j\to\infty} f_{\varepsilon_j}(z) + (1-\chi)H(0,z)$ for $z \in ¢\backslash X$. Then $f \in L^2_{1,loc}$, and $f(z) = H(0,z)$ for $z \in ¢\backslash X$. Note that since $f_{\varepsilon_j}(z) = 0$ for all $z \in X \cap S'$, $f(z) = 0$ for a.e. $z \in X \cap S'$. If necessary, we may redefine f on $X \cap S'$ so that $f(z) = 0$ for every $z \in X \cap S'$.

Set $g = \frac{{}^t\partial}{\partial \bar{z}} f$. Then $g \in L^2(X)$, and g is a representing function for 0 (see [2, p. 3]). If $z \notin X$, $g(z) = 0$. Clearly, $m((\text{supp } g) \cap S') = 0$. We have

$$\begin{aligned} \int_{A_n\cap X} |g|^2 dm &\le F \sum_{|\beta|\le 1} \int_{A_n\cap X} |D^{\beta}f|^2 dm \\ &\le F \sum_{|\beta+\lambda|\le 1} \sum_{k=0}^{\infty} \int_{A_n\cap X} |D^{\beta}h_k D^{\lambda}\psi_k|^2 dm. \end{aligned}$$

The integral $\int_{A_n\cap X} |D^{\beta}h_k\psi_k|^2 dm$ will be nonzero only for those k such that $A'_k \cap A_n \cap X \neq \phi$, i.e., $k = n-1, n, n+1$. Thus, by (*) and Lemma 4.2,

$$\int_{A_n \cap X} |g|^2 dm \le F \sum_{|\beta+\lambda|\le 1} \sum_{k=n-1}^{n+1} \int_{A_n \cap X} |D^\beta h_k D^\lambda \psi_k|^2 dm$$

$$\le F \sum_{k=n-1}^{n+1} 2^{2k} C_{1,2}(A_k' \backslash X).$$

This completes the proof of (i), (ii), and (iii).

We will use the next lemma to obtain representing functions for points near 0 on the line segment L. Let $0, X, S$, and g be as in the previous lemma, and let $c(y)$ be as defined in Section 2.

LEMMA 5.2. Let $0 \in X$ be represented by a function $v \in L^2(X)$. Suppose that ϕ is an admissible function and that $v(z)\phi(|z|)^{-1} \in L^2(X)$. Then for any $\varepsilon > 0$ there exists a δ such that if $|y| < \delta$ and $y \in L$, then $|c(y)| = |1 + y\hat{g}(y)| > 1 - \varepsilon$.

PROOF. Since the capacities Γ_2 and $C_{1,2}$ are equivalent, Theorem 3.1 implies that

$$\sum_{n=1}^{\infty} 2^{2n} \phi(2^{-n})^{-2} C_{1,2}(A_n \backslash X) < \infty.$$

To show that $c(y)$ is defined, we first note that

$$|y|\left| \int g\cdot(z-y)^{-1} dm\right| \le \phi(|y|)\psi(|y|) \int |g|\psi(|z-y|)^{-1}\phi(|z-y|)^{-1} dm.$$

where $\psi(r) = r\cdot\phi(r)^{-1}$. By definition of S' there is a constant k_1 such that $k_1|z-y| \ge |z|$ for any $y \in L$ and $z \in X\backslash S' - \{0\}$. Similarly, there is a constant k_2 such that $k_2|z-y| \ge |y|$ for any $y \in L$ and $z \in X\backslash S' - \{0\}$. Since ϕ and ψ are both increasing,

$$\phi(|z|)\phi(|z-y|)^{-1} \le k_1 \quad\text{and}\quad \psi(|y|)\psi(|z-y|)^{-1} \le k_2.$$

Hence

$$|y|\left| \int g\cdot(z-y)^{-1} dm\right| \le F\phi(|y|) \int |g|\cdot\phi^{-1} dm.$$

We claim that $g\cdot\phi^{-1} \in L^2(X)$ and therefore $g\cdot\phi^{-1} \in L^1(X)$. First observe that

$$\int |g|^2\cdot\phi^{-2} dm \le \sum_{n=1}^{\infty} \phi(2^{-n})^{-2} \int_{A_n \cap X} |g|^2 dm.$$

By Lemma 5.1 and the subadditivity of $C_{1,2}$ we get

$$\int |g|^2\phi^{-2}dm \le \sum_{n=1}^{\infty} \phi(2^{-n})^{-2}2^{2n}C_{1,2}(A_n\backslash X).$$

The capacity series converges. Thus, $\hat{g}(y)$ is defined. Since $\lim_{r\to 0}\phi(r) = 0$, we can choose for any given $\varepsilon > 0$ a $\delta > 0$ such that

$$|y\hat{g}(y)| = |y| \left| \int g\cdot(z-y)^{-1}dm\right| \le F\phi(|y|) \int |g|\cdot\phi^{-1}dm < \varepsilon$$

for $|y| < \delta$ and $y \in L$. It follows that $|c(y)| = |1 + y\hat{g}(y)| > 1 - \varepsilon$.

In the following theorem, X, 0, and L are just as they have been.

THEOREM 5.1. Let $0 \in \partial X$ be a BPE for $R^2(X)$ which is represented by function $v \in R^2(X)$. Suppose that ϕ is an admissible function and that $v(z)\phi(|z|)^{-1} \in L^2(X)$. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $y \in L \cap \Delta(0,\delta)$,

$$|f(y) - f(0)| \le \varepsilon\phi(|y|)||f||_2$$

for all $f \in R^2(X)$.

PROOF. Let $g \in L^2(X)$ be a representing function for 0 as in Lemma 5.1. Choose δ_1 by Lemma 5.2 so that for $y \in L$ and $|y| < \delta_1$, $|c(y)| > 1/2$. Then by Corollary 2.1,

$$\begin{aligned} f(y) - f(0) &= c(y)^{-1} \int [f - f(0)]z(z-y)^{-1}gdm \\ &= c(y)^{-1} \int [f - f(0)][1 + y(z-y)^{-1}]gdm \\ &= yc(y)^{-1} \int [f - f(0)](z-y)^{-1}gdm. \end{aligned}$$

Thus, for $y \in L$ and $|y| < \delta_1$

$$|f(y) - f(0)| \le 2|y| \int |f - f(0)||z-y|^{-1}|g|dm.$$

There exists a monotone, increasing function $\bar{\phi}$ such that $\lim_{r\to 0^+}\bar{\phi}(r) = 0$ and $\phi(|z|)^{-1}\bar{\phi}(|z|)^{-1}v(z) \in L^2(X)$ (see [6, p. 74]). Moreover, we may choose $\bar{\phi}$ so that the function $r\phi(r)^{-1}\bar{\phi}(r)^{-1}$ is also monotone increasing. Let $\Phi(r) = \phi(r)\cdot\bar{\phi}(r)$. Then recalling that $k_1|z-y| \ge |z|$ and $k_2|z-y| \ge |y|$ for $y \in L$ and $z \in X\backslash S' - \{0\}$, we have

$$|f(y) - f(0)| \leq F\Phi(|y|)||f||_2\{\sum_{n=1}^{\infty} \Phi(2^{-n})^{-2} \int_{A_n \cap X} |g|^2 dm\}^{1/2}.$$

If the sum of the infinite series is less than 1, the theorem is nearly proved. Suppose the sum is greater than or equal to 1. Then

$$|f(y) - f(0)| \leq F|(|y|)||f||_2 \sum_{n=1}^{\infty} 2^{2n}\Phi(2^{-n})^{-2}C_{1,2}(A_n\backslash X)$$
$$\leq F\bar{\phi}(|y|)\phi(|y|)||f||_2 \sum_{n=1}^{\infty} 2^{2n}\Phi(2^{-n})^{-2}C_{1,2}(A_n\backslash X).$$

Since the capacity series converges by Theorem 3.1, we may choose δ_2 such that for $|y| < \delta_2$ $F\bar{\phi}(|y|) \sum_{n=1}^{\infty} 2^{2n}\Phi(2^{-n})^{-2}C_{1,2}(A_n\backslash X) < \varepsilon$.

Then $|f(y) - f(0)| \leq \varepsilon\phi(|y|)||f||_2$ for $|y| < \min(\delta_1,\delta_2)$ and $y \in L$. This concludes the proof.

REMARKS. (i) If $0 \in \delta X$ is a BPE for $R^2(X)$, there always exists an admissible function ϕ as in the hypotheses of Theorem 5.1 (see [5, p. 74]).

(ii) The theorem may be extended by techniques of Wang [5] to include bounded point derivations of order s so that a statement similar to Theorem 1.1(ii) holds for $y \in L \cap \Delta(0,\delta)$.

(iii) For certain sets X a point $0 \in \partial X$ which is a BPE for $R^2(X)$ may not be the vertex of any sector having interior in Int X. Suppose, however, that 0 is a cusp for a curve whose interior is in Int X. Let L be a line segment which bisects the cusp at 0 and let C denote the interior of the cusp near 0. Then if $y \in L \cap C$ and $z \in X\backslash C$, $|y-z|\tau(|y|) \geq |y|$ where τ is a monotone decreasing function such that $\lim_{r\to 0^+}\tau(r) = \infty$. Depending on how rapidly τ approaches ∞ at 0 (or how rapidly the cusp "narrows"), we can show that functions in $R^2(X)$ satisfy an inequality similar to that of Theorem 5.1.

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