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ON THE EQUATION $x^2 + 2^a \cdot 3^b = y^n$

FLORIAN LUCA

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We find all positive integer solutions (x, y, a, b, n) of $x^2 + 2^a \cdot 3^b = y^n$ with $n \geq 3$ and coprime x and y.

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1. Introduction. The Diophantine equation $x^2 + C = y^n$, where x and y are positive integers, $n \geq 3$ and C is a given integer, has received considerable interest. The earliest reference seems to be an assertion by Fermat that he had shown that when $C = 2$, $n = 3$, the only solution is given by $x = 5$, $y = 3$; a proof was published by Euler in 1770. The first result for general n is due to Lebesgue [9] who proved that there are no solutions for $C = 1$. Ljunggren [10] solved this equation for $C = 2$, Nagell [13, 14] solved it for $C = 3$, 4, and 5 and Chao [5] proved that it has no solutions for $C = -1$. For an extensive list of references one should consult Cohn's beautiful paper [6] in which he develops a method by which he finds all solutions of the above equation for 77 of the values of $C \leq 100$. This equation was later solved for two additional values of $C \leq 100$ (namely, $C = 74$ and $C = 86$) by Mignotte and de Weger [12]. It is interesting to mention that the equation $x^2 + 7 = y^n$ is still unsolved.

In recent years, a different form of the above equation has been considered, namely, when C is no longer a fixed integer but a power of a fixed prime. Le [8] investigated the equation $x^2 + 2^m = y^n$. Arif and Muriefah solved the equation $x^2 + 3^m = y^n$ when m is odd (see [2]). They also gave partial results in the case when m is even (see [1]) but the general solution in the case m is even was found by Luca in [11].

For any nonzero integer k, let $P(k)$ be the largest prime dividing k. Let C_1 be any fixed positive constant. It follows, from the work of Bugeaud [4] and Turk [15], that if

$$x^2 + z = y^n \quad \text{with } (x, y) = 1,\ P(z) < C_1, \tag{1.1}$$

then $\max(|x|, |y|, n)$ is bounded by a constant computable in terms of C_1 alone.

In this paper, we find all solutions of (1.1) when $C_1 = 5$ and $z > 0$. More precisely, we find all solutions of the equation

$$x^2 + 2^a \cdot 3^b = y^n \quad \text{with } a, b \geq 0,\ n \geq 3,\ (x, y) = 1. \tag{1.2}$$

The proof uses the new result on the existence of primitive divisors of the Lucas numbers due to Bilu et al. [3] as well as a computational result of de Weger [7].

2. The result

Theorem 2.1. *All positive solutions of the equation*

$$x^2+2^a\cdot 3^b=y^n \quad \text{with } a,b\geq 0,\ n\geq 3,\ (x,y)=1 \tag{2.1}$$

have $n=4$ or $n=3$. For $n=4$, the solutions are

$$(x,y)=(7,3),(23,5),(7,5),(47,7),(287,17). \tag{2.2}$$

For $n=3$, the solutions are

$$\begin{aligned}(x,y)=&(5,3),(11,5),(10,7),(17,7),(46,13),(35,13),\\&(595,73),(955,97),(2681,193),(39151,1153).\end{aligned} \tag{2.3}$$

In the statement of the theorem we have listed only the values of x, y, and n as the values of the parameters a and b that can be determined from the prime factor decomposition of x^2-y^n once x, y, and n are given.

From Lebesgue's result, we know that the equation $x^2+1=y^n$ has no positive solutions for $n\geq 3$ and from the work of Arif, Muriefah, and Luca, we know that the only positive solutions of the equation $x^2+3^m=y^n$ with $(x,y)=1$ are $(x,y,m,n)=(10,7,5,3)$ and $(46,13,4,3)$. From now on, we assume that $a>0$. In particular, both x and y are odd.

3. The case $n\neq 3$ or 4. In this section, we show that it suffices to assume that $n\in\{3,4\}$. Indeed, assume that $n\neq 4$. We may certainly assume that n is an odd prime. If $n\neq 3$, it follows that $n\geq 5$. Write $2^a\cdot 3^b=dz^2$ where $d\in\{1,2,3,6\}$. Equation (2.1) can be written as

$$\left(x+i\sqrt{d}z\right)\left(x-i\sqrt{d}z\right)=y^n. \tag{3.1}$$

Since x is odd and dz^2 is even, it follows that the two ideals $[(x+i\sqrt{d}z)]$ and $[(x-i\sqrt{d}z)]$ are coprime in the ring of integers of $\mathbf{Q}(i\sqrt{d})$. Since the class number of $\mathbf{Q}(i\sqrt{d})$ is 1 or 2 and $n\geq 5$ is prime, it follows that there exists an integer u and a root of unity ε in $\mathbf{Q}(i\sqrt{d})$ such that

$$x+i\sqrt{d}z=\varepsilon u^n, \qquad x-i\sqrt{d}z=\overline{\varepsilon u}^n. \tag{3.2}$$

Since ε is a root of unity belonging to a quadratic extension of $\mathbf{Q}$, it follows that $\varepsilon^k=1$ for some $k\in\{1,2,3,4,6\}$. Since $n\geq 5$ is prime, it follows that up to a substitution one may assume that $\varepsilon=1$ in system (3.2). From (3.2) with $\varepsilon=1$, it follows that

$$2i\sqrt{d}z=u^n-\bar{u}^n. \tag{3.3}$$

Since certainly

$$\frac{u^n-\bar{u}^n}{u-\bar{u}}\in\mathbf{Z}, \tag{3.4}$$

we have that

$$P\left(\frac{u^n-\bar{u}^n}{u-\bar{u}}\right)<5. \tag{3.5}$$

From (3.5), we find that the Lucas number given by formula (3.4) has no primitive divisor. From [3], it follows that there are at most 10 pairs (u,n) satisfying inequality (3.5) and all of them appear in [3, Table 1]. A quick investigation reveals that none of the u's from [3, Table 1] belongs to $\mathbf{Q}(i\sqrt{d})$ for some $d\in\{1,2,3,6\}$, which is the desired contradiction.

4. The case $n=4$. Let $S=\{k\mid P(k)<5\}$. Then, we have the following preliminary result.

Lemma 4.1. *All solutions of the equation*

$$x^2=k\pm l\quad \text{with } k,l>0,\ k,l\in S,\ (k,l)=1 \tag{4.1}$$

are

$$\begin{aligned}(x,k,l)=&(1,2,1),(2,3,1),(3,8,1),(5,24,1),(7,48,1),\\ &(17,288,1),(1,4,3),(1,9,8),(5,16,9),(5,27,2),(7,81,32).\end{aligned} \tag{4.2}$$

Proof of Lemma 4.1. This lemma is a particular case of a result of de Weger [7, Chapter 7]. □

The proof of the theorem for $n=4$. Rewrite (2.1) as

$$(y^2-x)(y^2+x)=2^a\cdot 3^b. \tag{4.3}$$

Since $a>0$ and $(x,y)=1$, it follows that $(y^2-x,y^2+x)=2$. Thus,

$$y^2-x=k,\quad y^2+x=l,\quad \text{with } k,l>0,\ k,l\in S,\ (k,l)=2. \tag{4.4}$$

Hence,

$$y^2=\frac{k}{2}+\frac{l}{2}, \tag{4.5}$$

where $k/2$, $l/2\in S$ are positive and coprime. By Lemma 4.1, we obtain that (4.5) has only 6 solutions. Five of them lead to solutions (2.2) of (2.1). One of the solutions of (4.5) leads to

$$2^2+2^2\cdot 3=2^4, \tag{4.6}$$

which is not a convenient solution of (2.1) because $x=2$ and $y=2$ are not coprime.

The case $n=4$ is therefore settled.

5. The case $n=3$. We begin with another lemma.

Lemma 5.1. *The only solutions of the equation*

$$3x^2=k\pm l\quad \text{with } k,l>0,\ k,l\in S,\ (k,l)\in\{1,3\} \tag{5.1}$$

are

$$\begin{aligned}(x,k,l) = &(1,2,1),(1,4,1),(1,6,3),(2,9,3),\\ &(3,24,3),(5,72,3),(7,144,3),(17,864,3),\\ &(1,12,9),(1,27,24),(5,48,27),(5,81,6),(7,243,96).\end{aligned} \tag{5.2}$$

Proof of Lemma 5.1. This lemma too is a particular instance of the more general computation of de Weger [7, Chapter 7]. □

The proof of the theorem for $n = 3$. Write again $2^a \cdot 3^b = dz^2$ where $d \in \{1,2,3,6\}$. From arguments employed in Section 3, we know that there exist u and ε in $\mathbf{Q}(i\sqrt{d})$ such that $y = |u|^2$, ε is a root of unity and

$$x + i\sqrt{d}z = \varepsilon u^3, \qquad x - i\sqrt{d}z = \overline{\varepsilon u}^3. \tag{5.3}$$

Clearly,

$$2i\sqrt{d}z = \varepsilon u^3 - \overline{\varepsilon u}^3. \tag{5.4}$$

We distinguish two cases.

Case 1 ($\varepsilon = 1$)**.** Equation (5.4) reads

$$2i\sqrt{d}z = u^3 - \bar{u}^3. \tag{5.5}$$

Assume first that $u = a + ib\sqrt{d}$ with a and b integers. Then, we get

$$2i\sqrt{d}z = (a + ib\sqrt{d})^3 - (a - ib\sqrt{d})^3 \tag{5.6}$$

or

$$2i\sqrt{d}z = 2i\sqrt{d}b(3a^2 - db^2). \tag{5.7}$$

Hence, $b \mid z$ and

$$3a^2 = db^2 \pm \frac{z}{b}. \tag{5.8}$$

Let $k = db^2$ and $l = z/b$. Notice that $k,l \in S$. Moreover, notice that $(k,l) \in \{1,3\}$. Indeed, if $(k,l) \notin \{1,3\}$, it follows that there exists a prime p such that $p \mid (k,l,a)$. In particular, $p \mid db^2$ and $p \mid a$, therefore $p \mid a^2 + db^2 = y$. Since $p \mid z$ and $2^a \cdot 3^b = dz^2$, we come to $p \mid 2^a \cdot 3^b$. It follows now that $p \mid (y^3 - 2^a \cdot 3^b) = x^2$ and therefore $p \mid x$. This contradicts the fact that x and y are coprime. Now all solutions of (5.8) are given by Lemma 5.1. For example, the solution

$$3 \cdot 1^2 = 2^1 + 1 \tag{5.9}$$

gives either $a = 1$, $d = 2$, $b = 1$, and $z = 1$ or $a = 1$, $d = 1$, $b = 1$, and $z = 2$. The first possibility yields $y = a^2 + db^2 = 1 + 2 = 3$ and $dz^2 = 2$, which leads to the solution $3^3 = 2 + 5^2$ of (2.1). The second possibility gives $y = a^2 + db^2 = 2$ and $dz^2 = 4$, which leads to the solution $2^3 = 2^2 + 2^2$ of (2.1). This is not an acceptable solution, since $x = 2$ and $y = 2$ are not coprime.

All the solutions of (2.1) for the case $n=3$ except for $(x,y)=(10,7)$ are obtained in this way by identifying a, b, d, and z from (5.8) via Lemma 5.1.

When $d=3$, we also need to investigate the case in which

$$u=\frac{a+i\sqrt{3}b}{2} \tag{5.10}$$

for some odd integers a and b. From (5.5), we simply get that

$$16i\sqrt{3}z=\left(a+i\sqrt{3}b\right)^3-\left(a-i\sqrt{3}b\right)^3 \tag{5.11}$$

or

$$16i\sqrt{3}z=2i\sqrt{3}b\left(3a^2-3b^2\right). \tag{5.12}$$

It follows that b divides z and

$$3a^2=3b^2\pm\frac{8z}{b}. \tag{5.13}$$

From Lemma 5.1, we derive that (5.13) has only two convenient solutions, namely, $3\cdot 1^2=3\cdot 3^2-8\cdot 3$ and $3\cdot 7^2=3\cdot 3^4-8\cdot 12$. These lead to the solutions $(x,y)=(10,7)$ and $(595,73)$ of (2.1).

Case 2 ($\varepsilon\neq 1$)**.** It is easy to see that the only case in which one may not be able to set $\varepsilon=1$ in system (5.4) is when $d=3$. In this case, one may assume that $\varepsilon=(1+i\sqrt{3})/2$ and that $u=(a+i\sqrt{3}b)/2$ for some integers a and b such that $a\equiv b \pmod 2$. Then (5.4) becomes

$$2i\sqrt{3}z=\left(\frac{1+i\sqrt{3}}{2}\right)\cdot\left(\frac{a+i\sqrt{3}b}{2}\right)^3-\left(\frac{1-i\sqrt{3}}{2}\right)\cdot\left(\frac{a-i\sqrt{3}b}{2}\right)^3. \tag{5.14}$$

This equation is equivalent to

$$16z=a^3+3a^2b-9ab^2-3b^3. \tag{5.15}$$

Assume first that both a and b are odd. Then, from (5.15), it follows that

$$16z=\left(a^3-ab^2\right)+\left(3a^2b-3b^3\right)-8ab^2=\left(a^2-b^2\right)(a+3b)-8ab^2. \tag{5.16}$$

Since a and b are both odd, we obtain that $16\mid (a^2-b^2)(a+3b)$. Equation (5.16) forces $16\mid 8ab^2$, which is impossible.

Assume now that both a and b are even. Since $y=(a/2)^2+3(b/2)^2$ is odd, it follows that exactly one of the numbers $a/2$ and $b/2$ is even. Equation (5.15) now implies that

$$2z=\left(\frac{a}{2}\right)^3+3\left(\frac{a}{2}\right)^2\left(\frac{b}{2}\right)-9\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)^2-3\left(\frac{b}{2}\right)^3. \tag{5.17}$$

However, (5.17) is now impossible, because precisely one of the numbers $a/2$ and $b/2$ is even and the other one is odd. Hence, this case can never occur.

The theorem is therefore completely proved. □

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Florian Luca: Mathematical Institute, UNAM, Ap. Postal 61-3 (Xangari), CP 58 089 Morelia, Michoacán, Mexico

E-mail address: fuca@matmor.unam.mx