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COMPACTIFICATIONS OF CONVERGENCE SPACES

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ABSTRACT. This paper summarizes most of the results to date on convergence space compactifications, and establishes necessary and sufficient conditions for the existence of largest and smallest compactifications subject to various conditions imposed upon the compactifications.

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1. INTRODUCTION.

The subject of convergence space compactifications is now about ten years old, although some related concepts, such as Novak's sequential envelope [13], are of earlier vintage. Our goal is to summarize the results in this area which have been obtained to date, and to give further development to the subject. In the latter endeavor, we follow a path initiated by C. J. M. Rao, making use of ideas introduced by Ellen Reed and inspired by the completion theory of H. Kowalsky.

Convergence spaces were originally defined in terms of sequences by M. Frechet [5] in 1906. A compactification theory for convergence spaces had to await the development of convergence spaces defined by means of filters. The foundational papers for filter convergence spaces were written by G. Choquet [1] in 1948, H. Kowalsky [12] in 1954, and H. Fischer [4] in 1959.

In 1970, G. D. Richardson [21] and J. F. Ramaley and O. Wyler [15] published different versions of a "Stone-Čech compactification" for convergence spaces. In the former paper, each T_2 convergence space is embeded in a compact T_2 convergence space with the property that each map into a compact T_3 space can be lifted to the compactification space. Similar results are obtained in the latter paper, but with the following significant differences: the "compactification" space is T_3, but the injection map into this space is not an embedding. Thus the Ramaley-Wyler "compactification" is not a compactification in the sense that we use the term

The universal property for Richardson's compactification is not entirely satisfactory because the compactification space is usually not T_3. Indeed, R. Gazik [6] showed that this compactification is T_3 iff each non-convergent ultrafilter coincides with its own closure. Gazik's condition is also necessary and sufficient in order for Richardson's compactification of a completely regular topological space to be equivalent to the topological Stone-Čech compactification.

In 1972, the authors showed that a convergence space X can be embedded in a compact T_3 convergence space iff X has the same ultrafilter convergence as a completely regular topological space. Such convergence spaces are said to be completely regular. Each completely regular convergence space has a T_3 compactification with the same universal property which characterizes the topological

Stone-Čech compactification. Other formulations and proofs of essentially the same results were given independently in 1973 by C. H. Cook [3] and A. Cochran and R. Trail [2].

C. J. M. Rao [16], [17], and Vinod Kumar [24] investigated the conditions under which a space X has a largest T_2 and smallest T_2 and T_3 compactification. The summary of their results, along with some additions by the present authors, is the subject of Section 3.

In 1971, Ellen Reed [19] made a detailed study of Cauchy space completions. Motivated by Kowalsky's completion theory [12], she defined "relatively diagonal" and "relatively round" completions; in a later paper on proximity convergence spaces [20], she introduced a relatively round compactification called the "Σ-compactification". Compactifications satisfying these two conditions receive further attention in Sections 4 and 5 of this paper.

There are a number of directions from which the subject of compactifications can be approached. A recent paper by R. A. Herrman [7] gives a non-standard development of convergence space compactifications. Another possibility is to consider embeddings into spaces which are, in some sense, approximately compact; this technique is used in [8], [11], and [18].

Our approach, like that of Rao, is to study the conditions under which a space will have a largest and smallest compactification subject to certain conditions (including the aforementioned properties introduced by Reed) imposed on the compactification. Some of our main results are summarized at the end of the paper.

2. PRELIMINARIES.

Let F(X) denote the set of all filters on a set X. The term "ultrafilter" will be abbreviated "u.f."; the fixed u.f. generated by x is denoted $\dot{x}$.

A convergence space (X, →) is a set X and a relation → between F(X) and X subject to the following conditions:

(C_1) For each $x \in X$, $\dot{x} \to x$.

(C_2) If $F \to x$ and $G \geq F$, then $G \to x$.

(C_3) If $F \to x$ and $G \to x$, then $F \cap G \to x$.

Ordinarily, a convergence space $(X, \to)$ will be denoted only by the X; the term "space" will always mean "convergence space".

A space is *T_2* if each filter converges to at most one point; the assumption is made throughout this paper that all spaces are T_2 unless otherwise indicated. A space X is *T_3* if $cl_X F \to x$ whenever $F \to x$, where cl_X denotes the closure operator for X. A subset A of a space X is *compact* if each u.f. containing A converges to a point in A; if every convergent u.f. contains a compact set, then X is said to be *locally compact*.

Let $\nu_X(x)$ denote the *X-neighborhood filter* at a point $x \in X$; $\nu_X(x)$ is the intersection of all filters which converge to x. If $\nu_X(x) \to x$ for all $x \in X$, then X is called a *pretopological space*. For any space X, the finest pretopological (topological) space coarser than X is denoted $\pi X(\lambda X)$. A continuous function will be called a *map*.

A *compactification* $\kappa = (Y,k)$ of a space X consists of a compact space Y and an embedding map $k: X \to Y$ such that $cl_Y kX = Y$.

Since the term "compactification" is used so frequently in this paper we shall use the abbreviation "compn.".

If $\kappa_1 = (Y_1,k_1)$ and $\kappa_2 = (Y_2,k_2)$ are compns. of a space X, and there is a map f which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{k_1} & Y_1 \\ & {\scriptstyle k_2}\searrow & \downarrow{\scriptstyle f} \\ & & Y_2 \end{array}$$

commute, then κ_1 is said to be *larger* than κ_2 (written $\kappa_2 \leq \kappa_1$). If $\kappa_2 \leq \kappa_1$ and $\kappa_1 \leq \kappa_2$, then the two compns. are said to be *equivalent*.

In this section, we shall construct two compactifications which will play a key role in the remainder of the paper; they are the one-point compatification

$\hat{\kappa}$ and Richardson's compactification κ^*.

Let X be a space, and let $\hat{X} = X \cup \{a\}$, where $a \notin X$. Let $\hat{i}$ be the identity function from X into $\hat{X}$, assign to $\hat{X}$ the finest convergence structure subject to the following conditions: (1) If $x \in X$, then $\mathcal{F} \to x$ in $\hat{X}$ iff the restriction of $\mathcal{F}$ to X converges to x in X; (2) $\mathcal{F} \to a$ in $\hat{X}$ iff the restriction of $\mathcal{F}$ to X has no adherent point in X. Then $\hat{\kappa} = (\hat{X}, \hat{i})$ is called the <u>one-point compactification</u> of X. Although other non-equivalent "one-point compactifications" for X are possible, $\hat{\kappa}$ is the only one that we shall consider in this paper.

Let N_X denote the set of all non-convergent u.f.'s on a space X. If $A \subseteq X$, let $A^* = A \cup \{\mathcal{F} \in N_X : A \in \mathcal{F}\}$, and for each $\mathcal{G} \in F(X)$, let $\mathcal{G}^*$ be the filter on X^* generated by $\{G^* : G \in \mathcal{G}\}$. The convergence structure for X^* is defined as follows: (1) If $x \in X$, then $\mathcal{H} \to x$ in X^* iff there is $\mathcal{F} \to x$ in X such that $\mathcal{H} \geq \mathcal{F}^*$; (2) If $y = \mathcal{G} \in N_X$, then $\mathcal{H} \to y$ in X^* iff $\mathcal{H} \geq \dot{y} \cap \mathcal{G}^*$. Let i^* be the identity map of X into X^*. The following proposition is proved in [21].

PROPOSITION 2.1. If X is a space, then $\kappa^* = (X^*, i^*)$ is a compn. of X. If $f : X \to Y$ is a map, and Y is a compact T_3 space, then there is a map $\bar{f}$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i^*} & X^* \\ & \searrow^{f} & \downarrow \bar{f} \\ & & Y \end{array}$$

commutes.

We now introduce some less familiar convergence space properties which turn out to be important in the study of compns. A subset A of a space X is <u>bounded</u> if every u.f. containing A converges to some point in X; a space in which every convergent filter contains a bounded set is said to be <u>locally bounded</u>. The term "bounded" was suggested by Kasahara [9]; a further study of these concepts is given by C. Riecke [23]. The notion of "boundedness" has been studied by other authors under other names; for instance, H. Poppe [14] refers to the same concept as "weakly relatively compact".

A space X is said to be essentially bounded if, for each $\mathcal{F} \in N_X$, $\mathcal{F} \vee (\bigcap \{\mathcal{G} \in N_X : \mathcal{G} \neq \mathcal{F}\}) = \phi$. (In general, the statement "$\mathcal{F}_1 \vee \mathcal{F}_2 = \phi$" will mean that the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ contain disjoint sets.) X is said to be essentially compact if N_X is a finite set; this terminology is due to Vinod-Kumar [24].

PROPOSITION 2.2. A space X is essentially compact iff it is essentially bounded and locally bounded.

PROOF. An essentially compact space obviously has both properties. On the other hand, if N_X is an infinite set, then there is a free u.f. $\mathcal{F}$ on X such that $\mathcal{F} \geq \bigcap \{\mathcal{G} \in N_X : \mathcal{G} \neq \mathcal{F}\}$. If $\mathcal{F} \in N_X$, then X fails to be essentially bounded; if $\mathcal{F} \notin N_X$, then X fails to be locally bounded. ∎

Some additional characterizations of local boundedness and essential boundedness are given below.

PROPOSITION 2.3. The following statements about a space X are equivalent.

(1) X is locally bounded.

(2) If $\mathcal{F}$ is convergent in X, then $X \in \mathcal{F}^*$.

(3) If $\mathcal{F}$ is convergent in X, then $\mathcal{F} \vee (\bigcap N_X) = \phi$.

(4) X is open in X*.

(5) $\hat{\kappa} \leq \kappa^*$.

PROOF. (1) => (2). If $\mathcal{F} \to x$, then there is $F \in \mathcal{F}$ such that each u.f. containing F converges to a point in X. Consequently, $F^* = F$, and so $F \in \mathcal{F} \Rightarrow X \in \mathcal{F}^*$.

(2) => (3). If $\mathcal{F} \vee (\bigcap N_X) \neq \phi$, then for each $F \in \mathcal{F}$, there is $\mathcal{G}_F \in N_X$ such that $F \in \mathcal{G}_F$. But then, for each $F \in \mathcal{F}$, $\mathcal{G}_F \in F^*$, contrary to the assertion $X \in \mathcal{F}^*$.

(3) => (4). If $\mathcal{H}$ is an u.f. containing X* - X and converging in X* to a point x in X, then $\mathcal{H} \geq \mathcal{F}^*$ for some filter $\mathcal{F} \to x$ in X. It follows

that $\mathcal{F} \vee (\bigcap N_X) \neq \phi$.

(4) => (5). The canonical map from X^* into $\hat{X}$, which carries $X^* - X$ onto $a \in \hat{X}$, is clearly continuous if $X^* - X$ is a closed set.

(5) => (1). If X is not locally bounded, then there is $\mathcal{F} \to x$ in X such that $F^* \cap (X^* - X) \neq \phi$ for all $F \in \mathcal{F}$. Thus there is an u.f. containing $X^* - X$ which converges to x in X^*, and the canonical map from X^* into $\hat{X}$ fails to be continuous. Therefore, $\hat{\kappa} \not\leq \kappa^*$. ∎

PROPOSITION 2.4. The following statements about a space X are equivalent.

(1) X is essentially bounded.

(2) If $\mathcal{F} \in N_X$, then $X \cup \{\mathcal{F}\} \in \mathcal{F}^*$.

(3) $X^* - X$ is discrete.

PROOF. (1) => (2). If $\mathcal{F} \in N_X$ and X is essentially bounded, then there is $F \in \mathcal{F}$ such that $\mathcal{F}$ is the only member of N_X containing F. Thus $F^* = F \cup \{\mathcal{F}\} \in \mathcal{F}$, which implies $X \cup \{\mathcal{F}\} \in \mathcal{F}^*$.

(2) => (3). Since $\mathcal{F} \in N_X$ implies $\mathcal{F}^* = \mathcal{F} \cap \dot{\mathcal{F}}$, $\dot{\mathcal{F}}$ is the only filter containing $X^* - X$ which can converge in X^* to $\mathcal{F} \in X^* - X$.

(3) => (1). If $\mathcal{F} \in N_X$ is such that $\mathcal{F} \vee (\bigcap \{\mathcal{G} \in N_X : \mathcal{G} \neq \mathcal{F}\}) \neq \phi$, then one can construct a free u.f. containing $X^* - X$ which converges to $\mathcal{F}$ in X^*. ∎

If $\kappa_1 = (Y_1, k_1)$ is a compn. of a space X such that $\kappa_1 \geq \kappa^*$, then it is easy to see that in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{k_1} & Y_1 \\ & \searrow^{i^*} & \downarrow \theta \\ & & X^* \end{array}$$

, the map θ is one-to-one and onto X^*. Furthermore, if $\mathcal{G}$ is an u.f. on Y_1, then $\mathcal{G} \to y$ in Y_1 iff $\theta(\mathcal{G}) \to \theta(y)$ in X^*. Thus, there is no loss of generality in assuming that Y_1 and X^* have the same underlying set, an assumption which we shall use whenever convenient.

Finally, if $\kappa = (Y,k)$ is a compn. of X such that $\kappa^* \geq \kappa$, then we shall consistently denote by ϕ the function $\phi : X^* \to Y$ defined by $\phi(a) = y$

if there is $F \to a$ in X^* such that $X \in F$ and $k(i^*)^{-1} F \to y$ in Y. ϕ is always well-defined but not necessarily continuous; in later sections we shall refer to ϕ as the canonical function from X^* to Y.

3. T_2 AND T_3 COMPACTIFICATIONS.

In this section we seek to clarify, simplify, and extend results initially obtained in [16], [17], [22], and [24]. It should be noted that, for all of the results of this section, the convergence space axiom (C_3) can be replaced by the weaker axiom:

$$(C_3') \quad F \to x \text{ implies } F \cap \dot{x} \to x.$$

LEMMA 3.1. If $f: X \to Y$ is a map between spaces, A a dense subset of X (meaning $cl_X A = X$), and the restriction of f to A is a homeomorphism, then $f(X - A) \subset Y - f(A)$.

THEOREM 3.2. The following statements about a space X are equivalent.

(1) X has a smallest compn.

(2) X is open in each of its compns.

(3) X is essentially compact.

(4) X has a largest compn.

PROOF. The equivalence of (1) and (2) was established by Rao in [16].

(2) => (3). Assume that N_X is an infinite set. Let $Y = X \cup N_X$ be equipped with a convergence structure which agrees with X^* on filters containing X, and with the property that every free u.f. which contains N_X converges in Y to some fixed point x_0 in X. Then (Y, i^*) is a compn. of X, and $i^*(X) = X$ is clearly not open in Y.

(3) => (4). It is easy to verify that (X^*, j) is the largest compn. of an essentially compact space.

(4) => (2). Let $\kappa = (Y,k)$ be the largest T_2 compn. of X. Using Lemma 3.1 and the fact that $\hat{\kappa} \leq \kappa$, it follows that $k(X)$ is an open subset

of Y. Let (Z,g) be any T_2 compn. of X; then there is a map f which makes the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ & \searrow^{g} & \downarrow f \\ & & Z \end{array}$$

Since $\kappa^* \leq \kappa$, it is easy to verify that f must map Y onto Z. Making use of Lemma 3.1 and the fact that $Y - k(X)$ is closed and, consequently, compact, $f(Y - kX) = Z - g(X)$ is also compact, and therefore $g(X)$ is open in Z. ∎

Vinod-Kumar [24] questioned Rao's proof in [17] of the equivalence of statements (3) and (4) of Theorem 3.2; neither noticed the equivalence of statements (2) and (3). For an essentially compact space X, κ^* is the largest and $\hat{\kappa}$ is the smallest compn.

A space is defined to be <u>completely regular</u> if it is T_3 and has the same ultrafilter convergence as a completely regular topological space. The next theorem is proved in [22].

THEOREM 3.3. A space X has a largest T_3 compn. iff X is completely regular.

The largest T_3 compn. is constructed by making relatively minor modifications in the convergence of filters relative to the topological Stone-Čech compn.; details can be found in [22].

In the final theorem of this section, we add two alternate characterizations to that given by Rao for spaces having a smallest T_3 compn.

THEOREM 3.4. The following statements about a completely regular convergence space are equivalent.

(1) X has a smallest T_3 compn.

(2) πX is a locally compact topological space.

(3) X is a locally compact convergence space.

(4) X is open in each of its regular compns.

PROOF. The equivalence of (1) and (2) was proved by Rao in [16].

(2) => (3). Since X and πX have the same u.f. convergence, they have the same compact sets, and so every X-convergent filter contains a compact set.

(3) => (4). If $\kappa = (Z,k)$ is any regular compn. of X, and G is any u.f. on Z containing $Z - kX$, then there is an u.f. F on X such that $G \geq cl_Z k\, F$. Since X is locally compact, G cannot converge to a point in k(X), and therefore k(X) is open in Z.

(4) => (2). Since X is completely regular, (4) implies that πX is open in each of its compactifications; since πX is a topology, πX is locally compact. ∎

If X is a space satisfying any of the equivalent conditions of Theorem 3.4, then $\hat{\kappa}$ is the smallest T_3 compn.

4. RELATIVELY DIAGONAL AND RELATIVELY T_3 COMPACTIFICATIONS.

In the preceding section, we studied compns. subject to convergence space properties T_2 and T_3. In this section and the next, we deal with properties of the compns. themselves; these properties are not meaningful when applied to the underlying space. The concept of a strict compn. was introduced in [10], where it was shown that the strict T_3 compns. of a completely regular space X correspond in a one-to-one manner with a certain class of Cauchy structures compatible with X. Relatively diagonal and relative round compns. were introduced by Reed [19] as Cauchy space completion properties, and relatively T_3 is a new compn. property which is being introduced here for the first time. Before formally defining these terms, some additional notation is needed.

Let $\kappa = (Y,k)$ denote a compn. of a space X. A κ - selection function σ is a function $\sigma: Y \to F(Y)$ such that $\sigma(y) \to y$ in Y, $\dot{y} \geq \sigma(y)$, and $\sigma(y) = \dot{y}$ if $y \in k(X)$. Let $[\kappa]$ denote the set of all κ-selection functions. If $\sigma \in [\kappa]$, $A \subset Y$, and $F \in F(Y)$, then let:

$A^\sigma = \{y \in Y : A \in \sigma(y)\}$

$\mathcal{F}_\sigma = \{A \subset Y : A^\sigma \in \mathcal{F}\}$.

It is easy to see that $A^\sigma \subset A$ and $\mathcal{F}_\sigma \leq \mathcal{F}$.

Under the assumptions of the preceding paragraph, let A,B be subsets of Y, $\sigma \in [\kappa]$, and define $A <_\sigma B$ to mean $A \subset B$ and, for all $y \in Y$, $B \in \sigma(y)$ or else $Y - A \in \sigma(y)$. If $\mathcal{F} \in F(Y)$ and $\sigma \in [\kappa]$, define $r_\sigma \mathcal{F} = \{A \subseteq Y : \exists\, F \in \mathcal{F}$ such that $F <_\sigma A\}$.

Again, let $\kappa = (Y,k)$ be a compn. of X. If $A \subset Y$ and $F \in F(Y)$, define $p_\kappa A = A \cup (cl_Y A - kX)$, and let $p_\kappa \mathcal{F}$ be the filter on Y generated by sets of the form $p_\kappa F$ for $F \in \mathcal{F}$. (We denote these concepts by pA and $p\mathcal{F}$, respectively, if there is no possibility of confusion regarding the intended compn.)

For the purpose of formulating the following four definitions, we continue assuming that $\kappa = (Y,k)$ is a compn. of X.

DEFINITION 4.1. κ is a <u>strict</u> compn. of X if, whenever $\mathcal{F} \to y$ in Y, there is $\mathcal{G} \to y$ in Y such that $k(X) \in \mathcal{G}$ and $\mathcal{F} \geq cl_Y\, \mathcal{G}$.

DEFINITION 4.2. κ is a <u>relatively diagonal</u> compn. of X if, for each $\sigma \in [\kappa]$, $\mathcal{F} \to y$ in Y implies $\mathcal{F}_\sigma \to y$ in Y.

DEFINITION 4.3. κ is a <u>relatively round</u> compn. of X if, for each $\sigma \in [\kappa]$, $r_\sigma \mathcal{F} \to y$ in Y whenever $\mathcal{F} \to y$ in Y.

DEFINITION 4.4 κ is a <u>relatively T_3</u> compn. of X if κ is strict and $p\,\mathcal{F} \to y$ in Y whenever $\mathcal{F} \to y$ in Y.

PROPOSITION 4.5 Each relatively diagonal compn. is strict.

PROOF. Let $\kappa = (Y,k)$ be a relatively diagonal compn. of X, and let $\mathcal{F} \to y_0$ in Y. Let $\mu : Y \to F(X)$ be a function which associates, with each $y \in Y$, a filter $\mu(y) \in F(X)$ such that $\dot{y} \geq \mu(y)$, $k(\mu(y)) \to y$ in Y, and $\mu(y) = \dot{x}$ if $y = k(x)$ for $x \in X$. Given $A \subset X$, define $A_\mu = \{y \in Y : A \in \mu(y)\}$, and let $G = \{A \subset X : A_\mu \in F\}$. Finally, let $\sigma \in [\kappa]$ be defined by $\sigma(y) = k(\mu(y)) \cap \dot{y}$, for all $y \in Y$. By straightforward arguments one can show that $F \geq cl_Y k\, G$, and $kG \geq F_\sigma$. Since κ is a relatively diagonal compn., $F_\sigma \to y_0$ in Y. Therefore $k\,G \to y_0$ in Y, and κ is a strict compn. ∎

LEMMA 4.6. If $\kappa = (Y,k)$ is a compn. of a space X, $F \in F(Y)$, and $\sigma \in [\kappa]$, then $(pF)_\sigma \leq r_\sigma F \leq F_\sigma \leq F$.

PROOF. The assertion $F_\sigma \leq F$ is obvious. Let $A \in r_\sigma F$. Then there is $F \in F$ such that $F <_\sigma A$. If $y \in F$, then $Y - F \notin \sigma(y)$, and so $A \in \sigma(y)$, which implies $F \subseteq A^\sigma$. Consequently, $A \in F_\sigma$, and $r_\sigma F \leq F_\sigma$ is established. Finally, let $A \in (pF)_\sigma$. Then $A^\sigma \in pF$, and thus there is $F \in F$ such that $F \cup (cl_Y F - kX) \subset A^\sigma \subset A$. If $Y - F \notin \sigma(y)$ for $y \in Y - k(X)$, then there is an u.f. K on Y such that $F \in K$ and $K \to y$ in Y. This would imply $y \in (cl_Y F - k(X)) \subseteq A^\sigma$, and thus $A \in \sigma(y)$. Thus $F <_\sigma A$, implying $A \in r_\sigma F$, and the proof is complete. ∎

THEOREM 4.7. (1) A relatively round compactification is relatively diagonal. (2) A relatively T_3, relatively diagonal compn. is relatively round. (3) A T_3 compn. is relatively T_3.

PROOF. The first two assertions follow immediately from Lemma 4.6; the third is obvious. ∎

THEOREM 4.8. (1) For any space X, the compn. κ^* is relatively round and relatively T_3.

(2) For any space X, the compn. $\hat{\kappa}$ is relatively round.

(3) The compn. $\hat{\kappa}$ of a space X is relatively T_3 iff X is locally bounded.

PROOF. (1) One can routinely vertify that $A^* = p(A^*) = (A^*)^\sigma$ for any set $A \subseteq X$. From this result it follows that κ^* is relatively diagonal (which implies strict) and also relatively T_3. It then follows by Theorem 4.7 that

κ^* is also relatively round.

(2) For each $\sigma \in [\hat{\kappa}]$, one can routinely verify that $r_\sigma F = F$ for each filter F which converges in $\hat{X}$.

(3) If $F \in F(X)$, F an u.f., and $F \to x$ in X, then $\dot{a} \geq p(\hat{i}F)$ iff $F \geq \bigcap N_X$. if $G \to a$ in $\hat{X}$, then $p(G) = G \bigcap \dot{a}$. Thus the assertion follows by Proposition 2.3. ∎

THEOREM 4.9. A space X has a largest strict, relatively diagonal, or relatively round compn. iff X is essentially compact. In each case the largest compn., if it exists is equivalent to κ^*.

PROOF. If X is essentially compact, then κ^* is known to be the largest compn. of X, and the desired conclusion follows by theorem 4.8.

Conversely, assume that $\kappa = (Y,k)$ is the largest strict compn. of X. Since κ^* is strict, $\kappa^* \leq \kappa$, and in accordance with our remarks at the end of Section 2, we shall assume that κ^* and κ have the same underlying set and the same convergence relative to u.f.'s. From the fact that $\hat{k}$ is strict (indeed, relatively round), and Proposition 2.3, it follows that X must be locally bounded.

Next, assume that X is not essentially bounded. Then there is $F \in N_X$ such that $F \vee (\{\bigcap\{G \in N_X : G \neq F\}) \neq \phi$. Let $Z = X \cup \{a,b\}$; let j be the identity map from X into Z, and assign to Z a convergence structure which makes $\kappa' = (Z,j)$ a compn. of X subject to the conditions: $j(F) \to a$ in Z and $j(G) \to b$ in Z for $G \in N_X$ and $G \neq F$. One can show that κ' is a relatively round compn. on X, and one can show that the canonical function ϕ from Y into Z is not continuous. This argument shows that the existence of a largest strict compn. also requires that X be essentially bounded. Since we showed earlier X has to be locally bounded, it follows by Proposition 2.2 that X must be essentially compact. This, along with $\kappa^* \leq \kappa$, implies that κ is equivalent to κ^*.

Had we begun by assuming that κ is the largest relatively round or relatively diagonal compn., precisely the same argument can be used to conclude that X is essentially compact and κ equivalent to κ^*. ∎

THEOREM 4.10. The following statements about a space X are equivalent:

(1) X is locally bounded.

(2) X has a smallest strict compn.

(3) X has a smallest relatively diagonal compn.

(4) X has a smallest relatively round compn.

(5) X has a smallest relatively regular compn.

If X is locally bounded, then the smallest compn. subject to each of the specified conditions is $\hat{\kappa}$.

PROOF. (1) => (2). Since $\hat{\kappa}$ is a strict compn. of X, it is necessary only to show that the natural map $\psi:Y \to \hat{X}$ is continuous for any strict compn. $\kappa = (Y,k)$ of X. It is clear that ψ will be continuous if there is no u.f. $\mathcal{F} \in F(Y)$, where $Y - k(X) \in \mathcal{F}$ and $\mathcal{F} \to y$ in Y for some $y \in k(X)$. If such a filter $\mathcal{F}$ existed, then by the assumption of strictness there would be a filter $\mathcal{G} \in F(Y)$ such that $\mathcal{G} \to y$ in Y, $k(X) \in \mathcal{G}$, and $\mathcal{F} \geq cl_Y\mathcal{G}$. But the assumption that X is locally bounded guarantees that $k(X) \in cl_Y\mathcal{G}$, and so $\mathcal{F} \geq cl_Y\mathcal{G}$ is impossible. Thus $\psi:Y \to \hat{X}$ is continuous, and $\hat{\kappa} = (\hat{X},\hat{i})$ is the smallest strict compn. of X.

The same argument is valid if "strict" is replaced by "relatively round", "relatively regular", or "relatively diagonal". Thus condition (1) also implies conditions (3), (4), and (5).

(2) => (1). It is easy to show that the smallest strict compn. of X must be equivalent to $\hat{\kappa}$. By Proposition 2.3, the canonical map ϕ of X^* on $\hat{X}$ is continuous iff X is locally bounded. Since κ^* is also a strict compn.

of X, X must be locally bounded. This argument is also valid if "strict" is replaced by "relatively diagonal" or "relatively round". Thus conditions (3) and (4) also imply condition (1).

(5) => (1). Assume that X is not locally bounded, and let $\kappa = (Y,k)$ denote a relatively regular compn. of X such that $\kappa \leq \kappa^*$. Then $Y - k(X)$ must be an infinite set (otherwise, continuity of the canonical map from X^* onto Y would be violated). Let y_1, y_2 be arbitrary points in $Y - k(X)$, and let Z be the quotient space derived from Y by identifying the points y_1 and y_2. Then $\kappa' = (Z,k)$ is also a relatively regular compn. of X and it is clear that $\kappa \not\leq \kappa'$. Thus X can have no smallest relatively regular compn. when X is not locally compact. ∎

We next consider some lifting properties of certain types of maps relative to relatively T_3, relatively diagonal, and relatively round compns. However we first need some additional terminology.

Let $\kappa = (Y,k)$ be a compn. of a space X, and let $\mathcal{C}_\kappa$ denote the set of all filters $\mathcal{F} \in F(X)$ such that $k(\mathcal{F})$ converges in Y. $\mathcal{C}_\kappa$ is called the <u>κ-Cauchy structure</u> for X, and its members are called κ-Cauchy filters.

If X_1 and X_2 are spaces with compns., $\kappa_1 = (Y_1,k_1)$ and $\kappa_2 = (Y_2,k_2)$, respectively, then a map $f:X_1 \to X_2$ is said to be a <u>$\kappa_1\kappa_2$-Cauchy map</u> if $f(\mathcal{F}) \in \mathcal{C}_{\kappa_2}$ for each $\mathcal{F} \in \mathcal{C}_{\kappa_1}$.

THEOREM 4.11. If $f:X_1 \to X_2$ is a $\kappa_1\kappa_2$-Cauchy map, where κ_1 and κ_2 are relatively T_3 compns. of X_1 and X_2, respectively, then there is a unique map $\bar{f}$ which makes the following diagram commute.

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle k_2} \\ Y_1 & \xrightarrow[\bar{f}]{} & Y_2 \end{array}$$

PROOF. For each $y \in Y_1$, choose $G \in C_{\kappa_1}$ such that $k_1 G \to y$ in Y_1, and define $\bar{f}(y) = z$, where $k_2 f(G) \to z$ in Y_2. The assumption that f is a $\kappa_1\kappa_2$-Cauchy map, along with convergence space axiom C_3, are sufficient to show that $\bar{f}$ is a well-defined function. It remains to show that $\bar{f}$ is continuous.

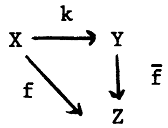
Let $F \to y$ in Y_1. Write F in the form $F = F_1 \cap F_2$, where $k_1X_1 \in F_1$ and $Y_1 - k_1X_1 \in F_2$. Let $\bar{f}(y) = z$. It is immediate that $\bar{f}(F_1) = k_2\, f\, k_1^{-1}(F_1) \to z$ in Y_2. Using the fact that κ_1 is strict, there is $G \in F(Y_1)$ such that $k_1X_1 \in G$, $F_2 \geq cl_{Y_1} G$, and $G \to y$ in Y_1. Note that $F_2 \geq p_{\kappa_1} G$, since $Y_1 - k_1\, X \in F_2$. Since $\bar{f}(G) \to z$ and κ_2 is a relatively T_3 compn. of X_2, it remains only to show that $\bar{f}(p_{\kappa_1} G) \geq p_{\kappa_2} \bar{f}(G)$. But this is easily established, and it follows that $\bar{f}(F_2) \to z$ in Y_2. Thus $\bar{f}(F) \to z$ in Y_2, and the proof is complete. ∎

If X is a space, κ any compn. of X, and i the identity map on X, then $i{:}X \to X$ is a $\kappa^*\kappa$ Cauchy map. Thus we obtain

COROLLARY 4.12. For any space X, κ^* is the largest relatively T_3 compn. of X.

Theorem 4.11 would not, in general, be a correct statement if "relatively T_3" were replaced by "relatively round", "relatively diagonal", or "strict"; otherwise, there would always be a largest compn. of any space X subject to these properties, contrary to Theorem 4.9. However the lifting theorem that follows applies to relatively diagonal and relatively round compns. as well as relatively T_3 compns; it generalizes the lifting property of the compn. κ^*.

THEOREM 4.13. Let X have a compn. $\kappa = (Y,k)$ which is relatively diagonal, relatively round, or relatively T_3. Let Z be a compact T_3 space, and $f{:}X \to Z$ a map with the property that $f(F)$ is convergent in Z for each $F \in C_\kappa$. Then there is a unique map $\bar{f}$ which makes the following diagram commute.



PROOF. If κ is relatively diagonal, then (Y,k) is equivalent (in the sense defined in [19]) to some member of the family of Cauchy space completions of $(X,\mathcal{C}_k)$ defined in [19]. Thus, for relatively diagonal and relatively round compns., the assertion follows from Theorem 4 of [19].

If κ is relatively T_3, then we can regard Z as a T_3 (and hence relatively T_3) compn. κ' of itself, where $\mathcal{C}_{\kappa'}$ is the set of all Z-convergent filters. Then the assumptions of the theorem imply that f is a $\kappa\kappa'$ - Cauchy map, and the conclusion follows as a corollary to Theorem 4.11. ■

We conclude this section by showing that relatively round compns. need not be relatively T_3, and vice versa. Indeed, it follows by Theorem 4.8 that for any space X which is not locally bounded, $\hat{\kappa}$ is relatively round but not relatively T_3. In the example that follows, we construct a strict T_3 compn. of a space X which is not relatively diagonal.

EXAMPLE 4.14. Let I be the unit interval [0,1] of the real line with its usual topology. Let (a_n) be a sequence in [0,1] which converges to 0 in I; let $\mathcal{H}$ be the filter on I generated by the sequence (a_n). Let Y be the space consisting of the set [0,1] with convergence defined as follows: (1) For $y \neq 0$, $\mathcal{F} \to y$ in Y iff $\mathcal{F} \to y$ in I; (2) $\mathcal{F} \to 0$ in Y iff there is a finite set of u.f.'s $\mathcal{G}_1,\ldots,\mathcal{G}_n$ converging to 0 in I such that $\mathcal{F} \geq \left(\bigcap_{i=1}^{n} \mathrm{cl}_I \mathcal{G}_i\right) \cap \mathcal{H}$. If X is the subspace of Y determined by the subset $[0,1] - \{a_n : n = 1, 2, \ldots\}$, and i the identity embedding of X into Y, then it follows that $\kappa = (Y,i)$ is a strict T_3 compn. of X.

Let $\sigma \in [\kappa]$ be the selection function defined as follows: (1) $\sigma(x) = \dot{x}$ for $x \in X$; (2) $\sigma(a_n) = \nu_Y(a_n)$, for $n = 1,2,\ldots\ldots$. If $A \in \mathcal{H}_\sigma$, then $A^\sigma \in \mathcal{H}$, and therefore A^σ contains all but finitely many of the a_n's. It follows that $\mathcal{H}_\sigma$ is not finer than any of the filters which converge to 0 in Y, and therefore $\mathcal{H}_\sigma \not\to 0$ in Y. But $\mathcal{H} \to 0$ in Y, and so κ is not relatively diagonal.

5. SIMPLE COMPACTIFICATIONS.

A compn. $\kappa = (Y,k)$ of a space X is said to be simple if κ is strict and, for each $y \in Y - X$, the neighborhood filter $\nu_Y(y) \to y$ in Y. A strict compn. $\kappa = (Y,k)$ will be called pretopological if Y is a pretopological space. Note that only pretopological spaces can have pretopological compns. We omit the straightforward proof of the first proposition.

PROPOSITION 5.1. The following statements hold for any (pretopological) space X.

(1) κ^* is simple (pretopological).

(2) $\hat{\kappa}$ is simple (pretopological) iff X is locally bounded.

PROPOSITION 5.2. A simple, relatively round compn. is relatively T_3.

PROOF. Let $\kappa = (Y,k)$ be a simple, relatively round compn. of a space X, let $\mathcal{H} \in F(Y)$, and let $\sigma \in [\kappa]$. Since κ is simple, we can assume without loss of generality that $\sigma(y) = \nu_Y(y)$ for $y \in Y - k(X)$. We shall show that $r_\sigma \mathcal{H} \leq p_\kappa \mathcal{H}$. Let $A \in r_\sigma \mathcal{H}$; then there is $H \in \mathcal{H}$ such that $H <_\sigma A$. If $y \in cl_Y H - k(X)$, then there is an u.f. $\mathcal{F} \in F(Y)$ such that $H \in \mathcal{F}$, and $\sigma(y) = \nu_Y(y) \leq \mathcal{F}$. Thus $Y - H \notin \sigma(y)$, and, since $H <_\sigma A$, it follows that $A \in \sigma(y)$. Consequently, $H \cup (cl_Y H - k(X)) \subset A$, and the proof is complete. ∎

COROLLARY 5.3. For any space X, κ^* is the largest simple, relatively round compn. of X.

PROOF. This is an immediate consequence of Corollary 4.12 and Propositions 5.1 and 5.2. ∎

For simple compns., the converse of Proposition 5.2 does not hold. The compn. κ constructed in Example 4.14 is simple and T_3, but not relatively round.

A pretopological compn. $\kappa = (Y,k)$ of X is said to be <u>relatively topological</u> if, for each $y \in Y - k(X)$, there is a base of sets for $\nu_Y(y)$ consisting of sets V such that $z \in V \cap (Y - k(X))$ implies $V \in \nu_Y(z)$. Sets V of this type will be called <u>κ - basic sets for y</u>.

PROPOSITION 5.4. Let $\kappa = (Y,k)$ be a pretopological compn. of a space X. Then κ is relatively topological iff κ is relatively diagonal.

PROOF. Let κ be relatively topological. To show that κ is relatively diagonal, it is sufficient to show that $\nu_Y(y) = (\nu_Y(y))_\sigma$, for each $y \in Y$ and $\sigma \in [\kappa]$. If $y \in k(X)$, this assertion is obvious. If $y \in Y - k(X)$, then it is easy to check that $V^\sigma = V$ for any set V which is κ-basic for y, and the desired equality is established.

Conversely, assume that κ is relatively diagonal, and define $\sigma \in [\kappa]$ by $\sigma(y) = \nu_Y(y)$ for all $y \in Y - k(X)$ and $\sigma(y) = \dot{y}$ for $y \in k(X)$. Then $\nu_Y(y) = (\nu_Y(y))_\sigma$ for all $y \in Y$, and sets of the form $\{V^\sigma : V \in \nu_Y(y)\}$ are κ-basic for all points $y \in Y - k(X)$. Thus κ is relatively topological. ∎

THEOREM 5.5. For any pretopological space X, κ^* is the largest relatively topological compn. of X.

PROOF. The fact that κ^* is relatively topological is an immediate consequence of Theorems 4.7 and 4.8, along with Proposition 5.1 and 5.4.

Let $\kappa = (Y,k)$ be a relatively topological compn. of X, and let $\phi : X^* \to Y$ be the canonical function. Let θ be an u.f. on X^* such that $\theta \to a$ in X^*. Then there is an u.f. $\mathcal{F}$ on X such that $\mathcal{F}^* \to a$ in X^* and $\theta \geq \mathcal{F}^*$. Assume $k(\mathcal{F}) \to y$ in Y; then $y = \phi(a)$ by definition of ϕ, and the proof will be completed by showing $\phi(\mathcal{F}^*) \to y$ in Y.

If $X \in \mathcal{F}^*$, then $\phi(\mathcal{F}^*) = \nu_Y(y)$ clearly follows. Suppose that $X \notin \mathcal{F}^*$; then for each $F \in \mathcal{F}$, choose $a_F \in F^* - X$, and let $\mathcal{H}$ be the filter on X^* generated by the net $(y_F)_{F \in \mathcal{F}}$. Let $\mathcal{K}$ be an u.f. finer than $\mathcal{H}$, and let z be the point in Y to which $\phi(\mathcal{K})$ converges. If V is a κ-basic neighborhood of z, then one can show that $k(F) \cap V \neq \phi$ for all $F \in \mathcal{F}$. Since $k(\mathcal{F})$ is an u.f. and Y is pretopological, $k(\mathcal{F}) \to z$ in Y, and hence $y = z$. Therefore, $\phi(\mathcal{K}) \to y$ in Y and, consequently, $\phi(\mathcal{H}) \to y$ in Y. It follows that the image under ϕ of any u.f. finer than $\mathcal{F}^*$ converges to y in Y, and therefore $\phi(\mathcal{F}^*) \to y$ in Y. ∎

PROPOSITION 5.6. If $\kappa = (Y,k)$ is a relatively T_3 compn. of X and Z the subspace $Y - X$ of Y, then πZ is a topological space. If κ is, in addition, a simple compn., then Z is a regular topological space.

PROOF. Let $A \subset Z$ and let $y \in cl_Z^2 A$. Then there is an u.f. $\mathcal{H} \to y$ in Y such that $cl_Z A \in \mathcal{H}$. Let $\mathcal{K}$ be an u.f. containing A such that $cl_Z \mathcal{K} \leq \mathcal{H}$. Assume that $\mathcal{K} \to t$ in Y. Since $Z \in \mathcal{K}$ and κ is relatively T_3, it follows that $cl_Z \mathcal{K} \to t$ in Y. Thus $t = y$, $\mathcal{K} \to y$, and $y \in cl_Z A$. Since the closure operator for Z is idempotent, πZ is a topological space. If κ is also simple, then Z is a pretopological, and hence topological, space; the regularity of Z is an easy consequence of the assumption that κ is relatively T_3. ∎

THEOREM 5.7. A (pretopological) space X has a smallest simple (pretopological) compn. iff X is locally bounded. The smallest simple (pretopological) compn., when it exists, is equivalent to $\hat{\kappa}$.

PROOF. The argument used to establish the equivalence of Conditions (1) and (5) in the proof of Theorem 4.10 can be applied to establish this result. ∎

We next turn to the problem of characterizing those spaces having a largest simple or pretopological compn. For the former property, the problem has not yet been solved in its full generality. A property slightly weaker than essential compactness is needed for the solution of the problem; this property is defined and discussed in the next paragraph.

A space X is defined to be <u>almost essentially compact</u> if there is at most one point in X^* to which a free filter containing $X^* - X$ converges in X^*. This property can also be characterized internally, albeit more clumsily, as follows: X is almost essentially compact iff, either X is locally bounded and at most one member F of N_X has the property that $F \vee (\bigcap\{G \in N_X : G \neq F\}) \neq \phi$, or else X is essentially bounded, and there is at most one point $x \in X$ such that, for some $F \to x$, $F \vee (\bigcap N_X) \neq \phi$.

THEOREM 5.8. If X is almost essentially compact, the κ^* is the largest simple compn. of X.

PROOF. Let $\kappa = (Y,k)$ be a simple compn. of X, and $\phi : X^* \to Y$ the canonical function. If X is essentially compact, the conclusion follows by Theorem 3.2, so assume that there is exactly one point $b \in X^*$ such that there is an u.f. $G \to b$ in X^* such that $X^* - X \in G$. Let $\phi(b) = z$; to establish continuity of ϕ, it is sufficient to show that $\phi(G) \to z$ in Y. If $\phi(G)$ is a free u.f., then $Y^* - k(X) \in \phi(G)$ by construction of ϕ; it is clear from the conditions imposed on X that z is the only point in Y to which a free u.f. containing $Y - k(X)$ can converge.

Suppose, on the other hand, that $\phi(G) = \dot{y}$ for some $y \in Y - k(X)$. Now $G \to b$ in X^* implies there is $F \in F(X)$ such that $k(F) \to b$ in X^* and $G \geq F^*$. Choose $G \in G$ such that $\phi(G) = \{y\}$; by Lemma 3.1, $y \in Y - k(X)$. For each $F \in F$, choose $H_F \in N_X$ such that $H_F \in F^* \cap G$. Then $\phi(H_F) = y$, which implies $k(H_F) \to y$ for all $F \in F$. Since κ is simple, $\bigcap\{k(H_F) : F \in F\} = k(\bigcap\{H_F : F \in F\}) \to y$. But $F \geq \bigcap\{H_F : F \in F\}$ implies $kF \to y$, and so $y = z$. ∎

THEOREM 5.9. Let X be a space which is locally bounded (pretopological). If X has a largest simple (pretoplogical) compn., then X is almost essentially compact.

PROOF. If X is not almost essentially compact, then there are at least two distinct points a, b in X^* such that free filters containing $X^* - X$ converges to a and b. If X is locally bounded, then necessarily a and b are in $X^* - X$. Thus, since κ^* is simple, the assumption that X is either pretopological or locally bounded leads to the conclusion that $\nu_{X*}(a) \to a$ and $\nu_{X*}(b) \to b$. Choose $A \in \nu_{X*}(a)$ and $B \in \nu_{X*}(b)$ such that $A \cap B = \phi$. Note that $A_1 = A - (X \cup \{a\})$ and $B_1 = B - (X \cup \{b\})$ are both infinite sets; with no loss of generality, assume that the cardinality of A_1 does not exceed that of B_1.

Now A_1 and B_1 both consist of free u.f.'s on X. Let the members of A_1 be indexed as follows: $A_1 = \{F_\alpha : \alpha \in I\}$; then under our cardinality assumption, we can index a subset $B_2 = \{G_\alpha : \alpha \in I\}$ of B_1 with the same index set I. Finally, we define a totally bounded Cauchy structure C on X consisting of: (1) all convergent filters on X; (2) all members of N_X not included in A_1 or B_2; (3) all filters finer then filters of the form $F_\alpha \cap G_\alpha$ for $\alpha \in I$.

Let Y be the set of C - equivalent classes. Let $\sigma : Y \to F(X)$ be the function defined as follows: (1) $\sigma([\dot{x}]) = \dot{x}$; (2) $\sigma([F]) = F$ if $F \in N_X - (A_1 \cup B_2)$; (3) $\sigma([F_\sigma \cap G_\alpha]) = G_\alpha$, all $\alpha \in I$. (Here, $[F]$ denotes the Cauchy equivalence class determined by $F \in C$.) Let $j : X \to Y$ be the natural injection function given by $j(x) = [\dot{x}]$. If Y is equipped with the complete Cauchy structure denoted by C_Γ in [19], where $\Gamma = \{\sigma\}$, then one can verify straightforwardly that $\kappa = (Y, j)$ is a simple (pretopological) compn. of X, and $\phi : X^* \to Y$ is not continuous. Since κ^* is the only possible candidate for a largest simple (pretopological) compn. of X, it follows that X has no largest simple (pretopological) compn. ∎

COROLLARY 5.10. A pretopological space X has a largest pretopological compn. iff X is almost essentially compact.

COROLLARY 5.11. A locally bounded space X has a largest simple compn. iff X is almost essentially compact.

6. SUMMARY.

A space X has a largest compn. iff X is essentially compact. The same condition is also necessary and sufficient for the existence of a largest strict, relatively diagonal, or relatively round compn. If X is pretopological (locally bounded), then X has a largest pretopological (simple) compn. iff X is almost essentially compact. Every space X has a largest relatively T_3 compn. and a largest simple relatively round compn. Every pretopological space has a largest relatively topological compn. In every case cited, the largest compn., when it exists, is equivalent to κ^*.

A space X has a smallest compn. iff X is essentially compact. A weaker condition, local boundedness, is necessary and sufficient for the existence of a smallest compn. subject to each of the following properties: strict, relatively diagonal, relatively round, relatively T_3, and simple. Local boundedness is also necessary and sufficient in order for a pretopological space to have a smallest pretopological compn. In each case cited, the smallest compn., when it exists, is equivalent to $\hat{\kappa}$.

REFERENCES

1. Choquet, G., "Convergences", Annales Univ. Grenoble 23 (1948) 57 - 112.

2. Cochran, A. C. and Trail, R. B., "Regularity and Complete Regularity for Convergence Spaces", Proc. of VPI Topology Conference, Springer Lecture Note Series No. 375 (1973) 64 - 70.

3. Cook, C. H., "Compact Pseudo-Convergence", Math. Ann. 202 (1973), 193 - 202.

4. Fischer, H. R., "Limesräume", Math. Ann. 137 (1959), 269 - 303.

5. Frechet, M., "Sur quelques points du calcul fonctionnel", Thèse, Paris 1906, et Rend. Circ. Mat. Palermo 22 (1906).

6. Gazik, R. J., "Regularity of Richardson's Compactification", Can. J. Math. 26 (1974), 1289 - 1293.

7. Herrmann, R. A., "A nonstandard Approach to Pseudotopological Compactifications", Notices A.M.S. 26 (1979) A121. Abstract #763-54-3.

8. Hong, S. S. and Nel, L. D., "E-Compact Convergence Spaces and E-Filters", Kyunkpook Math. J. 15 (1975), 183 - 189.

9. Kasahara, S., "Boundedness in Convergence Spaces", Proc. Reno Conference on Convergence Spaces (1976), 83 - 92.

REFERENCES

10. Kent, D. C., and Richardson, G. D., "Regular Completions of Cauchy Spaces", Pacific J. Math. 51 (1974), 483 - 490.

11. Kent, D. C., Richardson, G. D., and Gazik, R. J., "T-regular-closed Convergence Spaces", Proc. Amer. Math. Soc. 51 (1975).

12. Kowalsky, H. J., "Limesräume and Komplettierung:, Math. Nachr. 12 (1954), 301 - 340.

13. Novak, J., "On Convergence Spaces and Their Sequential Envelopes", Czech. Math. J. 15 (90) (1965), 74 - 100.

14. Poppe, H., Compactness in General Function Spaces, VEB Deutscher Verlag der Wissenschaften, Berlin, 1974.

15. Ramaley, J. F., and Wyler, O., "Cauchy Spaces II. Regular Completions and Compactifications", Math. Ann. 187 (1970), 187 - 199.

16. Rao, C. J. M., "On Smallest Hausdorff Compactification for Convergence Spaces", Proc. Amer. Math. Soc. 44 (1974), 225 - 230.

17. ____________, "On Largest Hausdorff Compactification for Convergence Spaces", Bull. Austral. Math. Soc. 12 (1975), 73 - 79.

18. ____________, "On m-Ultracompactifications", Proc. Reno Conference on Convergence Spaces", (1976), 184 - 187.

19. Reed, E. E., "Completions of Uniform Convergence Spaces", Math. Ann. 194 (1971) 83 - 108.

20. ____________, "Proximity Convergence Structures", Pacific J. Math.

21. Richardson, G. D., "A Stone-Čech Compactification for Limit Spaces", Proc. Amer. Math. Soc. 25 (1970), 403 - 404.

22. Richardson, G. D., and Kent D. C., "Regular Compactifications of Convergence Spaces", Proc. Amer. Math. Soc. 31 (1972), 571 - 573.

23. Riecke, C. V., "On Boundedness for Convergence Spaces", Bolletino U. M. I. (5) 15-B (1978), 49 - 59.

24. Vinod-Kumar, "On the Largest Hausdorff Compactification of a Hausdorff Convergence Space", Bull. Austral. Math. Soc. 16 (1977), 189 - 197.