ON MOMENT INEQUALITIES AND STOCHASTIC ORDERING FOR WEIGHTED RELIABILITY MEASURES

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ABSTRACT. We obtain stochastic inequalities, error bounds, and classification probability for a general class of distributions. We introduce the notion of variability ordering via the probability functional and comparisons made for the weighted and the original distributions. We present moment inequalities, comparisons, and applications.

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1. Introduction. Weighted distributions are of tremendous practical importance in various aspects of reliability, biometry, survival analysis and renewal theory to mention a few areas. In renewal theory the residual lifetime has a limiting distribution that is a weighted distribution with the weight function equal to the reciprocal of the hazard (failure) rate function. When observations are selected with probability proportional to their "length" the resulting distribution is referred to as a length-biased distribution. Length-biased distributions occur naturally in a wide variety of settings and are discussed by several authors including but not limited to Gupta and Akman [4], Zelen and Feinleib [6]. The problem of providing error bounds for exponential approximations to classes of life distributions in particular, the class of weighted distributions is addressed in this paper. Keilson [5] suggested a measure of departure from exponentiality within the class of completely monotone distributions (mixture of exponential distributions). These measures of departure are given in terms of $\rho = |\mu_2/2\mu^2 - 1|$, where $\mu_2 = E(X^2)$ and $\mu = E(X)$. This is due to the fact that the exponential distribution satisfies $\rho = 0$. Brown [2] obtained bounds for the class of increasing mean residual life (IMRL) functions.

The main objective of this paper is to obtain inequalities for weighted reliability measures, partial order via probability functional and compare reliability measures for weighted and in particular length-biased distributions. This paper is organized as follows. Section 2 contains some basic definitions, utility notions and comparisons. In Section 3 we present some moment inequalities for weighted reliability measures. In Section 4 some partial ordering via the probability functional are presented. The results are used to compare experiments for weighted distributions. Section 5 is concerned with comparisons for weighted and related distributions. The results are applied to length-biased mixtures of distributions.

2. Some definitions, utility notions, and comparisons. In this section, we present some definitions and useful notions. Let $\mathcal{F}$ be the set of absolutely continuous

distribution function satisfying

$$H(0) = 0, \qquad \lim_{x \to \infty} H(x) = 1, \qquad \sup\{x : H(x) < 1\} = \infty. \qquad (2.1)$$

Note that if the mean of a random variable in $\mathcal{F}$ is finite, it is positive.

Let X be a nonnegative random variable with reliability function $\bar{F}$ and probability density function (pdf) f, where $\bar{F}(x) = \int_x^\infty f(t)dt$. The weighted random variable has a pdf given by

$$f_W(x) = \frac{W(x)f(x)}{\delta^*}, \qquad (2.2)$$

where δ^* is a normalizing constant.

DEFINITION 2.1. A distribution function F is said to have increasing mean residual life (IMRL) on $(0, \infty)$ if $\mu = \int_0^\infty \bar{F}(x)dx < \infty$, $F(0) < 1$ and $E(X - x \mid X > x)$ is increasing in $x \geq 0$.

DEFINITION 2.2. A distribution function F is said to have increasing (decreasing) hazard rate on $[0, \infty)$, denoted by IHR (DHR), if $F(0-) = 0$, $F(0) < 1$ and $P(X > x + t \mid X > t) = \bar{F}(x + t)/\bar{F}(t)$ is decreasing (increasing) in $t \geq 0$ for each $x > 0$.

Note that if F has DHR and $\mu_X = \int_0^\infty \bar{F}(x)dx < \infty$, then F has IMRL.

DEFINITION 2.3. Let X and Y be two nonnegative random variables with probability density functions f and g, respectively. The random variable X is said to be larger than Y in monotone likelihood ratio ordering ($X \geq_{\mathrm{lr}} Y$) if $f(x)/g(x)$ is nondecreasing in $x \geq 0$.

The following definition is due to Ebrahimi and Pellerey [3].

DEFINITION 2.4. The uncertainty of residual life distribution $H(X;t)$, of a component at time t, is the entropy of the residual life random variable $(X - t \mid X > t)$, and is given by

$$
\begin{aligned}
H(X;t) &= -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \\
&= \log \bar{F}(t) - \{\bar{F}(t)\}^{-1} \int_t^\infty f(x) \log f(x) dx \qquad (2.3) \\
&= 1 - \{\bar{F}(t)\}^{-1} \int_t^\infty f(x) \log \{\lambda_F(x)\} dx,
\end{aligned}
$$

where $\lambda_F(x) = f(x)/\bar{F}(x)$.

The entropy of the weighted residual life random variable $(X_W - t \mid X_W > t)$ is given by

$$
\begin{aligned}
H(X_W;t) &= -\int_t^\infty \frac{f_W(x)}{\bar{F}_W(t)} \log \frac{f_W(x)}{\bar{F}_W(t)} dx \\
&= \log \bar{F}_W(t) - \{\bar{F}_W(t)\}^{-1} \int_t^\infty f_W(x) \log f_W(x) dx.
\end{aligned}
\qquad (2.4)
$$

Under length-biased sampling, $H(X_W;t)$ reduces to

$$H(X_l;t) = 1 - \{\bar{F}(t)V_F(t)\}^{-1} \int_t^\infty x f(x) \log\left(\frac{x\lambda_F(x)}{V_F(x)}\right) dx, \qquad (2.5)$$

where $V_F(t) = E(X - t \mid X > t) + t$. $H(X;t)$ is the expected uncertainty in the conditional distribution of $X - t$, given $X > t$ about the predictability of the remaining lifetime of the component that has survived for time t.

Let X_t denote the residual lifetime of a unit functioning at time t. Then as $t \to \infty$, X_t has the limiting probability density function

$$f_e(x) = \frac{\bar{F}(x)}{\mu_F}, \quad x \geq 0. \qquad (2.6)$$

The length-biased equilibrium survival function is given by

$$\bar{F}_{l_e}(x) = (\mu_F^2 + \sigma_F^2)^{-1} \int_x^\infty \bar{F}(u)\{u + \delta_F(u)\} du, \qquad (2.7)$$

where $\delta_F(u) = \int_x^\infty \bar{F}(t)dt / \bar{F}(u)$, $u \geq 0$, and σ_F^2 is the variance of F. Note that if the weight function $W(x)$ is increasing on $[0, \infty)$, then $Y \geq_{\mathrm{lr}} X$ and $\lambda_G(x) \leq \lambda_F(x)$ for all $x \geq 0$, so that $\bar{G}(x) \geq \bar{F}(x)$ for all $x \geq 0$.

PROPOSITION 2.5. *If $X \leq_{\mathrm{lr}} Y$, then $H(X;t) \leq H(Y;t)$, for all $t \geq 0$, where Y is the length-biased random variable with equilibrium distribution F_{l_e}.*

PROOF.   The length-biased equilibrium survival function is given by

$$\bar{F}_l(x) = \frac{\bar{F}(x)\delta_F(x)}{\mu_X}, \qquad (2.8)$$

where $\delta_F(x) = V_F(x) - x$, and $\mu_X = \int_0^\infty \bar{F}(x)dx$. Clearly, $\bar{F}_l(x) \geq \bar{F}(x)$, for all $x \geq 0$. This follows from the fact that $Y \geq_{\mathrm{lr}} X$, so that X has a DFR distribution. Consequently, $H(X;t) \leq H(Y;t)$, for all $t \geq 0$.   □

3. Moment inequalities for weighted reliability measures. In this section, we present some inequalities for weighted distributions. Let $\mathcal{F}$ be the set of absolutely continuous distribution function given by (2.1). The weighted survival function is given by

$$\bar{G}(x) = \delta^{*-1} \int_x^\infty W(y)dF(y), \qquad (3.1)$$

where $W(y)$ is a positive real function and $0 < \delta^* = E(W(X)) < \infty$.

It is well known that if G_1 and G_2 are absolutely continuous with respect to a σ-finite measure v, with Radon-Nikodym derivative g_1 and g_2, then

$$\int |g_2 - g_1|dv = 2\sup_{y} |G_1(\Delta) - G_2(\Delta)|, \qquad (3.2)$$

where y is the collection of Borel subsets of $[0, \infty)$. Indeed if $P(X = Y)$ is small, then g_1 and g_2 are close in $L_1(v)$ norm, where the distributions of X and Y are given by

G_1 and G_2, respectively. The weighted survival function $\bar{G}(x)$ can be written as

$$\frac{\bar{G}(x)}{\bar{F}(x)} = \frac{E[W(X) \mid X > x]}{\delta^*} \tag{3.3}$$

and the corresponding ratio of hazard functions is given by

$$\frac{\lambda_G(x)}{\lambda_F(x)} = \frac{W(x)}{E[W(X) \mid X > x]}. \tag{3.4}$$

The quantity $E[W(X) \mid X > x]$ is referred to as the weighted vitality function. Under size-biased sampling, $W(x) = x$ and

$$V_F(x) = E(X - x \mid X > x) + x, \tag{3.5}$$

where

$$E[X - x \mid X > x] = \int_x^\infty \frac{\bar{F}(y)\,dy}{\bar{F}(x)} \tag{3.6}$$

is the residual life function (MRLF).

It is clear that if $V_F(x)$ is increasing in x, then

$$\bar{G}_l(x) = \mu^{-1} \int_x^\infty y\,dF(y) \geq c^{-1}\bar{F}(x) \geq \bar{F}(x), \tag{3.7}$$

where $c = \bar{F}(0), \bar{G}(0) = 1$, and $c^{-1} = \bar{G}(0)/\bar{F}(0)$. Indeed if the weighted vitality function $E[W(X) \mid X > x]$ is increasing in x, then $\bar{G}(x)/\bar{F}(x)$ is increasing in x.

THEOREM 3.1 (see [1]). *If F has DMRL, then $S_k(x) \leq S_k(0)e^{-x/\mu}$, $k = 1, 2, \ldots$, and $S_k(x) \geq \mu S_{k-1}(0)e^{-x/\mu} - \mu S_{k-1}(0) + S_k(0)$, $k = 2, 3, \ldots$, where*

$$S_k(x) = \begin{cases} \bar{F}(x) & \text{if } k = 0, \\ \int_0^\infty \dfrac{\bar{F}(x+t)t^{k-1}dt}{(k-1)!} & \text{if } k = 1, 2, \ldots, \end{cases} \tag{3.8}$$

is a sequence of decreasing functions for which F possess moments of order J, that is, $\mu_k = E(X^k)$ exists, $k = 1, 2, \ldots, J$.

Let $S_{-1}(x) = f(x)$ be the pdf of F if it exists. Then $S_k(0) = \mu_k/k!$, and $S_k'(x) = -S_{k-1}(x)$, $k = 0, 1, 2, \ldots, J$. The ratio $S_{k-1}(x)/S_k(x)$ is a hazard function of a distribution function with survival function $S_k(x)/S_k(0)$. The inequalities in Theorem 3.1 are reversed if F has increasing mean residual life (IMRL).

THEOREM 3.2. *Let $\bar{G}(x)$ be an IHR weighted distribution function with monotone weight function. Then*

$$\int_0^\infty |\bar{G}(x) - \bar{G}_l(x)|\,dx \leq 2\mu(1 + \beta), \tag{3.9}$$

where $\beta = \mu - \mu_2/2\mu^2$, and $\bar{G}_l(x) = (1 + x/\mu)e^{-x/\mu}$, for $x \geq 0$.

PROOF. Let $A = \{x \mid \bar{G} \leq (1 + x/\mu) \exp(-x/\mu)\}$. Then we have for $x \geq 0$,

$$\int_0^\infty \left| \bar{G}(x) - \left(1 + \frac{x}{\mu}\right) e^{-x/\mu} \right| dx$$

$$\leq 2 \int_A \left(\left(1 + \frac{x}{\mu}\right) e^{-x/\mu} - \bar{G}(x) \right) dx$$

$$\leq 2 \int_0^\infty \left(\left(1 + \frac{x}{\mu}\right) e^{-x/\mu} - \bar{F}(x) \right) dx \tag{3.10}$$

$$= 2 \int_0^\infty \left(\left(1 + \frac{x}{\mu}\right) e^{-x/\mu} - \frac{S_1(x)}{\mu} \right) dx = 2 \left(\mu^2 + \mu - \frac{\mu_2}{2\mu} \right)$$

$$= 2\mu \left(1 + \mu - \frac{\mu_2}{2\mu} \right) = 2\mu(1 + \beta).$$

The first inequality is trivial and the second inequality is due to the stochastic order between $\bar{G}(x)$ and $\bar{G}_l(x)$ whenever $W(x)$ is increasing in $x \geq 0$. $\qquad\square$

THEOREM 3.3. *Let $\bar{G}(x)$ be a DHR weighted distribution function with monotone weight function $W(x) \geq 0$. Then for $\epsilon \geq \mu \geq 1$,*

$$\int_0^\infty |\bar{G}(x) - e^{-x/\mu}| \, dx \geq 2\mu e^{-\epsilon/\mu}(\mu - 1). \tag{3.11}$$

PROOF. Let $\epsilon \geq \mu$, then for $x \geq 0$, we have

$$\int_0^\infty |\bar{G}(x) - e^{-x/\mu}| \, dx = 2 \int_\epsilon^\infty \{\bar{G}(x) - e^{-x/\mu}\} dx$$

$$\geq 2 \int_\epsilon^\infty \{\bar{F}(x) - e^{-x/\mu}\} dx$$

$$= 2(\mu S_1(\epsilon) - \mu e^{-\epsilon/\mu})$$

$$\geq 2\mu \{S_1(0) e^{-\epsilon/\mu} - e^{-\epsilon/\mu}\} \tag{3.12}$$

$$= 2\mu e^{-\epsilon/\mu}(S_1(0) - 1)$$

$$= 2\mu e^{-\epsilon/\mu}(\mu - 1).$$

The first inequality is due to the fact that $\bar{G}(x)$ and $\bar{F}(x)$ are stochastic ordered whenever $W(x)$ is increasing in $x \geq 0$, and the second inequality follows from Theorem 3.1. $\qquad\square$

The next result is due in part to an application of the lemma given by Brown [2].

THEOREM 3.4. *Assume that the weighted vitality function $E[W(X) \mid X > x]$ is increasing in x, $0 < \delta^* = E[W(X)] < \infty$ and $\mu_F = \int_0^\infty \bar{F}(x) dx < \infty$, then*

$$\sup_x |\bar{G}(x) - \bar{F}(x)| \leq 1 - \frac{\delta^*}{\mu_G}, \tag{3.13}$$

where $\mu_G = \int_0^\infty \bar{G}(x) dx$.

PROOF. Since $E[W(X) \mid X > x] = \delta^* \bar{G}(x)/\bar{F}(x)$ is increasing, we obtain

$$
\begin{aligned}
\sup_x \left| \bar{G}(x) - \bar{F}(x) \right| &\leq 1 - \delta^{*-1} \int_0^\infty \left(\frac{\bar{F}(x)}{\bar{G}(x)} \right) W(x) f(x) dx \\
&= 1 - \delta^{*-1} \int_0^\infty \left(\frac{\bar{F}(x)}{\bar{G}(x)} \right)^2 \left(\frac{\bar{F}(x)}{\bar{G}(x)} \right) W(x) f(x) dx \\
&\leq 1 - \delta^{*-1} \mu_G \left(\int_0^\infty \frac{W(x) f(x) dx}{\mu_G} \right)^2 \\
&= 1 - \frac{\delta^*}{\mu_G}.
\end{aligned}
\tag{3.14}
$$

$\square$

THEOREM 3.5. *Under length or size-biased sampling*

$$
\sup_x \left| \bar{G}_l(x) - \bar{F}(x) \right| \leq 1 - \frac{\mu^2}{\mu_2} = \frac{\sigma_F^2}{(\sigma_F^2 + \mu_F^2)},
\tag{3.15}
$$

where $\mu_2 = E(X^2)$ and $\sigma_F^2 = \mathrm{Var}(X)$ is the variance of X.

PROOF. This follows from the fact that

$$
E_G(X^r) = \frac{E_F(X^{r+1})}{\mu_F},
\tag{3.16}
$$

$r \geq 1$ and δ^* reduces to $\mu_F = \int_0^\infty \bar{F}(x) dx$. $\square$

4. Partial ordering via probability functional. Let (X, y) be a random vector, where X lies on the real line and is observable. Let y be 0 or 1. We assume the conditional probability density function of X, given $y = 1$ and $y = 0$ are $g_1(x) = f(y \mid x = 1)$ and $g_0(x) = f(y \mid x = 0)$, respectively. Consider a simple classification procedure defined as a partition of $C = (C_0, C_1)$ in $\mathbb{R}$, such that $y_1 = i$ if and only if $X \in C_i$, $i = 0, 1$. The mean probability of true classification is

$$
P(Y_1 = y) = \frac{1}{2} P(X \in C_0 \mid y = 0) + \frac{1}{2} P(X \in C_1 \mid y = 1).
\tag{4.1}
$$

For this setting, the optimal Bayes rule $A = (A_0, A_1)$ that maximizes the mean probability of true classification is of the form

$$
A_0 = \{x \mid g_0(x) \geq g_1(x)\}, \qquad A_1 = \{x \mid g_1(x) > g_0(x)\}.
\tag{4.2}
$$

The true probability of classification is given by

$$
\Delta = \frac{1}{2} \int_{\mathbb{R}} \max(g_0(x), g_1(x)) dx = \frac{1}{2} + \frac{1}{4} \int_{\mathbb{R}} |g_0(x) - g_1(x)| dx.
\tag{4.3}
$$

For the weighted distribution with weight function $W(x) \geq 0$ and $0 < E[W(X)] < \infty$, the probability functional (4.3) reduces to

$$\frac{1}{2} \int_0^\infty \max\left(\frac{W(x)f(x)}{E(W(X))}, f(x)\right) dx$$

$$= \frac{1}{2} + \frac{1}{4\delta} \{E\{\max(W(X) - \delta, 0)\} + E\{\max(\delta - W(X), 0)\}\}, \tag{4.4}$$

where $0 \neq \delta = E[W(X)]$.

Let $g_{W_i}(x) = W_i(x) f_i(x) / \delta_i$, $i = 1, 2$, then

$$\int_0^\infty \max(g_{W_1}(x), f_1(x)) dx \geq \int_0^\infty \max(g_{W_2}(x), f_2(x)) dx \tag{4.5}$$

if and only if

$$E \left| \left\{ \frac{W_1(X)}{\delta_1} \right\} - 1 \right| \geq E \left| \left\{ \frac{W_2(X)}{\delta_2} \right\} - 1 \right|, \tag{4.6}$$

where $0 \neq \delta_i = E[W_i(X)]$, $i = 1, 2$.

Let (X_i, y_i) be random vectors and g_{0i} and g_{1i}, $i = 1, 2$, be given by (4.3).

DEFINITION 4.1. We say the random vector (X_1, y_1) is more varied than the random vector (X_2, y_2) if

$$\Delta_1 = \frac{1}{2} + \frac{1}{4} \int_{\mathbb{R}} |g_{01}(x) - g_{11}(x)| dx \geq \Delta_2 = \frac{1}{2} + \frac{1}{4} \int_{\mathbb{R}} |g_{02}(x) - g_{22}(x)| dx. \tag{4.7}$$

This is denoted by $(X_1, y_1) \overset{\text{mv}}{\geq} (X_2, y_2)$.

Now suppose one of the hypotheses given below is true. The hypotheses are

$$H_0 : (X_1 \mid y_1 = 0) \sim g_0, \qquad (X_1 \mid y_1 = 1) \sim g_1,$$

$$H_1 : (X_1 \mid y_1 = 0) \sim g_1, \qquad (X_1 \mid y_1 = 1) \sim g_0. \tag{4.8}$$

Let π_i denote the prior probability that H_i is true and $R(\pi_i)$ the Bayes risk when the experiment $X_1 \mid y_1 = i$, $i = 0, 1$ is the case. The next theorem allows for the comparison of two experiments $X_1 \mid y_1 = 1$ and $X_1 \mid y_1 = 0$ in terms of Bayes risk.

THEOREM 4.2.

$$R_{X_1 \mid y_1 = 1}(\pi_0) \geq R_{X_1 \mid y_1 = 0}(\pi_0) \tag{4.9}$$

if and only if

$$\int_{\mathbb{R}} \max(g_0(x), y g_1(x)) dx \geq \int_{\mathbb{R}} \max(y g_0(x), g_1(x)) dx, \tag{4.10}$$

where $y = \alpha_0 \pi_0 / \alpha_1 \pi_1$, $\pi_1 > 0$ and $\alpha_i > 0$ is the loss if H_i is true and not accepted.

PROOF. Let $A = \{x : \gamma g_0(x) \le g_1(x)\}$ and d_i the Bayes decision to accept H_i. Then

$$
\begin{aligned}
R_{X_1|y_1=1}(\pi_0) &= \alpha_0 \pi_0 P(d_1 \mid H_0) + \alpha_0 \pi_0 P(d_0 \mid H_1) \\
&= \alpha_0 \pi_0 \int_A g_1(x)dx + \alpha_1 \pi_1 \int_{A^c} g_0(x)dx \\
&= \alpha_1 \pi_1 \int_{\mathbb{R}} \max(\gamma g_0(x), g_1(x))dx,
\end{aligned}
\tag{4.11}
$$

$$
R_{X_1|y_1=1}(\pi_0) = \alpha_1 \pi_1 \int_{\mathbb{R}} \max(g_0(x), \gamma g_1(x))dx,
$$

where $\gamma = \alpha_0 \pi_0 / \alpha_1 \pi_1$, $\pi_1 > 0$. The result now follows. If $\gamma = 1$ or g_0 and g_1 are mutually disjoint, then $R_{X_1|y_1=1}(\pi_0) = R_{X_1|y_1=0}(\pi_0)$. The optimal choice of experiment depends only on g_0, g_1 and the value of γ.   □

5. Comparisons for weighted and related models. It is important to compare experiments for some family of models. For any such comparisons, interest should be in comparisons that are compatible and practically possible. In this light, one might be inclined to investigate and compare the possibility of sampling or selection of experiment from weighted distribution as opposed to the parent or original distribution. In a similar setting comparison might be restricted to a class of distributions including possibly distributions with monotone likelihood ratios, comparisons via some informational measures such as informational energy functions, Fisher, Shannon or Kullback-Leibler information.

Let the conditional density of X, given $y = 1$, differ from $g_0(x)$ by a shift, that is, $g_1(x) = g_0(x - c)$. Let $H(g_0, c) = \int_{\mathbb{R}} |g_0(x) - g_0(x - c)|dx$, $c > 0$.

PROPOSITION 5.1.

$$
\inf_{g_0 \in G} H(g_0, c) \le c(a)^{-1/2},
\tag{5.1}
$$

where G is the class of probability density functions on $\mathbb{R}$ with constant variance.

PROOF. Consider the family of distribution functions with mean zero and finite support on the interval $[-\sqrt{a}, \sqrt{a}]$, $a > 0$, and the random variable U such that $P(U = -\sqrt{a}) = P(U = \sqrt{a}) = 1/2$. Then

$$
E|U - c| \ge E|X - c|,
\tag{5.2}
$$

for all $c > 0$ and by the assumption

$$
H(g_0, c) = c(a)^{-1/2}.
\tag{5.3}
$$

Consequently,

$$
\inf_{g_0 \in G} H(g_0, c) \le c(a)^{-1/2}.
\tag{5.4}
$$

□

Let $f(x)$ and $f_l(x)$ denote the original and length-biased probability density functions, respectively. The true probability of classification in this regard is given by

$$
\begin{aligned}
\Delta_l &= \frac{1}{2} \int_0^\infty \max\left(f(x), f_l(x)\right) dx = \frac{1}{2} + \frac{1}{4} \int_0^\infty |f(x) - x f(x)/\mu_X|\, dx \\
&= \frac{1}{2} + \frac{1}{4\mu_X} \int_0^\infty |X - \mu_X|\, dx = \frac{1}{2} + \frac{1}{4\mu_X} E\{|X - \mu_X|\}.
\end{aligned}
\tag{5.5}
$$

It is clear that

$$
\int_0^\infty |f(x) - f_l(x)|\, dx = \frac{1}{4\mu_X} \int_0^\infty (1 - F_{X - \mu_X}(u))\, du,
\tag{5.6}
$$

due to the fact that

$$
\begin{aligned}
E|X - \mu_X| &= E\{\max(X - \mu_X, 0)\} + E\{\max(\mu_X - X, 0)\} \\
&= \int_0^\infty (1 - F_{X-\mu_X}(u))\, du + \int_{-\infty}^0 F_{X-\mu_X}(u)\, du.
\end{aligned}
\tag{5.7}
$$

We have the following result.

THEOREM 5.2. *Let X_1 and X_2 be two nonnegative random variables with probability density functions g_1 and g_2 and corresponding length-biased probability density functions g_{l_1} and g_{l_2}, respectively. Assume $0 < E|X_i| < \infty$, and $\mu_{X_1} = \mu_{X_2}$, then*

$$
\Delta_{l_1} = \frac{1}{2} \int_0^\infty \max\left(g_1(x), g_{l_1}(x)\right) dx \geq \Delta_{l_2} = \frac{1}{2} \int_0^\infty \max\left(g_2(x), g_{l_2}(x)\right) dx
\tag{5.8}
$$

if and only if

$$
E|X_1 - \mu_{X_1}| \geq E|X_2 - \mu_{X_2}|.
\tag{5.9}
$$

Let $\{F_\theta : \theta \in \Theta\}$. The survival function $\bar{H}$, given by

$$
\bar{H}(x) = \int_\Theta \bar{F}_\theta(x)\, dK(\theta),
\tag{5.10}
$$

is a mixture of $\bar{F}_\theta$, where K is the mixing distribution. Assume that $\bar{F}_\theta(x)$ has density $f(x)$, then the density of h is given by

$$
h(x) = \int_\Theta f_\theta(x)\, dK(\theta).
\tag{5.11}
$$

A sufficient condition for dilation is that $\mu_{X_1} = \mu_{X_2}$ and $g_1 - g_2$ has two sign changes, zero disregarded and order of the sign sequence is $+, -, +$, this is denoted by $X_1 \overset{\text{dil}}{\leq} X_2$, where g_1 and g_2 are the probability functions of X_1 and X_2, respectively.

THEOREM 5.3. *Let X_1 and X_2 be two nonnegative random variables with probability density functions g_1 and g_2 and corresponding length-biased probability density functions g_{l_1} and g_{l_2}, respectively. Consider the mixtures given by*

$$
h_1(x) = \int_\Theta g_{1\theta}(x)\, dK_1(\theta), \qquad h_2(x) = \int_\Theta g_{2\theta}(x)\, dK(\theta),
\tag{5.12}
$$

respectively. Then

$$\int_0^\infty \big| h_1(x) - h_{l_1}(x) \big| \, dx \geq \int_0^\infty \big| h_2(x) - h_{l_2}(x) \big| \, dx \qquad (5.13)$$

if $E|X_1 - \mu_{X_1}| \geq E|X_2 - \mu_{X_2}|$ *and* $K_1 \overset{\mathrm{dil}}{\geq} K_2$.

PROOF. Note that

$$
\begin{aligned}
\int_0^\infty \max\big(h(x), h_l(x)\big) dx &= \int_0^\infty \max\left(\int_\Theta f_\theta(x) dK(\theta), \int_\Theta f_{l_\theta}(x) dK(\theta)\right) dx \\
&\geq \int_\Theta \left(\int_0^\infty \max\big(f_\theta(x), f_{l_\theta}(x)\big) dx\right) dK(\theta),
\end{aligned}
\qquad (5.14)
$$

where $f_{l_\theta}(x)$ is the length-biased probability density function corresponding to $f_\theta(x)$, and from Theorem 5.2 the result follows. $\qquad\square$

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