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THE GALOIS EXTENSIONS INDUCED BY IDEMPOTENTS IN A GALOIS ALGEBRA

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Let B be a Galois algebra with Galois group G, $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$, e_g the central idempotent such that $BJ_g = Be_g$, and $e_K = \sum_{g\in K, e_g\neq 1} e_g$ for a subgroup K of G. Then Be_K is a Galois extension with the Galois group $G(e_K)$ $(= \{g \in G \mid g(e_K) = e_K\})$ containing K and the normalizer $N(K)$ of K in G. An equivalence condition is also given for $G(e_K) = N(K)$, and Be_G is shown to be a direct sum of all Be_i generated by a minimal idempotent e_i. Moreover, a characterization for a Galois extension B is shown in terms of the Galois extension Be_G and $B(1-e_G)$.

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1. Introduction. The Boolean algebra of idempotents for commutative Galois algebras plays an important role (see [1, 3, 6]). Let B be a Galois algebra with Galois group G and $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$. Then, in [2], it was shown that the ideal $BJ_g = Be_g$ for some central idempotent e_g. By using the Boolean algebra of central idempotents $\{e_g\}$ in the Galois algebra B, the following structure theorem of B was shown. There exist some subgroups H_i of G and minimal idempotents of $\{e_i \mid i = 1,2,\dots,m$ for some integer $m\}$ such that $B = \oplus\sum_{i=1}^m Be_i \oplus B(1-\sum_{i=1}^m e_i)$ where Be_i is a central Galois algebra with Galois group H_i for each $i = 1,2,\dots,m$, and $B(1-\sum_{i=1}^m e_i)$ is $C(1-\sum_{i=1}^m e_i)$, a commutative Galois algebra with Galois group induced by and isomorphic with G in case $1 \neq \sum_{i=1}^m e_i$ where C is the center of B. Let $(B_a;\dotplus,\cdot)$ be the Boolean algebra generated by $\{0, e_g \mid g \in G\}$ where $e\cdot e' = ee'$ and $e \dotplus e' = e+e'-ee'$ for any e and e' in B_a. In the present paper, we study the Galois extension Be_K where $e_K = \sum_{g\in K, e_g\neq 1} e_g \in B_a$ for a subgroup K of G. Let $G(e) = \{g \in G \mid g(e) = e\}$ for a central idempotent e. Then it will be shown that $K \subset N(K) \subset G(e_K)$ and Be_K is a Galois extension with Galois group $G(e_K)$ where $N(K)$ is the normalizer of K in G. A necessary and sufficient condition for $G(e_K) = N(K)$ is also given so that Be_K is a Galois extension of $(Be_K)^K$ with Galois group K, and $(Be_K)^K$ is a Galois extension of $(Be_K)^{G(e_K)}$ with Galois group $G(e_K)/K$. Let $S(K) = \{H \mid H$ is a subgroup of G and $e_H = e_K\}$. Then the map $S(K) \to e_K$ from $\{S(K) \mid K$ is a subgroup of $G\}$ to B_a is one-to-one. In particular, when $K = G$, we derive an expression for B, $B = Be_G \oplus B(1-e_G)$ such that $Be_G = \oplus\sum_{i=1}^m Be_i$, a direct sum of central Galois algebras with Galois subgroup H_i, and $B(1-e_G) = B(1-\sum_{i=1}^m e_i) = C(1-e_G)$ which is a commutative Galois algebra with Galois group induced by and isomorphic with G. Moreover, a characterization for a Galois extension B is shown in terms of the Galois extension Be_G and $B(1-e_G)$.

2. Definitions and notation. Let B be a ring with 1, C the center of B, G an automorphism group of B of order n for some integer n, and B^G the set of elements in B fixed under each element in G. We call B a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1,2,\ldots,m\}$ for some integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. We call B a Galois algebra over B^G if B is a Galois extension of B^G which is contained in C, and B a central Galois extension if B is a Galois extension of C. Throughout this paper, we will assume that B is a Galois algebra with Galois group G. Let $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$. In [2], it was shown that $BJ_g = Be_g$ for some central idempotent e_g of B. We denote by $(B_a; \dotplus, \cdot)$ the Boolean algebra generated by $\{0, e_g \mid g \in G\}$ where $e \cdot e' = ee'$ and $e \dotplus e' = e + e' - ee'$ for any e and e' in B_a. Throughout, $e + e'$ for $e, e' \in B_a$ means the sum in the Boolean algebra $(B_a; \dotplus, \cdot)$ and a monomial e in B_a is $\Pi_{g \in S} e_g \neq 0$ for some $S \subset G$.

3. Galois extensions generated by idempotents. Let K be a subgroup of G. The idempotent $\sum_{g \in K, e_g \neq 1} e_g \in B_a$ is called the group idempotent of K denoted by e_K. Let $G(e) = \{g \in G \mid g(e) = e\}$ for $e \in B_a$. Then we will show that $K \subset G(e_K)$ and e_K generates a Galois extension Be_K with Galois group $G(e_K)$. A necessary and sufficient condition for $G(e_K) = N(K)$ is also given where $N(K)$ is the normalizer of K in G. Thus some consequences for the Galois extension Be_K can be derived when K is a normal subgroup of G or $K = G$.

Lemma 3.1. *For any $g, h \in G$,*
(1) $g(e_h) = e_{ghg^{-1}}$.
(2) $e_h = 1$ *if and only if* $e_{ghg^{-1}} = 1$.

Proof. (1) It is easy to check that $g(J_h) = J_{ghg^{-1}}$, so $Bg(e_h) = g(Be_h) = g(BJ_h) = Bg(J_h) = BJ_{ghg^{-1}} = Be_{ghg^{-1}}$. Thus $g(e_h) = e_{ghg^{-1}}$.
(2) It is clear by (1). □

Theorem 3.2. *Let K be a subgroup of G, $e_K = \sum_{g \in K, e_g \neq 1} e_g$, and $G(e_K) = \{g \in G \mid g(e_K) = e_K\}$. Then*
(1) *K is a subgroup of $G(e_K)$ and*
(2) *$B = Be_K \oplus B(1 - e_K)$ such that Be_K and $B(1 - e_K)$ are Galois extensions with Galois group induced by and isomorphic with $G(e_K)$.*

Proof. (1) For any $g \in K$, by Lemma 3.1,

$$\begin{aligned} g(e_K) &= g\left(\sum_{\substack{k \in K \\ e_k \neq 1}} e_k\right) = \sum_{\substack{k \in K \\ e_k \neq 1}} g(e_k) \\ &= \sum_{\substack{k \in K \\ e_k \neq 1}} e_{gkg^{-1}} = \sum_{\substack{gkg^{-1} \in gKg^{-1} \\ e_{gkg^{-1}} \neq 1}} e_{gkg^{-1}} = e_{gKg^{-1}}. \end{aligned} \tag{3.1}$$

Since $g \in K$, $gKg^{-1} = K$. Hence $g(e_K) = e_K$, and so $g \in G(e_K)$.

(2) We first claim that for any $e \neq 0$ in B_a, Be is a Galois extension with Galois group induced by and isomorphic with $G(e)$. In fact, since B is a Galois extension with Galois group G, there exists a G-Galois system for B $\{a_i, b_i$ in B, $i = 1,2,\ldots,m\}$ for some

integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Hence $\sum_{i=1}^{m}(a_i e)g(b_i e) = e\delta_{1,g}$ for each $g \in G(e)$. Therefore, $\{a_i e, b_i e$ in Be, $i = 1,2,\ldots,m\}$ is a $G(e)$-Galois system for Be, and $e = \sum_{i=1}^{m}(a_i e)(b_i e - g(b_i e))$ for each $g \neq 1$ in $G(e)$. But $e \neq 0$, so $g|_{Be} \neq 1$ whenever $g \neq 1$ in $G(e)$. Thus, Be is a Galois extension with Galois group induced by and isomorphic with $G(e)$. Statement (2) is a particular case when $e = e_K$ and $e = 1 - e_K$, respectively. □

The proof of Theorem 3.2(2) suggests an equivalence condition for a Galois extension B.

Theorem 3.3. *The extension B is a Galois extension with Galois group $G(e)$ for a central idempotent e of B if and only if $B = Be \oplus B(1-e)$ such that Be and $B(1-e)$ are Galois extensions with Galois group induced by and isomorphic with $G(e)$. In particular, B is a Galois algebra with Galois group $G(e)$ for a central idempotent e of B, if and only if $B = Be \oplus B(1-e)$ such that Be and $B(1-e)$ are Galois algebras with Galois group induced by and isomorphic with $G(e)$.*

Proof. (⇒) Since B is a Galois extension with Galois group $G(e)$, $B = Be \oplus B(1-e)$ such that Be and $B(1-e)$ are Galois extensions with Galois group induced by and isomorphic with $G(e)$ by the proof of Theorem 3.2(2).

(⇐) Let $\{a_j^{(1)}; b_j^{(1)} \in Be \mid j = 1,2,\ldots,n_1\}$ be a $G(e)$-Galois system for Be and let $\{a_j^{(2)}; b_j^{(2)} \in B(1-e) \mid j = 1,2,\ldots,n_2\}$ be a $G(e)$-Galois system for $B(1-e)$. Then we claim that $\{a_j^{(i)}; b_j^{(i)} \mid j = 1,2,\ldots,n_i,\ i = 1,2\}$ is a $G(e)$-Galois system for B. In fact, $\sum_{i=1}^{2}\sum_{j=1}^{n_i} a_j^{(i)} b_j^{(i)} = e + (1-e) = 1$. Moreover, for each $g \neq 1$ in $G(e)$—noting that $g \neq 1$ in $G(e)$ if and only if $g|_{Be} \neq 1$ and $g|_{B(1-e)} \neq 1$ by hypothesis—we have that $\sum_{j=1}^{n_i} a_j^{(i)} g(b_j^{(i)}) = 0$, $i = 1,2$, so $\sum_{i=1}^{2}\sum_{j=1}^{n_i} a_j^{(i)} g(b_j^{(i)}) = 0$. Therefore $\{a_j^{(i)}; b_j^{(i)} \mid j = 1,2,\ldots,n_i,\ i = 1,2\}$ is a $G(e)$-Galois system for B, and so B is a Galois extension with Galois group $G(e)$.

Next, it is clear that $B^{G(e)} \subset C$ if and only if $(Be)^{G(e)} \subset Ce$ and $(B(1-e))^{G(e)} \subset C(1-e)$, so by the above argument, B is a Galois algebra with Galois group $G(e)$ for a central idempotent e of B if and only if $B = Be \oplus B(1-e)$ such that Be and $B(1-e)$ are Galois algebras with Galois group induced by and isomorphic with $G(e)$. □

Corollary 3.4. *An algebra B is a Galois algebra with Galois group G if and only if $B = Be_G \oplus B(1-e_G)$ such that Be_G and $B(1-e_G)$ are Galois algebras with Galois group induced by and isomorphic with G.*

Proof. By Theorem 3.2(1), $G(e_G) = G$, so the corollary is immediate by Theorem 3.3. □

Now let $S(K) = \{H \mid H$ is a subgroup of G and $e_H = e_K\}$ and $\alpha : S(K) \to e_K$. It is easy to see that α is a bijection from $\{S(K) \mid K$ is a subgroup of $G\}$ to the set of group idempotents in B_a.

We are interested in an equivalence condition for K such that $G(e_K) = N(K)$. We need the following lemma.

Lemma 3.5. *Let K be a subgroup of G, then for a $g \in G$, $g \in G(e_K)$ if and only if $gKg^{-1} \in S(K)$.*

Proof. Suppose $g \in G(e_K)$, then

$$
\begin{aligned}
e_K = g(e_K) = g\left(\sum_{\substack{k\in K\\ e_k\neq 1}} e_k\right) &= \sum_{\substack{k\in K\\ e_k\neq 1}} g(e_k) \\
&= \sum_{\substack{k\in K\\ e_k\neq 1}} e_{gkg^{-1}} = \sum_{\substack{gkg^{-1}\in gKg^{-1}\\ e_gkg^{-1}\neq 1}} e_{gkg^{-1}} = e_{gKg^{-1}}.
\end{aligned}
\tag{3.2}
$$

Thus $gKg^{-1} \in S(K)$. On the other hand, suppose $gKg^{-1} \in S(K)$. Then

$$
\begin{aligned}
g(e_K) = g\left(\sum_{\substack{k\in K\\ e_k\neq 1}} e_k\right) &= \sum_{\substack{k\in K\\ e_k\neq 1}} g(e_k) \\
&= \sum_{\substack{k\in K\\ e_k\neq 1}} e_{gkg^{-1}} = \sum_{\substack{gkg^{-1}\in gKg^{-1}\\ e_gkg^{-1}\neq 1}} e_{gkg^{-1}} = e_{gKg^{-1}} = e_K.
\end{aligned}
\tag{3.3}
$$

Thus $g \in G(e_K)$. □

Theorem 3.6. *$G(e_K) = N(K)$ if and only if $S(K)$ contains exactly one conjugate of the subgroup K.*

Proof. ($\Rightarrow$) For any $g \in G$ such that $gKg^{-1} \in S(K)$, $g \in G(e_K)$ by Lemma 3.5. But $G(e_K) = N(K)$ by hypothesis, so $g \in N(K)$. Hence $gKg^{-1} = K$. Thus $S(K)$ contains exactly one conjugate of the subgroup K.

($\Leftarrow$) For any $g \in N(K)$, $gKg^{-1} = K$, so $gKg^{-1} \in S(K)$. Hence $g \in G(e_K)$ by Lemma 3.5. Thus $N(K) \subset G(e_K)$. Conversely, for each $g \in G(e_K)$, $gKg^{-1} \in S(K)$ by Lemma 3.5, so $gKg^{-1} = K$ by hypothesis. Thus $g \in N(K)$. This implies that $G(e_K) = N(K)$. □

Corollary 3.7. *Assume that the order of G is a unit in B. If $S(K)$ contains exactly one conjugate of the subgroup K, then Be_K is a Galois extension of $(Be_K)^K$ with Galois group K and $(Be_K)^K$ is a Galois extension of $(Be_K)^{G(e_K)}$ with Galois group $G(e_K)/K$.*

Proof. By Theorem 3.2(2), Be_K is a Galois extension with Galois group $G(e_K)$. Hence Be_K is a Galois extension of $(Be_K)^K$ with Galois group K for K is a subgroup of $G(e_K)$ by Theorem 3.2(1). Moreover, by hypothesis, the order of G is a unit in B, so the order of K is a unit in Be_K. Since $S(K)$ contains exactly one conjugate of the subgroup K, K is a normal subgroup of $G(e_K)$ by Theorem 3.6. Thus $(Be_K)^K$ is a Galois extension of $(Be_K)^{G(e_K)}$ with Galois group $G(e_K)/K$. □

Next are some consequences for an abelian group G or $K = G$.

Corollary 3.8. *If B is an abelian extension with Galois group G (i.e., G is abelian) of an order invertible in B, then for any subgroup K of G, Be_K is a Galois extension of $(Be_K)^K$ with Galois group K and $(Be_K)^K$ is a Galois extension of $(Be_K)^{G(e_K)}$ with Galois group $G(e_K)/K$.*

When $K = G$, we derive an expression for B by using the set $\{e_i \mid i = 1,2,\dots,m\}$ of minimal idempotents in B_a. This gives detail descriptions of the components Be_G and $B(1-e_G)$ as given in Corollary 3.4.

Theorem 3.9. *Let B be a Galois algebra with Galois group G. Then $B = Be_G \oplus B(1-e_G)$ such that $Be_G = \oplus\sum_{i=1}^{m} Be_i$ where each Be_i is a central Galois algebra with Galois group H_i for some subgroup H_i of G and $B(1-e_G) = C(1-e_G)$ which is a commutative Galois algebra with Galois group induced by and isomorphic with G in case $e_G \neq 1$ where $\{e_i \mid i = 1,2,\dots,m\}$ are given in* [5, Theorem 3.8].

Proof. Since $e_i = \Pi_{h\in H_i} e_h$ where H_i is the maximal subset (subgroup) of G such that $\Pi_{h\in H_i} e_h \neq \{0\}$ or $e_i = (1-\sum_{j=1}^{t} e_j)\Pi_{h\in H_i} e_h$ where H_i is the maximal subset (subgroup) of G for some $t < i$ such that $(1-\sum_{j=1}^{t} e_j)\Pi_{h\in H_i} e_h \neq \{0\}$ (see [5, Theorem 3.8]), we have that $e_i(\sum_{g\in G, e_g\neq 1} e_g) = e_i$ for each i. Thus $\sum_{i=1}^{m} e_i \leq \sum_{g\in G, e_g\neq 1} e_g$. Noting that $e_g(1-\sum_{i=1}^{m} e_i) = 0$ for each $g \neq 1$ in G (see [5, Theorem 3.8]), we have that $(\sum_{g\in G, e_g\neq 1} e_g)(1-\sum_{i=1}^{m} e_i) = 0$, that is, $(\sum_{g\in G, e_g\neq 1} e_g)(\sum_{i=1}^{m} e_i) = \sum_{g\in G, e_g\neq 1} e_g$. Hence $\sum_{g\in G, e_g\neq 1} e_g \leq \sum_{i=1}^{m} e_i$. Thus $\sum_{g\in G, e_g\neq 1} e_g = \sum_{i=1}^{m} e_i$, that is, $e_G = \sum_{i=1}^{m} e_i$. But then by [5, Theorem 3.8], $B = \oplus\sum_{i=1}^{m} Be_i \oplus B(1-\sum_{i=1}^{m} e_i) = Be_G \oplus B(1-e_G)$ such that $B(1-e_G) = C(1-e_G)$ which is a commutative Galois algebra with Galois group induced by and isomorphic with G, and $Be_G = \oplus\sum_{i=1}^{m} Be_i$ such that each Be_i is a central Galois algebra with Galois group H_i for some subgroup H_i of G where $\{e_i \mid i = 1,2,\dots,m\}$ are minimal idempotents of B_a. □

4. A relationship between idempotents.

In this section, we show a relationship between the set of idempotents $\{e_g \mid g \in G\}$ and the set of minimal elements in B_a, and give an equivalence condition for a monomial idempotent e_S $(= \sum_{g\in S, e_g\neq 1} e_g)$ where S is a subset of G, and a monomial e in B_a is $\Pi_{g\in S} e_g \neq 0$ for some $S \subset G$.

Theorem 4.1. *Let S be a subset of G. Then there exists a unique subset Z_S of the set $\{1,2,\dots,m\}$ such that $e_S = \sum_{i\in Z_S} e_i$.*

Proof. Since $C = \oplus\sum_{i=1}^{m} Ce_i \oplus Cf$ (see [5, Theorem 3.8]), $e_S = \sum_{i=1}^{m} c_i e_i + cf$ for some $c_i, c \in C$. It can be checked that e_i are minimal elements of B_a, so $e_S e_i = e_i$ or $e_S e_i = 0$. Let $Z_S = \{i \mid e_S e_i = e_i\}$. Then for each $i \in Z_S$, $e_i = e_S e_i = c_i e_i$, and for each $i \notin Z_S$, $0 = e_S e_i = c_i e_i$. Hence $e_S = \sum_{i\in Z_S} e_i + cf$. Moreover, since $e_g f = 0$ for each $g \neq 1$ in G (see [5, Theorem 3.8]), we have that $0 = e_S f = (\sum_{i\in Z_S} e_i + cf)f = cf$. Hence $e_S = \sum_{i\in Z_S} e_i$. The uniqueness of Z_S is clear. □

Next is a description of the components Be_K and $B(1-e_K)$ for a subgroup K of G as given in Theorem 3.2.

Corollary 4.2. *For any subgroup K of G, $B = Be_K \oplus B(1-e_K)$ such that $Be_K = \sum_{i\in Z_K} Be_i$ and $B(1-e_K) = B(1-\sum_{i\in Z_K} e_i)$ which are Galois extensions with Galois group induced by and isomorphic with $G(e_K)$.*

Proof. It is an immediate consequence of Theorems 3.2(2) and 4.1. □

In [4], let K be a subgroup of G. Then K is called a nonzero subgroup of G if $\prod_{k\in K} e_k \neq 0$, and K is called a maximal nonzero subgroup of G if $K \subset K'$ where K' is a nonzero subgroup of G such that $\prod_{k\in K} e_k = \prod_{k\in K'} e_k$, then $K = K'$. It was shown that the set of monomials in B_a and the set of maximal nonzero subgroups of G are in a one-to-one correspondence (see [4, Theorem 3.2]). Also, any maximal nonzero subgroup $K = H_e = \{g \in G \mid e \le e_g\}$ where $e = \Pi_{k\in K} e_k$ and H_e is a normal subgroup of $G(e)$ (see [4, Lemma 3.3]). Next is a characterization of a monomial idempotent e_S $(= \sum_{g\in S, e_g \ne 1} e_g)$ for a subset of G.

Theorem 4.3. *Let S be a subset of G such that $e_S = \sum_{g\in S, e_g\ne 1} e_g \ne 0, 1$. Then e_S is a monomial if and only if $e_j \le e_S$ whenever $H_{e_S} \subset H_{e_j}$ for an atom e_j.*

Proof. $(\Rightarrow)$ By [4, Theorem 3.2], $e \to H_e$ is a one-to-one correspondence between the set of monomials in B_a and the set of maximal nonzero subgroups of G. Noting that $e = \Pi_{g\in H_e} e_g$ when e is a monomial, we have for any monomials e and e', $H_e \subset H_{e'}$ implies that $e \ge e'$. Thus, $e_j \le e_S$ whenever $H_{e_S} \subset H_{e_j}$ for an atom e_j because e_S is a monomial by hypothesis.

$(\Leftarrow)$ By Theorem 4.1, $e_S = \sum_{e_i\in Z_S} e_i$ where $Z_S = \{e_i \mid e_i \le e_S\}$. Let $e = \Pi_{g\in H_{e_S}} e_g$. Then $e_S \le e$ and $H_{e_S} = H_e$. Suppose $e_S \ne e$. Then $e_S = \sum_{e_i\in Z_S} e_i < e = \sum e_j$ where $\sum_{e_i\in Z_S} e_i$ is a direct summand of $\sum e_j$ by Theorem 4.1. It is easy to check that $H_{e_S} = \cap_{e_i\in Z_S} H_{e_i} = H_e = \cap H_{e_j}$. Therefore there exists some $e_j \notin Z_S$, that is, $e_j \not\le e_S$ such that $H_{e_S} \subset H_{e_j}$. This is a contradiction. Thus $e_S = e$, which is a monomial. □

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