Internat. J. Math. & Math. Sci.
Vol. 2 (1979) 45-60

ON THE OVERCONVERGENCE OF CERTAIN SERIES

M. BLAMBERT and R. PARVATHAM

Institut Fourier
Mathématiques Pures
Boite postale 116
38402 ST MARTIN D'HERES
FRANCE

(Received April 27, 1978)

ABSTRACT. In this work, we consider certain class of exponential series with polynomial coefficients and study the properties of convergence of such series. Then we consider a subclass of this class and prove certain theorems on the overconvergence of such a series, which allow us to determine the conditions under which the boundary of the region of convergence of this series is a natural boundary for the function f defined by this series.

KEY WORDS AND PHRASES. LC-Dirichletian element, L-Dirichletian element, Convergence, Overconvergence.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 30A16, 30A64.

1. INTRODUCTION.

Let us consider the following LC-dirichletian element

$$\{f\} : \sum_{1}^{\infty} P_n(x) \exp -\lambda_n s \ , \tag{1.1}$$

where $P_n(s) = \sum_{j=0}^{m_n} a_{nj} s^j$, a_{nj} are complex constants with $a_{n,m_n} \neq 0$, $s = \sigma + i\tau$,

$(\sigma,\tau) \in \mathbf{R}^2$ and (λ_n) is a sequence of complex numbers such that $(|\lambda_n|)$ is a D-sequence. That is to say $(|\lambda_n|)$ is a sequence of positive real numbers satisfying

$$0 < |\lambda_1| < |\lambda_2| < \dots , \lim_{n\to\infty} |\lambda_n| = \infty . \tag{1.2}$$

Let

$$L = \limsup \left\{ \frac{\log n}{|\lambda_n|} / n \in \mathbb{N}-\{0\} \right\} \tag{1.3}$$

$$A_n = \mathrm{Max} \left\{ |a_{nj}| / j \in (0,1,\dots,m_n) \right\} \tag{1.4}$$

and

$$\beta^* = \limsup \left\{ \frac{m_n}{|\lambda_n|} / n \in \mathbb{N}-\{0\} \right\} . \tag{1.5}$$

Let $\mathcal{E}_n$ be the set of points of $\mathbb{C}$ which are zeros of $P_n(s)$ and $\mathcal{E} = \bigcup_n \mathcal{E}_n$. Let us denote by $\mathcal{E}^d$ the derived set of $\mathcal{E}$ and $\mathcal{E}_\infty = \left\{ s \in \mathbb{C} \mid \exists_{(n_j)} P_{n_j}(s) = 0 \right\}$ where (n_j) is an infinite subsequence of $\mathbb{N}-\{0\}$ depending on s ; let $\mathcal{E}^* = \mathcal{E}^d \cup \mathcal{E}_\infty$. $\mathcal{E}^*$ is a closed set. Let us suppose that $\mathbb{C}-\mathcal{E}^*$ is non empty. We put

$$\forall_{s\in\mathbb{C}-\mathcal{E}^*} \quad \delta(n,s) = -\frac{\log |P_n(s)\exp(-\lambda_n s)|}{|\lambda_n|} , \text{ for sufficiently large } n ,$$

$$\delta_*(s) = \liminf \left\{ \delta_n(s) / n \in \mathbb{N}-\{0\} \right\}$$

$$\forall_{\alpha\in\mathbf{R}} \quad \mathcal{D}_{*\alpha} = \{ s \in \mathbb{C}-\mathcal{E}^* / \delta_*(s) > \alpha \} .$$

In this paper, using a technique similar to that used by M. Blambert and J. Simeon [2] , we prove two lemmas for a LC-dirichletian element which enable us to discuss the properties of absolute convergence and uniform convergence for (1.1) in $\mathbb{C}-\mathcal{E}^*$ exclusively. Then we prove Jentzsch's theorem for a L-dirichletian element that is for element of the type (1.1) where λ_n are positive real numbers satisfying (1.2) $\left((\lambda_n)\right.$ is a D-sequence$\left.\right)$ and a theorem on the overconvergence for a L-dirichletian element.

2. MAIN RESULTS.

DEFINITION. - It is said that a function is sub-lipschitzian on an open set, if it is lipschitzian on each compact subset of that open set.

LEMMA 1. - Let $\mathcal{K}$ be any compact subset of $\mathbb{C}$. Then the following assertions are true.

(1) $\forall_{\mathcal{K}\subset\mathbb{C}-\mathcal{E}^*}\ \exists_{n'}\ \forall_{n\geq n'}$ the function $\mathcal{K}\ni s \to \delta(n,s)$ is lipschitzian.

(2) If $\beta^* < \infty$, and if there exists a $s_o \in \mathbb{C}-\mathcal{E}^*$ such that $|\delta_*(s_o)| < \infty$ then the function δ_* is sub-lipschitzian on $\mathbb{C}-\mathcal{E}^*$.

PROOF. Let $\forall_{\mathcal{K}\subset\mathbb{C}-\mathcal{E}^*}\ \epsilon_{\mathcal{K}} = \mathrm{dist}(\mathcal{K},\mathcal{E}^*)$. Then it is easy to see that

$$\forall_{s\in\mathcal{K}}\ \forall_{\epsilon\in]0,\epsilon_{\mathcal{K}}[}\ \exists_{n'}\ \forall_{n\geq n'}\ \{j\in(1,2,\ldots,m_n) \Rightarrow \alpha_{nj} \notin d_{s,\epsilon}\},$$

where $d_{s,\epsilon}$ is the open disc centred at s and of radius ϵ and (α_{nj}), $j\in\{1,2,\ldots,m_n\}$, is the sequence of zeros of $P_n(s)$ (with its order of multiplicity is taken into account). More precisely let us show that,

$$\forall_{\epsilon\in]0,\epsilon_{\mathcal{K}}[}\ \exists_{n'}\ \forall_{n\geq n'}\ \forall_{s\in\mathcal{K}}\ \{j\in(1,2,\ldots,m_n) \Rightarrow \alpha_{nj}\notin d_{s,\epsilon}\}.$$

Let $G_\epsilon = \bigcup_{s\in\mathcal{K}} d_{s,\epsilon}$. It is evident that $\overline{G}_\epsilon$ the closure of G_ϵ is a compact subset of $\mathbb{C}-\mathcal{E}^*$. Let $\epsilon'\ni]0,\epsilon_{\mathcal{K}}-\epsilon[$ where $\epsilon\in]0,\epsilon_{\mathcal{K}}[$. The set of discs $d_{s,\epsilon'}$ indexed by s on $\overline{G}_\epsilon$ is an open covering of $\overline{G}_\epsilon$. Hence we have a finite subcovering ;

$$\exists_{\overline{G}_\epsilon\supset(s_1,\ldots,s_k)}\ \bigcup_{j=1}^{k} d_{s_j,\epsilon'} \supset \overline{G}_\epsilon .$$

Let $s\in\mathcal{K}$ and $s'\in d_{s,\epsilon}$; hence $s'\in G_\epsilon$. Then $s'\in\bigcup_{j=1}^{k} d_{s_j,\epsilon'}$ which implies that $\exists_{j^*\in(1,\ldots,k)}\ s'\in d_{s_{j^*},\epsilon'}$. Now

$$\forall_{j\in(1,\ldots,k)}\ \exists_{n'(=n_j)}\ \forall_{n\geq n'}\ \forall_{s\in d_{s_j,\epsilon'}}\ P_n(s)\neq 0$$

and hence

$$\forall_{n\geq \mathrm{Max}\{n_j/j\in(1,\ldots,k)\}}\ \forall_{j\in(1,2,\ldots,k)}\ \forall_{s\in d_{s_j,\epsilon'}}\ P_n(s)\neq 0,$$

which gives $\forall_{n\geq\mathrm{Max}\{n_j/j\in(1,\ldots,k)\}}\ P_n(s')\neq 0$. As s is arbitrary on $\mathcal{K}$ and s' is arbitrary on $d_{s,\epsilon}$ we have

$$\underset{\epsilon\in]0,\epsilon_{\mathcal{K}}[}{\forall}\ \underset{n'(=n_\epsilon)}{\exists}\ \underset{n\geq n'}{\forall}\ \underset{s\in\mathcal{K}}{\forall}\ \{j\in(1,\dots,k) \Rightarrow \alpha_{nj} \notin d_{s,\epsilon}\} .$$

From which we have

$$\underset{n\geq n'(=n_\epsilon)}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ \log|P_n(s)|-\log|P_n(s')| \leq \sum_{j=1}^{m_n} \log\left\{1+\frac{|s-s'|}{|s'-\alpha_{nj}|}\right\} \leq \sum_{j=1}^{m_n} \log\left\{1+\frac{|s-s'|}{\epsilon}\right\}.$$

Under the above conditions related to n , s and s' with $s \neq s'$,

$$|\delta(n,s)-\delta(n,s')| \leq |s-s'| + \frac{1}{|\lambda_n|}\sum_{j=1}^{m_n} \log\left\{1+\frac{|s-s'|}{\epsilon}\right\}$$

$$\leq |s-s'| + \frac{|s-s'|}{\epsilon|\lambda_n|}\sum_{j=1}^{m_n}\left\{\log\left(1+\frac{|s-s'|}{\epsilon}\right)\Big/\frac{|s-s'|}{\epsilon}\right\}$$

$$\leq |s-s'| + \frac{m_n|s-s'|}{\epsilon|\lambda_n|}\sup\left\{\frac{\log(1+x)}{x} / x>0\right\} ;$$

as $\sup\left\{\frac{\log(1+x)}{x}/x>0\right\} = 1$, $|\delta(n,s)-\delta(n,s')| \leq |s-s'|\left\{1+\frac{m_n}{\epsilon|\lambda_n|}\right\}$. Putting $\mu_{\epsilon,n} = 1+\frac{m_n}{\epsilon|\lambda_n|}$,

$$\underset{\epsilon\in]0,\epsilon_{\mathcal{K}}[}{\forall}\ \underset{n'}{\exists}\ \underset{n\geq n'}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta(n,s)-\delta(n,s')| \leq \mu_{\epsilon,n}|s-s'| ,$$

which proves the first part of the lemma.

Now let $\mu^*_\epsilon = \limsup_{n\to\infty} \mu_{\epsilon,n} = 1+\beta^*/\epsilon$ with $\epsilon\in]0,\epsilon_{\mathcal{K}}[$; as $\underset{s_o\in\mathbb{C}-\mathcal{E}^*}{\exists}\ \delta_*(s_o)<\infty$

$$\underset{\epsilon\in]0,\epsilon_{\mathcal{K}}[}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta_*(s)-\delta_*(s')| \leq \mu^*_\epsilon|s-s'|$$

and

$$\underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta_*(s)-\delta_*(s')| \leq \mu^*_{\epsilon_{\mathcal{K}}}|s-s'|$$

where

$$\mu^*_{\epsilon_{\mathcal{K}}} = \mathrm{Inf}\{\mu^*_\epsilon/\epsilon\in]0,\epsilon_{\mathcal{K}}[\} = 1+\frac{\beta^*}{\epsilon_{\mathcal{K}}} .$$

Hence

$$\underset{\mathcal{K}\subset\mathbb{C}-\mathcal{E}^*}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta_*(s)-\delta_*(s')| \leq \mu^*_{\epsilon_{\mathcal{K}}}|s-s'| ,$$

which completes the proof of the lemma.

Under the condition (2) of Lemma 1 , δ_* is continuous on $\mathbb{C}-\mathcal{E}^*$

which implies that $\mathcal{D}_{*\alpha}$ is an open subset of $\mathbb{C}-\mathcal{E}^*$; but $\mathcal{D}_{*\alpha}$ can have several connected components.

LEMMA 2. - When $\beta^*<\infty$, then

$$\forall_{\alpha\in\mathbb{R}} \left\{ \mathcal{D}_{*\alpha} \neq \phi \Rightarrow \forall_{\mathcal{K}\subset\mathcal{D}_{*\alpha}} \ \forall_{\beta'>\beta^*} \ \exists_{n'} \ \forall_{n\geq n'} \ \forall_{s\in\mathcal{K}} \ |P_n(s)\exp(-\lambda_n s)| < \exp(-|\lambda_n|(\alpha-\beta')) \right\} .$$

PROOF. Let $\alpha \in \mathbb{R}$ such that $\mathcal{D}_{*\alpha} \neq \phi$ (otherwise the lemma is trivial) and let $\mathcal{K}$ be a compact subset of $\mathcal{D}_{*\alpha}$. We can easily see that

$$\forall_{s\in\mathbb{C}-\mathcal{E}^*} \ \forall_{\epsilon\in]0,\mathrm{dist}(s,\mathcal{E}^*)[} \ \exists_{n'(=n_{s,\epsilon})} \ \forall_{n\geq n'} \ \forall_{s'\in\overline{d_{s,\epsilon}}} \ P_n(s') \neq 0$$

where $\overline{d_{s,\epsilon}}$ is the closed disc centred at s and of radius ϵ . Hence

$$\forall_{s\in\mathbb{C}-\mathcal{E}^*} \ \forall_{\epsilon\in]0,\epsilon_{\mathcal{K}}[} \ \exists_{n'(=n_{s,\epsilon})} \ \forall_{n\geq n'} \ \forall_{s'\in\overline{d_{s,\epsilon}}} \ P_n(s') \neq 0 .$$

Let us consider the compact subset $\mathcal{K}_\epsilon = \overline{\bigcup_{s\in\mathcal{K}} d_{s,\epsilon}}$, of $\mathbb{C}-\mathcal{E}^*$. As $\mathrm{dist}(\mathcal{K}_\epsilon,\mathcal{E}^*) > 0$, we have from lemma 1,

$$\forall_{\epsilon'\in]0,\mathrm{dist}(\mathcal{K}_\epsilon,\mathcal{E}^*)[} \ \exists_{n'(=n_{\epsilon'})} \ \forall_{n\geq n'} \ \forall_{(s,s')\in\mathcal{K}_\epsilon\times\mathcal{K}_\epsilon} \ |\delta(n,s)-\delta(n,s')| \leq \mu_{\epsilon',n}|s-s'|$$

where $\mu_{\epsilon',n} = 1 + \dfrac{m_n}{\epsilon'|\lambda_n|}$. In particular,

$$\forall_{n\geq n_{\epsilon'}} \ \forall_{s\in\mathcal{K}} \ \forall_{s'\in d_{s,\epsilon}} \ |\delta(n,s)-\delta(n,s')| \leq \mu_{\epsilon',n}|s-s'| ,$$

and hence

$$\delta(n,s') \geq \delta(n,s) - \mu_{\epsilon',n}|s-s'| .$$

Further $\forall_{\beta'>\beta^*} \ \exists_{n'(=n_{\beta'})} \ \forall_{n\geq n'} \ \dfrac{m_n}{|\lambda_n|} < \beta'$ and

$$\forall_{n\geq\max\{n_{\epsilon'},n_{\beta'}\}=n_1} \ \forall_{s\in\mathcal{K}} \ \forall_{s'\in d_{s,\epsilon}} \ \delta(n,s') \geq \delta(n,s) - |s-s'|(1+\frac{\beta'}{\epsilon'}) .$$

Since $\mathcal{K}$ is a compact subset of $\mathcal{D}_{*\alpha}$, $\forall_{s\in\mathcal{K}} \ \exists_{n'(=n_s)} \ \forall_{n\geq n'} \ \delta(n,s') > \alpha$; finally we have

$$\forall_{s\in\mathcal{K}} \ \forall_{\beta'>\beta^*} \ \forall_{\epsilon'\in]0,\mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\epsilon)[} \ \exists_{n'} \ \forall_{n\geq n'} \ \forall_{s'\in d_{s,\epsilon}} \ \delta(n,s') > \alpha - |s-s'|(1+\frac{\beta'}{\epsilon'}) \quad (2.1)$$

where ε is arbitrary in $]0,\varepsilon_{\mathcal{K}}[$. The set of discs $d_{s,\varepsilon}$ indexed by s on $\mathcal{K}$ is an open covering for $\mathcal{K}$ and hence $\exists_{\mathcal{K}\supset(s_1\dots s_k)} \bigcup_{j=1}^{k} d_{s_j,\varepsilon} \supset \mathcal{K}$. Further we have $\forall_{s\in\mathcal{K}} \exists_{j'\in(1\dots k)} s\in d_{s_{j'},\varepsilon}$. Using (2.1) for the particular pair $(s_{j'},s)$, we have

$$\forall_{\beta'>\beta^*} \forall_{\varepsilon'\in]0,\mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\varepsilon)[} \exists_{n'(=n_{s_{j'},\beta',\varepsilon'})} \forall_{n\geq n'} \delta(n,s) > \alpha - |s-s_{j'}|(1+\frac{\beta'}{\varepsilon'}) .$$

Let $n'' = \mathrm{Max}\{n_{s_{j'},\beta',\varepsilon'} | j\in(1\dots k)\}$ and as $|s-s_j| < \varepsilon$, we have

$$\forall_{\beta'>\beta^*} \forall_{\varepsilon'\in]0,\mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\varepsilon)[} \exists_{n''} \forall_{n\geq n''} \delta(n,s) > \alpha - \varepsilon(1+\frac{\beta'}{\varepsilon'}) .$$

Choosing $\varepsilon = \varepsilon' < \frac{\varepsilon_{\mathcal{K}}}{2}$ we have $\frac{\mathrm{dist}(\mathcal{K},\mathcal{E}^*)}{2} < \mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\varepsilon)$ and

$$\forall_{\beta'>\beta^*} \forall_{\varepsilon\in]0,\frac{\varepsilon_{\mathcal{K}}}{2}[} \exists_{n''} \forall_{n\geq n''} \delta(n,s) > \alpha - \varepsilon - \beta'$$

where s is any arbitrary point of $\mathcal{K}$ and n'' does not depend on s . Hence

$$\forall_{\beta'>\beta^*} \forall_{\varepsilon\in]0,\frac{\varepsilon_{\mathcal{K}}}{2}[} \exists_{n''} \forall_{n\geq n''} \forall_{s\in\mathcal{K}} \delta(n,s) > \alpha-\varepsilon-\beta' .$$

As β' is arbitrary and strictly greater than β^* , we have

$$\forall_{\mathcal{K}\subset\mathcal{D}_{*\alpha}} \forall_{\beta'>\beta^*} \exists_{n'} \forall_{n\geq n'} \forall_{s\in\mathcal{K}} \delta(n,s) > \alpha - \beta'$$

and hence

$$\forall_{\mathcal{K}\subset\mathcal{D}_{*\alpha}} \forall_{\beta'>\beta^*} \exists_{n'} \forall_{n\geq n'} \forall_{s\in\mathcal{K}} |P_n(s)\exp -\lambda_n s| < \exp(-|\lambda_n|(\alpha-\beta')) .$$

THEOREM 1. - When $\beta^* < \infty$, $L < \infty$, the LC-dirichletian element $\{f\}$ converges absolutely on $\mathcal{D}_{*,L+\beta^*}$ and uniformly on any compact subset of $\mathcal{D}_{*,L+\beta^*}$.

PROOF. Let us suppose that $\mathcal{D}_{*,L+\beta^*}$ is non empty. Let $\mathcal{K}_o$ be a compact subset of $\mathcal{D}_{*,L+\beta^*}$. We know that $\exists_{\alpha>L+\beta^*} \mathcal{K}_o \subset \mathcal{D}_{*\alpha}$. Let

$\beta' \in]\beta^*, \alpha-L[$. From Lemma 2 we have,

$$\exists_{n'} \ \forall_{n\geq n'} \ \forall_{s\in K_o} \ |P_n(s)\exp(-\lambda_n s)| < \exp\{-|\lambda_n|(\alpha-\beta')\} ,$$

where $\alpha - \beta' > L$. Hence

$$\sum_{n=n'}^{\infty} |P_n(s)\exp(-\lambda_n s)| < \sum_{n=n'}^{\infty} \exp\{-|\lambda_n|(\alpha-\beta')\}$$

and the series on the right hand side is convergent which proves that $\{f\}$ converges absolutely and uniformly on K_o . Since K_o is any arbitrary compact subset of $\mathcal{D}_{*,L+\beta^*}$, $\{f\}$ converges uniformly on any compact subset of $\mathcal{D}_{*,L+\beta^*}$ and absolutely on $\mathcal{D}_{*,L+\beta^*}$.

REMARK 1. By the following method, we obtain a bigger set of absolute convergence for $\{f\}$. Let $\mathcal{D}_{*L}$ be supposed to be non-empty and $L < \infty$. Then $\forall_{s\in\mathcal{D}_{*L}} \ \exists_{\varepsilon_s>0} \ \delta_*(s) > L+\varepsilon_s$; $\exists_{n'_s} \ \forall_{n\geq n'_s} \ \delta(n,s) > L+\varepsilon_s$ and

$\forall_{n\geq n'_s}$ $-\mathrm{Log}\,|P_n(s)\exp(-\lambda_n s)| > (L+\varepsilon_s)|\lambda_n|$. Hence

$$\sum_{n=n'_s}^{\infty} |P_n(s)\exp -\lambda_n s| < \sum_{n=n'_s}^{\infty} \exp\{-(L+\varepsilon_s)|\lambda_n|\}$$

and as the series on the right hand side converges, the series (1.1) converges absolutely on $\mathcal{D}_{*L}$. In this result, we have no restriction on β^* .

REMARK 2. $\{f\}$ diverges on $\mathbb{C}-\mathcal{E}^*-\bar{\mathcal{D}}_{*o}$. If $s \in \mathbb{C}-\mathcal{E}^*-\bar{\mathcal{D}}_{*o}$, then $\delta_*(s) < 0$ and $\exists_{\alpha\in\mathbb{R}_o^+} \ \delta_*(s) < -\alpha$. Hence $\forall_{s\in\mathbb{C}-\mathcal{E}^*-\bar{\mathcal{D}}_{*o}} \ \exists_{\alpha>0} \ \exists_{(n_j)} \ \delta(n_j,s) < -\alpha$ where (n_j) is an infinite subsequence of $\mathbb{N}-\{0\}$. Therefore

$$|P_{n_j}(s)\exp(-\lambda_{n_j} s)| > \exp(\alpha|\lambda_{n_j}|) > 1$$

and which shows that $\{f\}$ diverges on $\mathbb{C}-\mathcal{E}^*-\bar{\mathcal{D}}_{*o}$. When $L = 0$, we have convergence of the series (1.1) in $\mathcal{D}_{*o} \subset \mathbb{C}-\mathcal{E}^*$ and divergence in $\mathbb{C}-\mathcal{E}^*-\bar{\mathcal{D}}_{*o}$. We do not discuss the property of convergence of the series in $\mathcal{E}^*$.

From here onwards we consider a L-dirichletian element,

$$\{f\} : \sum_1^{\infty} P_n(s)\exp(-\lambda_n s) \tag{2.2}$$

where (λ_n) is a D-sequence (here λ_n are positive real numbers).

DEFINITION. It is said that a D-sequence (λ_n) is of the type (Λ) if the following conditions are satisfied :

i) the Dirichlet series $\sum_{j=1}^{\infty} \exp(-\lambda_n s)$ converges on $P_o = \{s \in \mathbb{C} \mid \sigma > 0\}$. (this gives that $\forall_{n \in \mathbb{N}-\{0\}} \sum_{j=n}^{\infty} \exp(-s(\lambda_j - \lambda_n))$ converges on P_o. Let $\theta_n(s)$ be its sum at the point s);

ii) $\forall \eta > 0$ the sequence of functions (θ_n) where $\theta_n : P_o \ni s \to \theta_n(s)$ is bounded on $\overline{P}_\eta = \{s \in \mathbb{C} / \sigma \geq \eta\}$;

iii) $\forall \eta > 0$ the sequence of functions (θ^*_n) where $\theta^*_n : P_o \ni s \to \sum_{j=1}^{n} \exp(-s(\lambda_n - \lambda_j))$ is bounded on $\overline{P}_\eta$.

EXAMPLE. - If (λ_n) is a D-sequence and $\exists_{\mu > 0} \operatorname{Inf}(\lambda_{n+1} - \lambda_n) = \mu$, then it is easy to see that (λ_n) is of the type (Λ) .

If the D-sequence (λ_n) is of the type (Λ) , then we can easily show that $L = 0$.

Now let us prove Jentzsch's theorem for L-dirichletian element. This theorem for Dirichlet series with complex exponents was proved by T.M. Gallie [3] . First let us consider the associated Dirichlet series of $\{f\}$.

$$\{f_A\} : \sum_1^{\infty} A_n \exp(-\lambda_n s)$$

where A_n is defined by (1.4). Let

$$\sigma_p^{f_A} = \operatorname{Inf}\{\sigma \in \mathbb{R} / \lim |A_n \exp(-\lambda_n s)| = 0 \ , \ n \to \infty\}$$

be the abscisse of pseudo convergence of $\{f_A\}$. Then we know that

$$\sigma_p^{f_A} = \limsup_{n \to \infty} \left\{\frac{\log A_n}{\lambda_n}\right\} ;$$

when $L = 0$, $\sigma_p^{f_A}$ is the same as $\sigma_c^{f_A}$, the abscisse of convergence of $\{f_A\}$.

Let n and n' be two natural numbers such that $n' \geq n$. Let $E_{n,n'}$ denote the set, indexed by (n,n'), of points of $\mathbb{C}$ which are zeros of the LC-dirichletian polynomial

$$S_{n,n'}(S) = \sum_{j=n}^{n'} P_j(s)\exp(-s\lambda_j) \; ;$$

let E denote the union of all sets $E_{n,n'}$ corresponding to all pairs (n,n') and E_∞ be the set formed by the points which are zeros for an infinity of polynomials $S_{n,n'}(s)$. Let us put $E^* = E^d \cup E_\infty$ where E^d is the derived set of E. E^* is a closed subset of $\mathbb{C}$. It is evident that $E \supset \mathcal{E}$ and $E_\infty \supset \mathcal{E}_\infty$ and hence $E^* \supset \mathcal{E}^*$. We suppose in what follows that $\mathbb{C}-E^* \neq \phi$ (which implies $\mathbb{C}-\mathcal{E}^* \neq \phi$). Then we have

THEOREM 2. - When the D-sequence (λ_n) is of the type (Λ), $\sigma_c^{f_A} < \infty$ and $\beta^* < \infty$, then we have $(\mathrm{Fr}(\mathcal{D}_{*o}) \cap \mathbb{C}-\mathcal{E}^*) \subset E^*$.

PROOF. Let us suppose that the theorem is not true. Then there exists a point $b \in (\mathrm{Fr}(\mathcal{D}_{*o}) \cap \mathbb{C}-\mathcal{E}^*)$ and a disc $d(b,\rho)$ centred at b of radius $\rho > 0$, included in $\mathbb{C}-\mathcal{E}^*$ such that

$$\exists_{n_o} \; \forall_{n' \geq n \geq n_o} \; \forall_{s \in d(b,\rho)} \; S_{n,n'}(s) \neq 0 \, .$$

We have $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+|s|)^{m_n} |\exp(-\lambda_n s)|$ and

$\forall_{\beta' > \beta^*} \; \exists_{n'_o (=n_{\beta'})} \; \forall_{n \geq n'_o} \; (m_n/\lambda_n) < \beta'$. Let us take a certain $\beta' > \beta^*$ and put

$\omega = \beta' \,\mathrm{Log}[1+\sup\{|s| / s \in d(b,\rho)\}] - \mathrm{Inf}\{\sigma / s \in d(b,\rho)\}$ and hence

$\forall_{n \geq n'_o} \; \forall_{s \in d(b,\rho)} \; |P_n(s)\exp(-\lambda_n s)| < A_n \exp(\omega\lambda_n)$. From the definition of $\sigma_c^{f_A}$ we have $\forall_{\sigma' > \sigma_c^{f_A}} \; \exists_{n''_o (=n_{\sigma'})} \; \forall_{n \geq n''_o} \; A_n < \exp(\sigma'\lambda_n)$. Hence putting $n_1 = \mathrm{Max}(n_o, n'_o, n''_o)$, we get $\forall_{n \geq n_1} \; \forall_{s \in d(b,\rho)} \; |P_n(s)\exp(-\lambda_n s)| < \exp((\omega+\sigma')\lambda_n)$.

Let $S_{n_1,n}(s) = \sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s)$ and $\forall_{s \in d(b,\rho)} \; T_{n_1,n}(s) = (S_{n_1,n}(s))^{1/\lambda_n}$; $[S_{n_1,n}(s)]^{1/\lambda_n}$ is defined to be equal to $\exp((1/\lambda_n)\,\mathrm{Log}\, S_{n_1,n}(s))$ where

$\operatorname{Im} \operatorname{Log} S_{n_1,n}(s) \in]-\pi,\pi]$. For each integer $n \geq n_1$ the function $T_{n_1,n} : d(b,\rho) \ni s \to T_{n_1,n}(s)$ is holomorphic on $d(b,\rho)$. We have

$$\forall_{s\in d(b,\rho)} \quad |T_{n_1,n}(s)| = |(\sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s))^{1/\lambda_n}| \leq \{\exp(\lambda_n(\omega+\sigma') + \log n)\}^{1/\lambda_n}$$
$$= \exp(\omega+\sigma')\exp(\frac{\log n}{\lambda_n}) \ .$$

Since (λ_n) is of the type (Λ) which implies $L = 0$, we have $\lim_{n\to\infty}\exp(\frac{\log n}{\lambda_n}) = 1$. Hence the sequence of functions $(T_{n_1,n})$, $n \geq n_1$, is bounded and hence normal on $d(b,\rho)$.

Let $\mathcal{K}$ be a compact subset of $d(b,\rho)$ such that $\operatorname{Int} \mathcal{K} \cap \mathcal{D}_{*o} \neq \phi$. From any extracted subsequence of $(T_{n_1,n})$ we can extract a subsequence which converges uniformly on $\mathcal{K}$ and the limit function is holomorphic on the Int $\mathcal{K}$.

Let $\mathcal{K}_1$ be a compact subset of $d(b,\rho) \cap \mathcal{D}_{*o}$ such that $\operatorname{Int}\mathcal{K} \cap \operatorname{Int}\mathcal{K}_1 \neq \phi$. Then we have $\forall_{s\in\mathcal{K}_1} \lim_{n\to\infty} T_{n_1,n} = 1$. Now $\mathcal{K} \cup \mathcal{K}_1$ is a compact subset of $d(b,\rho)$. Then the subsequence extracted from the arbitrarily extracted subsequence of $(T_{n_1,n})$ converges uniformly on $\mathcal{K}\cup\mathcal{K}_1$ to a limit function holomorphic in $\operatorname{Int}(\mathcal{K}\cup\mathcal{K}_1)$ and continuous on the boundary of $\mathcal{K}\cup\mathcal{K}_1$ and takes the value one at each point of $\mathcal{K}_1$. Hence the limit function takes the value one at each point of $\mathcal{K}\cup\mathcal{K}_1$. This results that the sequence $(T_{n_1,n})$ converges to the same limit function on $\mathcal{K}\cup\mathcal{K}_1$.

As $\mathcal{K}$ is any arbitrary compact subset of $d(b,\rho)$ and $\mathcal{K}_1$ is any arbitrary compact subset of $d(b,\rho) \cap \mathcal{D}_{*o}$ such that $\operatorname{Int}\mathcal{K} \cap \operatorname{Int}\mathcal{K}_1 \neq \phi$, we have

$$\forall_{s\in d(b,\rho)} \quad \lim_{n\to\infty} T_{n_1,n}(s) = 1 \ .$$

Let $s_o \in d(b,\rho) \cap (\mathbb{C}-\mathcal{E}^*-\overline{\mathcal{D}}_{*o})$. Then

$$\forall_{\epsilon>0} \quad \exists_{n'_1(=n_{s_o,\epsilon})\geq n_1} \quad \forall_{n\geq n'_1} \quad |\sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s)| < (1+\epsilon)^{\lambda_n}$$

and hence

$$\forall_{n>n_1'} \ |P_n(s_o)\exp(-\lambda_n s_o)| = |S_{n_1,n}(s_o) - S_{n_1,n-1}(s_o)| < 2(1+\epsilon)^{\lambda_n}$$

which gives

$$-\frac{\mathrm{Log}\,|P_n(s_o)\exp(-\lambda_n s_o)|}{\lambda_n} > \frac{-\mathrm{Log}\,2}{\lambda_n} - \mathrm{Log}(1+\epsilon) \ ;$$

$\delta_*(s_o) \geq 0$ as ϵ is arbitrary. Hence we arrive at a contradiction that $s_o \in \bar{\mathcal{D}}_{*o} \cap \mathbb{C}-\mathcal{E}^*$ which establishes the result.

Finally, let us prove a theorem on the overconvergence of $\{f\}$ defined by (7). Before proving the theorem let us note that

REMARK 3. Let $\bar{\Delta}$ be any compact subset of $\mathbb{C}-\mathcal{E}^*$ and (λ_n) be a D-sequence of the type (Λ). We have $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+|s|)^{m_n}\exp(-\sigma\lambda_n)$. If $s \in \bar{\Delta}$, then $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+m_\Delta)^{m_n}\exp(m_\Delta \lambda_n)$ where $m_\Delta = \sup\{|s| / s \in \bar{\Delta}\}$. As $\bar{\Delta}$ is a compact set, m_Δ is finite ; for sufficiently large n we have

$$\frac{\mathrm{Log}\,|P_n(s)\exp(-\lambda_n s)}{\lambda_n} \leq \frac{\mathrm{Log}\,A_n}{\lambda_n} + \frac{m_n}{\lambda_n}\mathrm{Log}(1+m_\Delta) + m_\Delta \ ;$$

$$\delta_*(s) \geq -\sigma_c^{f_A} - \beta^* \mathrm{Log}(1+m_\Delta) - m_\Delta \ .$$

Hence $\forall_{\epsilon>0} \ \bar{\Delta} \subset \mathcal{D}_{*,\alpha_o-\epsilon}$ with $\alpha_o = -\sigma_c^{f_A} - \beta^* \mathrm{Log}(1+m_\Delta) - m_\Delta$. If

$\beta^* < \dfrac{-\sigma_c^{f_A} - m_\Delta}{1+\mathrm{Log}(1+m_\Delta)}$, we have $\bar{\Delta} \subset \mathcal{D}_{*\beta^*}$.

THEOREM 3. - When (λ_n) is a D-sequence of the type (Λ), $\beta^* < \infty$ and $\mathcal{D}_{*\beta^*} \neq \phi$ if there exist an infinite subsequence (n_ν), $\nu \in \mathbb{N}$, of $\mathbb{N}-\{0\}$ and a sequence of strictly positive numbers (θ_ν) such that

$$\lim_{\nu\to\infty} \theta_\nu = +\infty$$

and

$$\forall_{\nu \in \mathbb{N}} \ \lambda_{n_\nu+1} > (1+\theta_\nu)\lambda_{n_\nu} \tag{2.3}$$

then the sequence $\{S_{n_\nu}(s)\}$, $\nu \in \mathbb{N}$, where $S_{n_\nu}(s) = \sum_{j=1}^{n_\nu} P_j(s)\exp(-\lambda_j s)$, converges at each point s of any open simply connected subset (whose intersection with $\mathcal{D}_{*,\beta^*}$ in non empty) of an open set included in $\mathbb{C}-\mathcal{E}^*$ in which the function f defined by $\{f\}$ is holomorphic.

PROOF. Let us choose 3 bounded domains Δ_1, Δ_2 and Δ_3 in the following manner : $\bar{\Delta}_1 \subset \Delta_2$, $\bar{\Delta}_2 \subset \Delta_3$, $\bar{\Delta}_3 \subset \mathbb{C}-\mathcal{E}^*$, $\bar{\Delta}_1 \subset \mathcal{D}_{*\beta^*}$ and $\bar{\Delta}_3$ is included in an open subset of $\mathbb{C}-\mathcal{E}^*$ in which the function f defined by $\{f\}$ is holomorphic. Further let $Fr(\Delta_1)$, $Fr(\Delta_2)$ and $Fr(\Delta_3)$ satisfy a condition of Hadamard's type, namely

$$\underset{b\in]0,1[}{\exists} \quad \mathrm{Log}\, M_2 \le b\, \mathrm{Log}\, M_1 + (1-b)\, \mathrm{Log}\, M_3$$

where $\underset{i\in(1,2,3)}{\forall} \quad M_i = \mathrm{Max}\{|f(s)|\,/\,s\in Fr(\Delta_i)\}$.

It is easy to see that $\underset{\alpha>\beta^*}{\exists} \quad \bar{\Delta}_1 \subset \mathcal{D}_{*\alpha}$. Let us consider the set $I_1 = \{\alpha/\mathcal{D}_{*\alpha} \supset \bar{\Delta}_1\}$. I_1 is non empty and is an interval. Let $\alpha_{\Delta_1} = \sup I_1$. Then $\alpha_{\Delta_1} > \beta^*$ and $\underset{\varepsilon>0}{\forall} \quad \mathcal{D}_{*,\alpha_{\Delta_1}-\varepsilon} \supset \bar{\Delta}_1$. We can easily show that, $\alpha_{\Delta_1} = \mathrm{Inf}\{\delta_*(s)\,|\,s\in\bar{\Delta}_1\}$ which implies that α_{Δ_1} is a finite number. Hence from lemma 2,

$$\underset{\beta'\in]\beta^*,\alpha_{\Delta_1}[}{\forall} \quad \underset{n_1}{\exists} \quad \underset{n\ge n_1}{\forall} \quad \underset{s\in Fr(\Delta_1)}{\forall} \quad |P_n(s)\exp(-\lambda_n s)| < \exp(-\lambda_n(\alpha_{\Delta_1}-\beta')) \;;$$

hence for $n \ge n_1$

$$\sum_{j=n+1}^{\infty} |P_j(s)\exp(-\lambda_j s)| < \sum_{j=n+1}^{\infty} \exp\{-\lambda_j(\alpha_{\Delta_1}-\beta^*)\}$$

$$= \exp\{-\lambda_{n+1}(\alpha_{\Delta_1}-\beta')\} \sum_{j=n+1}^{\infty} \exp\{-(\alpha_{\Delta_1}-\beta')(\lambda_j-\lambda_{n+1})\} \;.$$

Since (λ_n) is a D-sequence of the type (Λ) and $\alpha_{\Delta_1}-\beta' > 0$, there exists a finite number strictly positive $B(\beta')$ such that

$$\underset{n\in\mathbb{N}}{\forall} \quad \sum_{j=n+1}^{\infty} |\exp\{-(\lambda_j-\lambda_{n+1})s| \le B(\beta') \quad \text{where } \Re e\, s \ge \alpha_{\Delta_1}-\beta' \;;$$

thus we have for each $n \ge n_1$

$$\sum_{j=n+1}^{\infty} |P_j(s)\exp(-\lambda_j s)| < B(\beta')\exp\{-\lambda_{n+1}(\alpha_{\Delta_1} - \beta')\} \ . \tag{2.4}$$

Now let $I_2 = \{\alpha \in \mathbf{R} \mid \mathcal{D}_{*\alpha} \supset \bar{\Delta}_3\}$. We have

$$\forall_{s\in\bar{\Delta}_3} \ \exists_{n'(=n_s)} \ \forall_{n\geq n'} \ \delta(n,s) \geq \frac{-\operatorname{Log} A_n}{\lambda_n} - \frac{m_n}{\lambda_n}\operatorname{Log}(1+|s|) + \sigma \ .$$

Let $m_{\Delta_3} = \sup\{|s| \,/\, s \in \bar{\Delta}_3\}$. Then

$$\forall_{n\geq n'} \ \delta(n,s) \geq \frac{-\operatorname{Log} A_n}{\lambda_n} - \frac{m_n}{\lambda_n}\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3}$$

$$\delta_*(s) \geq -\sigma_c^{f_A} - \beta^*\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3} \ ,$$

which shows that $\bar{\Delta}_3 \subset \mathcal{D}_{*\alpha}$ with $\alpha < -\sigma_c^{f_A} - \beta^*\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3}$, and hence $I_2 \neq \phi$ and is an interval in $\mathbf{R}$. Let $\alpha_{\Delta_3} = \sup I_2$. Then $\forall_{\varepsilon>0} \ \mathcal{D}_{*,\alpha_{\Delta_3}-\varepsilon} \supset \bar{\Delta}_3$. We can easily show that $\alpha_{\Delta_3} = \operatorname{Inf}\{\delta_*(s) \mid s \in \bar{\Delta}_3\}$ which implies that α_{Δ_3} is a finite number. Once again, from lemma 2, we get

$$\forall_{\beta'>\beta^*} \ \exists_{n_2} \ \forall_{n\geq n_2} \ \forall_{s\in Fr(\Delta_3)} \ |P_n(s)\exp(-\lambda_n s)| < \exp\{-\lambda_n(\alpha_{\Delta_3} - \beta')\}$$

which gives

$$\forall_{s\in Fr(\Delta_3)} \ \sum_{j=1}^{n(\geq n_2)} |P_j(s)\exp(-\lambda_j s)| = \sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)| + \sum_{j=n_2}^{n} |P_j(s)\exp(-\lambda_j s)|$$

$$\leq \max\left\{\sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)| \,/\, s\in Fr(\Delta_3)\right\} + \sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) \ .$$

Let us choose $\beta' > \beta^*$ such that $\alpha_{\Delta_3} - \beta' \neq 0$. Now we examine the two cases.

<u>Case 1</u>. - If $\alpha_{\Delta_3} - \beta' > 0$, then

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) = \exp(\lambda_n(\alpha_{\Delta_3} - \beta'))\sum_{j=n_2}^{n} \exp\{-(\alpha_{\Delta_3} - \beta')(\lambda_j + \lambda_n)\} < B''(\beta')\exp(\lambda_n(\alpha_{\Delta_3} - \beta'))$$

where $B''(\beta')$ is the sum of the series $\sum_{j=0}^{\infty} \exp(-2(\alpha_{\Delta_3} - \beta')\lambda_j)$.

Case 2. - If $\alpha_{\Delta_3}-\beta' < 0$, then

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) = \sum_{j=n_2}^{n} \exp(\lambda_j|\alpha_{\Delta_3}-\beta'|) = \exp(\lambda_n|\alpha_{\Delta_3}-\beta'|)\sum_{j=n_2}^{n} \exp(-(\lambda_n-\lambda_j)|\alpha_{\Delta_3}-\beta'|)$$

Since the D-sequence (λ_n) is of the type (Λ) there exists a finite number strictly positive $B'(\beta')$ such that

$$\forall_{n\in \mathbb{N}-\{0\}} \sum_{j=1}^{n} \exp\{-(\lambda_n-\lambda_j)|\alpha_{\Delta_3}-\beta'|\} \le B'(\beta')$$

which implies that

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) \le B'(\beta')\exp(\lambda_n|\alpha_{\Delta_3}-\beta'|) \ .$$

On putting $B'''(\beta') = \operatorname{Max}\{B'(\beta'),B''(\beta')\}$ we have

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) \le B'''(\beta')\exp(\lambda_n|\alpha_{\Delta_3}-\beta'|) \ . \tag{2.5}$$

Using the generalized form of Hadamard three circle theorem [4] we have

$$\exists_{b\in]0,1[} \quad \operatorname{Log} M_{2,\nu} \le b \operatorname{Log} M_{1,\nu} + (1-b)\operatorname{Log} M_{3,\nu} \tag{2.6}$$

where

$$M_{i,\nu} = \operatorname{Max}\{|R_{n_\nu}(s)|/s\in \operatorname{Fr}(\Delta_i)\} \ , \ i = 1,2,3$$

with

$$R_{n_\nu}(s) = f(s) - \sum_{j=1}^{n_\nu} P_j(s)\exp(-\lambda_j s) \ .$$

From (2.4) we have for $n_\nu \ge n_1$

$$M_{1,\nu} \le B(\beta')\exp\{-(\lambda_{n_\nu+1})(\alpha_{\Delta_1}-\beta')\} < B(\beta')\exp\{-(1+\theta_\nu)\lambda_{n_\nu}(\alpha_{\Delta_1}-\beta')\} \tag{2.7}$$

because of (2.3) . On putting

$$B_o = \operatorname{Max}\{|f(s)|/s\in \operatorname{Fr}(\Delta_3)\} + \operatorname{Max}\left\{\sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)|/s\in \operatorname{Fr}(\Delta_3)\right\}$$

we have from (2.5) for $n_\nu \ge n_2$,

$$M_{3,\nu} \le B_o + B'''(\beta')\exp(\lambda_n|\alpha_{\Delta_3}-\beta'|) \ .$$

Let $B'_o(\beta') = \operatorname{Max}(B_o,B'''(\beta')$. Then for $n_\nu \ge n_2$,

$$M_{3,\nu} \le B'_o(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) \ . \qquad (2.8)$$

Then using (2.7) and (2.8) in (2.6) we get, for $n_\nu \ge \max\{n_1, n_2\}$

$$\operatorname{Log} M_{2,\nu} \le b \operatorname{Log} B(\beta') + (1-b) \operatorname{Log} B'_o(\beta') + \{-b(1+\theta_\nu)(\alpha_{\Delta_1} - \beta') + (1-b)|\alpha_{\Delta_3} - \beta'|\}\lambda_{n_\nu}$$

Since $\lim_{\nu\to\infty} \theta_\nu = \infty$, we have $\lim -b(1+\theta_\nu)(\alpha_{\Delta_1} - \beta') + (1-b)|\alpha_{\Delta_3} - \beta'| = -\infty$, $\nu \uparrow \infty$ and hence $\lim_{\nu\to\infty} \operatorname{Log} M_{2,\nu} = -\infty$ which proves the theorem.

When the polynomial $P_n(s)$ reduces to a complex number $a_{n,o}$, we get the famous Ostrowaski's theorem [1] for Dirichlet series. Our theorem contains G.L. Lunt'z theorem [5] as a particular case when $P_n(s) = a_n s^{m_n}$.

COROLLARY. - In theorem 3 if we replace (2.3) by the condition that there exists a sequence (θ_n) of strictly positive numbers such that $\lim_{n\to\infty} \theta_n = \infty$ and $\exists_{n'} \ \forall_{n \ge n'} \ \lambda_{n+1} > (1+\theta_n)\lambda_n$, then each point of $(\mathrm{Fr}\,\mathcal{D}_{*o}) \cap \mathbb{C} - \mathcal{E}^*$ is a singular point for f defined by (2.2). In particular if $(\mathrm{Fr}\,\mathcal{D}_{*o}) \subset \mathbb{C} - \mathcal{E}^*$ then $\mathrm{Fr}\,\mathcal{D}_{*o}$ is a natural boundary for f .

PROOF. Let us suppose that the corollary is false. Then there exists a point $b \in (\mathrm{Fr}\,\mathcal{D}_{*o}) \cap \mathbb{C} - \mathcal{E}^*$ and a disc $d(b,\rho)$ centred at b and of radius $\rho > 0$ on which f is holomorphic. As a result of theorem 3 the sequence (S_n) converges on $d(b,\rho)$. From remark 2 $\{f\}$ diverges on $\mathbb{C} - \mathcal{E}^* - \overline{\mathcal{D}}_{*o}$. There exists necessarily points common to $\mathbb{C} - \mathcal{E}^* - \overline{\mathcal{D}}_{*o}$ and $d(b,\rho)$. For these points there is a contradiction which establishes the corollary.

ACKNOWLEDGENT. - The second author would like to thank French Government for financial support.

REFERENCES

1. BERNSTEIN V. - Leçons sur les progrès récents de la théorie des séries de Dirichlet, Gauthier-Villars, Paris, 1933.

2. BLAMBERT M. and SIMEON J. - Sur une technique d'étude des propriétés de convergence des séries de Dirichlet à exposants complexes, An. Fac. Ci. Univ. Porto, 56 (1973), 1-33.

3. GALLIE T.M. - Mandelbrojt's inequality and Dirichlet series with complex exponents, Trans. Amer. Mat. Soc. 90 (1959), 57-72.

4. GOLUSIN G.M. - Geometric theory of functions of a complex variable, Providence, American Mathematical Soc. 1969, Translations of Mathematical monographs 26.

5. LUNTZ G.L. - On the over convergence of certain series, Izv. Ann. Armjan, S.S.R. Ser Fiz. Mat. Nauk. 15 (5), (1962), 11-26.

Laboratoire de Mathématiques Pures - Institut Fourier
dépendant de l'Université Scientifique et Médicale de Grenoble
associé au C.N.R.S.
B.P. 116
38402 ST MARTIN D'HERES (France)