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MATRIX SPLITTING PRINCIPLES

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ABSTRACT. The systematic analysis of convergence conditions, used in comparison theorems proven for different matrix splittings, is presented. The central idea of this analysis is the scheme of condition implications derived from the properties of regular splittings of a monotone matrix $A = M_1 - N_1 = M_2 - N_2$. An equivalence of some conditions as well as an autonomous character of the conditions $M_1^{-1} \geq M_2^{-1} \geq 0$ and $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$ are pointed out. The secondary goal is to discuss some essential topics related with existing comparison theorems.

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1. Introduction. The main objective of this expository paper is to present the systematic analysis of convergence conditions derived from their implications for the regular splitting case and discussed in the subsequent sections. The secondary goal is to survey, compare and further develop properties of matrix splittings in order to present more clearly some aspects related with the results known in the literature.

Consider the iterative solution of the linear equation system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a nonsingular matrix and $x, b \in \mathbb{C}^n$.

Traditionally, a large class of iterative methods for solving (1.1) can be formulated by means of the splitting

$$A = M - N \quad \text{with } M \text{ nonsingular}, \tag{1.2}$$

and the approximate solution $x^{(t+1)}$ is generated, as follows

$$Mx^{(t+1)} = Nx^{(t)} + b, \quad t \geq 0 \tag{1.3}$$

or equivalently

$$x^{(t+1)} = M^{-1}Nx^{(t)} + M^{-1}b, \quad t \geq 0, \tag{1.4}$$

where the starting vector $x^{(0)}$ is given.

The iterative method is convergent to the unique solution

$$x = A^{-1}b \tag{1.5}$$

for each $x^{(0)}$ if and only if $\varrho(M^{-1}N) < 1$, which means that the used splitting of $A = M - N$ is convergent. The convergence analysis of the above method is based on

the spectral radius of the iteration matrix $\varrho(M^{-1}N)$. For large values of t, the solution error decreases in magnitude approximately by a factor of $\varrho(M^{-1}N)$ at each iteration step; the smaller is $\varrho(M^{-1}N)$, the quicker is the convergence. Thus, the evaluation of an iterative method focuses on two issues: M should be chosen as an easily invertible matrix and $\varrho(M^{-1}N)$ should be as small as possible.

General properties of a splitting of $A = M - N$ (not necessary convergent), useful for proving comparison theorems, are given in the following theorem [20].

THEOREM 1.1. *Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$. If A and M are nonsingular matrices, then*

$$M^{-1}NA^{-1} = A^{-1}NM^{-1}, \tag{1.6}$$

the matrices $M^{-1}N$ and $A^{-1}N$ commute, and the matrices NM^{-1} and NA^{-1} also commute.

PROOF. From the definition of splitting of A, it follows that

$$M^{-1} = (A+N)^{-1} = A^{-1}(I+NA^{-1})^{-1} = (I+A^{-1}N)^{-1}A^{-1} \tag{1.7}$$

or

$$A^{-1} = M^{-1} + M^{-1}NA^{-1} = M^{-1} + A^{-1}NM^{-1} \tag{1.8}$$

which implies the equality (1.6) and hence,

$$M^{-1}NA^{-1}N = A^{-1}NM^{-1}N \quad \text{or} \quad NM^{-1}NA^{-1} = NA^{-1}NM^{-1}. \tag{1.9}$$

□

From the above theorem, the following results can be deduced.

COROLLARY 1.2. *Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$. If A and M are nonsingular matrices, then by the commutative property, both matrices $M^{-1}N$ and $A^{-1}N$ (or NM^{-1} and NA^{-1}) have the same eigenvectors.*

LEMMA 1.3. *Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$. If A and M are nonsingular matrices, and λ_s and τ_s are eigenvalues of the matrices $M^{-1}N$ and $A^{-1}N$, respectively, then*

$$\lambda_s = \frac{\tau_s}{1+\tau_s}. \tag{1.10}$$

PROOF. By Corollary 1.2 we have

$$M^{-1}Nx_s = \lambda_s x_s, \tag{1.11}$$

$$A^{-1}Nx_s = \tau_s x_s. \tag{1.12}$$

The last equation can be written as

$$A^{-1}Nx_s = (I - M^{-1}N)^{-1}M^{-1}Nx_s = \tau_s x_s \tag{1.13}$$

or equivalently

$$M^{-1}Nx_s = \frac{\tau_s}{1+\tau_s}x_s \tag{1.14}$$

and the result (1.10) follows by the comparison of (1.11) with (1.14). □

From the above lemma, the following results can be concluded.

COROLLARY 1.4. *Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$, where A and M are nonsingular matrices, and λ_s and τ_s are eigenvalues of the matrices $M^{-1}N$ and $A^{-1}N$, respectively.*

(a) *If $\tau_s \in \mathbb{R}$, then the corresponding eigenvalue $\lambda_s \in \mathbb{R}$, and conversely.*

(b) *If $\tau_s \in \mathbb{C}$, then the corresponding eigenvalue $\lambda_s \in \mathbb{C}$, and conversely.*

REMARK 1.5. Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$, where A and M are nonsingular matrices, and λ_s and τ_s are eigenvalues of the matrices $M^{-1}N$ and $A^{-1}N$, respectively. If the eigenvalue spectrum of $A^{-1}N$ is real, that is, $\tau_s \in \mathbb{R}$ for all $1 \le s \le n$, then the corresponding eigenvalues $\lambda_s \in \mathbb{R}$, and conversely.

THEOREM 1.6. *Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$, where A and M are nonsingular matrices, and λ_s and τ_s are eigenvalues of the matrices $M^{-1}N$ and $A^{-1}N$, respectively; and let the eigenvalue spectrum of $A^{-1}N$ be complex with the conjugate complex eigenvalues $\tau_s = a_s + b_s i$ and $\tau_{s+1} = a_s - b_s i$. The splitting is convergent if and only if*

$$\left[\frac{a_s(1+a_s)+b_s^2}{(1+a_s)^2+b_s^2}\right]^2 + \left[\frac{b_s}{(1+a_s)^2+b_s^2}\right]^2 < 1 \tag{1.15}$$

for all $s = 1,3,5,\ldots,n-1$.

PROOF. By the relation (1.10), we have

$$\begin{aligned}
\lambda_s &= \frac{a_s + b_s i}{1 + a_s + b_s i} = \frac{a_s(1+a_s)+b_s^2}{(1+a_s)^2+b_s^2} + \frac{b_s}{(1+a_s)^2+b_s^2} i, \\
\lambda_{s+1} &= \frac{a_s - b_s i}{1 + a_s - b_s i} = \frac{a_s(1+a_s)+b_s^2}{(1+a_s)^2+b_s^2} - \frac{b_s}{(1+a_s)^2+b_s^2} i.
\end{aligned} \tag{1.16}$$

Since for a convergent splitting

$$\varrho(M^{-1}N) = \max_s |\lambda_s| < 1 \tag{1.17}$$

hence, inequality (1.15) follows immediately. □

Two particular cases of Theorem 1.6 are presented in the following lemmas.

LEMMA 1.7. *Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$, where A and M are nonsingular matrices, and λ_s and τ_s are eigenvalues of the matrices $M^{-1}N$ and $A^{-1}N$, respectively. If the eigenvalue spectrum of $A^{-1}N$ is real, then the splitting is convergent if and only if $\tau_s > -1/2$ for all $1 \le s \le n$.*

PROOF. Since in this case $\lambda_s \in \mathbb{R}$ and $\tau_s \in \mathbb{R}$, then inequality (1.17) is satisfied when

$$-1 < \lambda_s = \frac{\tau_s}{1+\tau_s} < 1 \tag{1.18}$$

which implies that $\tau_s > -1/2$ for all $1 \le s \le n$. □

LEMMA 1.8. *Let $A = M - N$ be a splitting of $A \in \mathbb{C}^{n\times n}$, where A and M are nonsingular matrices, and λ_s and τ_s are eigenvalues of the matrices $M^{-1}N$ and $A^{-1}N$, respectively. If the eigenvalue spectrum of $A^{-1}N$ is complex with the purely imaginary eigenvalues $\tau_s = b_s i$ and $\tau_{s+1} = -b_s i$, then the splitting is convergent.*

PROOF. It is evident that for this case inequality (1.15) reduces to the form

$$\frac{b_s^2}{1+b_s^2} < 1 \tag{1.19}$$

which is satisfied for an arbitrary $b_s \in \mathbb{R}$. □

In the convergence analysis of iterative methods, the Perron-Frobenius theory of nonnegative matrices plays an important role. This theory provides many theorems concerning the eigenvalues and eigenvectors of nonnegative matrices and its main results are recalled in the following two theorems [14].

THEOREM 1.9. *If $A \geq 0$, then*

(1) *A has a nonnegative real eigenvalue equal to its spectral radius,*
(2) *To $\varrho(A) > 0$, there corresponds an eigenvector $x \geq 0$,*
(3) *$\varrho(A)$ does not decrease when any entry of A is increased.*

THEOREM 1.10. *Let $A \geq 0$ be an irreducible matrix. Then*

(1) *A has a positive real eigenvalue equal to its spectral radius,*
(2) *To $\varrho(A) > 0$, there corresponds an eigenvector $x > 0$,*
(3) *$\varrho(A)$ increases when any entry of A increases,*
(4) *$\varrho(A)$ is a simple eigenvalue of A.*

In the case of nonnegative matrices the following theorem holds.

THEOREM 1.11. *Let $A = M - N$ be a splitting of $A \in \mathbb{R}^{n\times n}$, where A and M are nonsingular matrices. If both matrices $M^{-1}N$ and $A^{-1}N$ are nonnegative, then the splitting is convergent and*

$$\varrho(M^{-1}N) = \frac{\varrho(A^{-1}N)}{1+\varrho(A^{-1}N)}. \tag{1.20}$$

Moreover, all complex eigenvalues τ_s of the matrix $A^{-1}N$, if they exist, satisfy the inequality

$$\left[\frac{a_s(1+a_s)+b_s^2}{(1+a_s)^2+b_s^2}\right]^2 + \left[\frac{b_s}{(1+a_s)^2+b_s^2}\right]^2 \leq \varrho^2(M^{-1}N) < 1 \tag{1.21}$$

and all real eigenvalues τ_t of the matrix $A^{-1}N$, if they exist, satisfy the inequality

$$-\frac{\varrho(M^{-1}N)}{1+\varrho(M^{-1}N)} \leq \tau_t \leq \frac{\varrho(M^{-1}N)}{1-\varrho(M^{-1}N)}. \tag{1.22}$$

PROOF. From Theorem 1.9, we conclude that both nonnegative matrices $M^{-1}N$ and $A^{-1}N$ have nonnegative dominant eigenvalues λ_1 and τ_1, respectively, equal to their spectral radii, and by Corollary 1.2 corresponding to the same nonnegative

eigenvector $x_1 \geq 0$. Hence, by Lemma 1.3 the result (1.20) follows immediately. The same result (1.20) has been obtained by Varga [14] for regular splittings forty years ago.

Since $|\lambda_s| \leq \lambda_1 = \varrho(M^{-1}N)$ for all $2 \leq s \leq n$, then the complex eigenvalues satisfy inequality (1.15) as is shown in (1.21).

In the case of real eigenvalues λ_t, inequality (1.18) is replaced by

$$-\varrho(M^{-1}N) \leq \lambda_t = \frac{\tau_t}{1+\tau_t} \leq \varrho(M^{-1}N) \tag{1.23}$$

which is satisfied when inequality (1.22) holds. □

Remark 1.12. Since $\varrho(M^{-1}N) = \varrho(NM^{-1})$ and $\varrho(A^{-1}N) = \varrho(NA^{-1})$, Theorem 1.11 holds when only both matrices NM^{-1} and NA^{-1} are nonnegative.

Historically, the idea of matrix splittings has its scientific origin in the regular splitting theory introduced in 1960 by Varga [14] and extended in 1973 by the results of the author's thesis [15] (recalled in [20]). These first results, given as comparison theorems for regular splittings of monotone matrices and proven under natural hypotheses by means of the Perron-Frobenius theory of nonnegative matrices [14], have been useful tools in the convergence analysis of some iterative methods for solving systems of linear equations [15, 16, 17, 18, 20, 26].

Further extensions for regular splittings have been obtained by Csordas and Varga [4] in 1984, and from this time a renewed interest with comparison theorems, proven under progressively weaker hypotheses for different splittings, is permanently observed in the literature. These new results lead to successive generalizations, but on the other hand are accompanied with an increased complexity with the verification of hypotheses; therefore, some comparison theorems may have a more theoretical than practical significance. Theorems proven under different hypotheses, for a few types of splittings of monotone matrices representing a large class of applications, are reviewed in [22].

The Varga's definition of regular splitting became the standard terminology in the literature, whereas other splittings are usually defined as a matter of author's taste. The definitions of splittings, with progressively weakening conditions and consistent from the viewpoint of names, are collected in the following definition.

Definition 1.13. Let $M, N \in \mathbb{R}^{n\times n}$. Then the decomposition $A = M - N$ is called:

(a) a *regular splitting of* A if $M^{-1} \geq 0$ and $N \geq 0$,
(b) a *nonnegative splitting of* A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$, and $NM^{-1} \geq 0$,
(c) a *weak nonnegative splitting of* A if $M^{-1} \geq 0$, and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*),
(d) a *weak splitting of* A if M is nonsingular, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
(e) a *weaker splitting of* A if M is nonsingular, and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*),
(f) a *convergent splitting of* A if $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$.

The above definition of splittings is a modification of that given in [22, 23, 24, 25]. The splittings defined in the successive items extend successively a class of splittings of $A = M - N$ for which the matrices N and M^{-1} may lose the properties of

nonnegativity. Distinguishing both types of weak nonnegative and weaker splittings leads to further extensions allowing us to analyze cases when $M^{-1}N$ may have negative entries, even if NM^{-1} is a nonnegative matrix, for which Lemma 2.4 may be used as well. Climent and Perea [3], after the inspection of the author's work [20], have the same conclusion.

The definition assumed in item (b) is equivalent to the definition of weak regular splitting of A, introduced originally by Ortega and Rheinboldt [11]. However, it is necessary to mention that some authors (Berman and Plemmons [2], Elsner [5], Marek and Szyld [9], Song [12]), using the same name "weak regular splitting," restrict this definition to its weaker version based on the conditions $M^{-1} \geq 0$ and $M^{-1}N \geq 0$ without the condition $NM^{-1} \geq 0$, and corresponding to a weak nonnegative splitting of the first type. The neglect of the condition $NM^{-1} \geq 0$ in the definition of weak regular splitting leads to a confusion with the interpretation of some comparison theorems, and it will be discussed in detail in the next sections. It should be remarked that the use of the Ortega-Rheinboldt's terminology "weak regular" in item (b) causes a confusion in using the splitting name in item (c); therefore, it seems that assuming the terms "nonnegative" and "weak nonnegative" allows us to avoid this confusion. The term "weak" has been introduced by Marek and Szyld [9] for the case of the first type weaker splitting of A of Definition 1.13, but it is again called as a splitting of positive type by Marek [8] and nonnegative splitting by Song [12]. It seems that the proposed terminology in items (d) and (e), by an analogy to items (b) and (c), allows us again to avoid a confusion in splitting names.

It is evident that, with the above definition of splittings, the following corollary holds.

Corollary 1.14. *A regular splitting is a nonnegative splitting, a nonnegative splitting is a weak nonnegative splitting and a weak splitting, a weak nonnegative splitting and weak splitting are a weaker splitting, but the contrary is not true.*

The majority of comparison theorems have been proven for matrix problems, and their extension and/or generalization can be done in a trivial way in many cases, but it may also lead to incorrect results, as will be shown later. Generalizations are desired results for further developments, and they strengthen the meaning of the original result inspiring to its generalization. The author finds the Varga's Theorem 2.1 (given in the next section) as the most fundamental result in the convergence analysis of iteration methods, and just this result was an inspiration for many generalizations [1, 2, 3, 8, 11, 12] (see also item (4) of Theorem 3.1).

Marek and Szyld [9] generalized some earlier results for general Banach spaces and rather general cones. Climent and Perea [3], following the Marek and Szyld's approach [9], extended some author's results [20] as well as Csordas and Varga results [4] to bounded operators in the general Banach space and rather general cones and in the Hilbert space. However, results obtained in both the above papers are illustrated only by matrix problems for which original results are fulfilled. The Climent and Perea paper [3], developed with Daniel Szyld's assistance, seems to be an incapable attempt of improving some author's results [20], therefore, in the present paper a special attention is paid to some of their results and conclusions.

As can be seen in the example of the Lanzkron-Rose-Szyld's theorem [7], discussed in detail in Section 3, the properties of some splittings are not sufficiently examined.

In Section 2 basic results for regular splittings are given together with the derivation of the scheme of condition implications. Results obtained for nonnegative, weak nonnegative, weak and weaker splittings are discussed in Sections 3 and 4, respectively. Finally, supplementary discussion about the utility of conditions is presented in Section 5.

2. Regular splittings and the scheme of condition implications. At the beginning we recall two basic results of Varga [14].

THEOREM 2.1. *Let $A = M - N$ be a regular splitting of A. If $A^{-1} \geq 0$, then*

$$\varrho(M^{-1}N) = \frac{\varrho(A^{-1}N)}{1+\varrho(A^{-1}N)} < 1. \tag{2.1}$$

Conversely, if $\varrho(M^{-1}N) < 1$, then $A^{-1} \geq 0$.

THEOREM 2.2. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $N_2 \geq N_1$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{2.2}$$

In particular, if $N_2 \geq N_1$ with $N_2 \neq N_1$, and if $A^{-1} > 0$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{2.3}$$

Theorem 2.2 allows us to compare spectral radii of iteration matrices only in the Jacobi and Gauss-Seidel methods [14]. The advantageous convergence properties of prefactorization AGA algorithms developed by the author, observed in numerical experiments [15, 16, 18, 20, 26], encouraged to further studies, the result of which is the following theorem.

THEOREM 2.3. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{2.4}$$

In particular, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{2.5}$$

This theorem, published in the external report [15] and recalled in [20], have been popularized in 1984 by Csordas and Varga [4] as the "useful but little known results of Woźnicki," and six years later it was recognized by Marek and Szyld [9] as "the less known result of Woźnicki," and recently by Climent and Perea [3] as "the lesser known ones introduced by Woźnicki in his unpublished dissertation (1973), although these results are being cited by Csordas and Varga." It seems to be an unusual or even funny matter that Theorem 2.3, on the one hand recognized as a less known result,

on the other hand became the subject of extensions or generalizations done just by the authors mentioned above as well as other authors.

One of the important applications of Theorem 2.3 is the generalization of the theorem of Stein-Rosenberg for iterative prefactorization methods in an irreducible case [15, 20].

It is easy to verify that for regular splittings of a monotone matrix A (i.e., $A^{-1} \geq 0$),

$$A = M_1 - N_1 = M_2 - N_2, \tag{2.6}$$

the assumption of Theorem 2.2

$$N_2 \geq N_1 \geq 0 \tag{2.7}$$

implies the equivalent condition

$$M_2 \geq M_1 \tag{2.8}$$

but the last inequality implies the condition of Theorem 2.3, that is,

$$M_1^{-1} \geq M_2^{-1} \geq 0. \tag{2.9}$$

From inequality (2.7), one obtains the inequality $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$. Since by relation (1.20), $\varrho(M^{-1}N)$ is a monotone function with respect to $\varrho(A^{-1}N)$, the result of Theorem 2.2 follows immediately.

In case of the proof of Theorem 2.3, condition (2.9) can be expressed as follows

$$(I + A^{-1}N_1)^{-1}A^{-1} \geq A^{-1}(I + N_2A^{-1})^{-1} \tag{2.10}$$

which, after relevant multiplications, is equivalent to

$$A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1} \geq 0. \tag{2.11}$$

From the above inequality, one obtains

$$A^{-1}N_2A^{-1}N_1 \geq (A^{-1}N_1)^2 \geq 0, \tag{2.12}$$

$$(A^{-1}N_2)^2 \geq A^{-1}N_1A^{-1}N_2 \geq 0. \tag{2.13}$$

Hence,

$$\varrho^2(A^{-1}N_2) \geq \varrho(A^{-1}N_1A^{-1}N_2) = \varrho(A^{-1}N_2A^{-1}N_1) \geq \varrho^2(A^{-1}N_1) \tag{2.14}$$

which gives us

$$\varrho(A^{-1}N_2) \geq \varrho(A^{-1}N_1) \tag{2.15}$$

and by result (1.20), the inequality

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2) \tag{2.16}$$

can be deduced.

In case of the strict inequality in (2.9), similar considerations lead to the strict inequality in (2.16) [15, 20].

On the other hand, from inequality (2.7), one obtains

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0, \tag{2.17}$$

which implies inequalities (2.11), (2.12), and (2.13), and additionally

$$A^{-1}N_1A^{-1}N_2 \geq \left(A^{-1}N_1\right)^2 \geq 0, \tag{2.18}$$

$$\left(A^{-1}N_2\right)^2 \geq A^{-1}N_2A^{-1}N_1 \geq 0. \tag{2.19}$$

Inequality (2.8) gives us

$$A^{-1}M_2 \geq A^{-1}M_1 \geq 0, \tag{2.20}$$

since for each splitting of A

$$A^{-1}M = I + A^{-1}N, \tag{2.21}$$

hence, it is evident that both conditions (2.17) and (2.20) are equivalent.

Each of the above conditions (except (2.13) and (2.19) as is discussed later) leads to proving inequality (2.16); however, as can be shown in simple examples of regular splittings, the reverse implications may fail. Thus, the above inequalities are progressively weaker conditions which are used as hypotheses in comparison theorems to provide successive generalizations of the results.

The scheme of implications of the above conditions is demonstrated in Figure 2.1. The equivalence of conditions (A) and (B) follows immediately from relation (2.6). Both conditions (D) and (E) are equivalent by relation (2.21). Conditions (C) and (D) imply conditions (G) and (F) which are equivalent by relation (2.21). Condition (C) implies indirectly only conditions (H1) and (H2), whereas condition (E) implies directly all conditions (H1), (H2), (H3), and (H4). It is evident that condition (D) implies that conditions (K1), (K2), (K3), and (K4) equivalent to conditions (L1), (L2), (L3), and (L4), respectively.

It seems to be interesting to ask, does a dependence exist between condition (C), playing the essential role in the conjugate type iterative solvers [19], and condition (E)? To give the answer to the above question, we consider for the following example of a matrix [20]

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}, \quad \text{where } A^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{2.22}$$

some regular splittings of $A = M_i - N_i$ given below.

$$M_1 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}, \qquad N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.23}$$

where

$$M_1^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad M_1^{-1}N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad A^{-1}N_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{2.24}$$

where

$$M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad M_2^{-1}N_2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \qquad A^{-1}N_2 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad N_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{2.25}$$

where

$$M_3^{-1} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad M_3^{-1}N_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \qquad A^{-1}N_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$M_4 = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}, \qquad N_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.26}$$

where

$$M_4^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \qquad M_4^{-1}N_4 = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}, \qquad A^{-1}N_4 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.27}$$

As can be easily noticed for $M_1^{-1} \geq M_2^{-1}$ ($M_2 \not\geq M_1$), $\varrho(M_1^{-1}N_1) = 1/2 < \varrho(M_2^{-1}N_2) = 1/\sqrt{2}$, whereas $A^{-1}N_2 \not\geq A^{-1}N_1$; and for $A^{-1}N_4 \geq A^{-1}N_3$, $\varrho(M_3^{-1}N_3) = 1/2 < \varrho(M_4^{-1}N_4) = 2/3$, whereas $M_3^{-1} \not\geq M_4^{-1}$. Thus, the above regular splitting examples show us that both conditions (C) and (E) have an autonomous character, and there is no even a precursor relation between them.

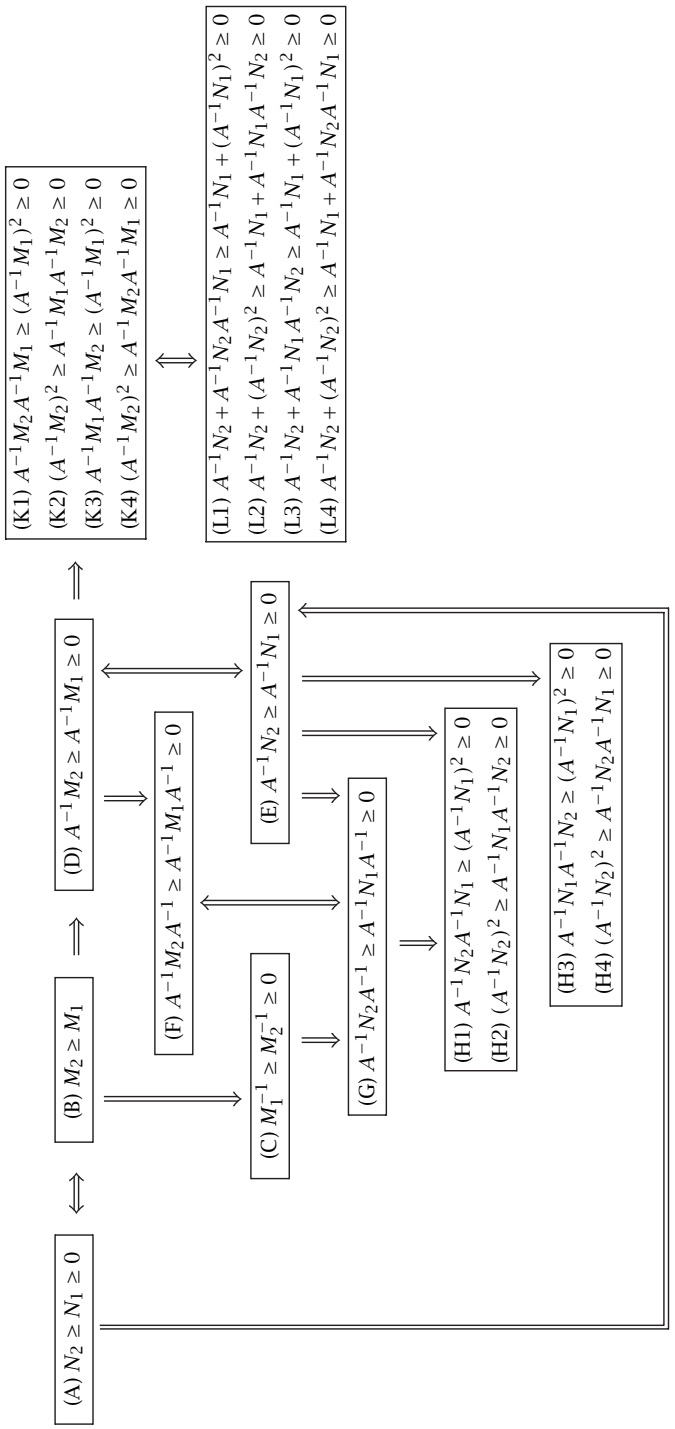

FIGURE 2.1. Scheme of condition implications for regular splittings of $A = M_1 - N_1 = M_2 - N_2$, where $A^{-1} \geq 0$.

Some results for condition (C) and regular splittings of monotone matrices, derived with a different fineness of block partitions, have been recently obtained in [21].

It is evident that the scheme of condition implications given in Figure 2.1 could be derived at the properties of regular splittings of a monotone matrix A, characterized by the conditions $M^{-1} \geq 0$ and $N \geq 0$. However, particular conditions of this scheme can be used for different types of splittings. For example, for regular splittings Csordas and Varga [4], assuming condition (G) and following the methodology used in the proof of Theorem 2.3, represented by the inequalities from (2.10) to (2.15), show inequality (2.16). In the case of weaker splittings of the first type, condition (E) has been considered by Marek [8] and the equivalent condition (D) by Song [12, 13]; conditions (H1), (H2), (H3), and (H4) have been originally used by Beauwens [1] as separate conditions, but it appears that only (H1) and (H3) can be used as such separate conditions [1, (Erratum)] [12, 25].

In the literature there are many comparison theorems proven under the hypotheses presented in Figure 2.1 as well as more composed hypotheses. For instance, for the regular splitting case Csordas and Varga [4] consider the hypothesis

$$(A^{-1}N_2)^j A^{-1} \geq (A^{-1}N_1)^j A^{-1} \geq 0, \quad j \geq 1 \tag{2.28}$$

derived from condition (G); for the case of weaker splittings, Miller and Neumann [10] analyze the condition

$$(A^{-1}N_2)^j (A^{-1}N_1)^l \geq (A^{-1}N_1)^{j+l} \geq 0, \quad j \geq 1,\ l \geq 1 \tag{2.29}$$

derived from condition (H1) and Song [12, 13] considers some conditions of the type of

$$(A^{-1}M_2)^j (A^{-1}M_1)^l \geq (A^{-1}M_1)^{j+l} \geq 0, \quad j \geq 1,\ l \geq 1 \tag{2.30}$$

derived from conditions (K1), (K2), (K3), and (K4).

Only Csordas and Varga [4] present a simple example of regular splittings satisfying inequality (2.28) with $j > 1$, but their example satisfies much simple hypotheses (f) and (h) of Lemma 3.4 given in the next section. Song [12, 13] illustrates his results only in examples of regular splittings, with $j = 1$ and $l = 1$, for which the Varga Theorem 2.2 is satisfied.

Another class of conditions, based on the knowledge of the eigenvectors of $M_1^{-1}N_1$ and $M_2^{-1}N_2$ corresponding to $\varrho(M_1^{-1}N_1)$ and $\varrho(M_2^{-1}N_2)$, respectively, have been introduced by Marek and Szyld [9].

Finally, it should be noted that the conditions of regular splitting of a monotone matrix $A = M - N$

$$M^{-1} \geq 0, \tag{2.31}$$

$$N \geq 0 \tag{2.32}$$

imply

$$M^{-1}N \geq 0, \tag{2.33}$$

$$A^{-1}N \geq 0 \tag{2.34}$$

and the extra conditions

$$NM^{-1} \geq 0, \tag{2.35}$$

$$NA^{-1} \geq 0 \tag{2.36}$$

which are important in convergence analysis as well. Thus, the principle of regular splitting is based on the six above conditions.

Both matrices $M^{-1}N$ and NM^{-1} (as well as $A^{-1}N$ and NA^{-1}) have the same eigenvalues because they are similar matrices. It may occur that for the splittings (2.6) none of the conditions given in Figure 2.1 is not satisfied but the following lemma holds.

Lemma 2.4 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $M_2^{-1}N_2 \geq N_1M_1^{-1} \geq 0$ (or $N_2M_2^{-1} \geq M_1^{-1}N_1 \geq 0$) or $A^{-1}N_2 \geq N_1A^{-1} \geq 0$ (or $N_2A^{-1} \geq A^{-1}N_1 \geq 0$), then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{2.37}$$

3. Nonnegative and weak nonnegative splittings. The first extension of the regular splitting case is due to Ortega and Rheinboldt [11] who introduced the class of weak regular splittings, based on the conditions (2.31), (2.33), and (2.35), for which Theorem 2.2 [11] and Theorem 2.3 (see as well, Theorem 3.5) hold. However, as it was already mentioned, some authors [2, 5, 9, 12], using the same name "weak regular splitting," restrict this definition to its weaker version based on conditions (2.31) and (2.33) only. It seems that this simplification of the Ortega-Rheinboldt definition is due to Berman and Plemmons [2] and it leads to a confusion in the interpretation of some comparison theorems. It is rather an unusual case that the same name "weak regular splitting" is used in the literature for two different definitions, it should be at least distinguished as the Ortega-Rheinboldt's weak regular splitting and the Berman-Plemmons' weak regular splitting. It seems that this unclear definition of weak regular splitting is eliminated by using the terminology of items (b) and (c) of Definition 1.13

Conditions ensuring that a splitting of $A = M - N$ will be convergent are unknown in a general case. As was pointed out in [20], the splittings defined in the first three items of Definition 1.13 are convergent if and only if $A^{-1} \geq 0$.

The properties of weak nonnegative splittings, extensively analyzed in [20] for the conditions of implication scheme demonstrated in Figure 2.1, are summarized in the following theorem.

Theorem 3.1 (see [20]). *Let $A = M - N$ be a weak nonnegative splitting of A. If $A^{-1} \geq 0$, then*

(1) $A^{-1} \geq M^{-1}$,
(2) $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$,
(3) *if $M^{-1}N \geq 0$, then $A^{-1}N \geq M^{-1}N$ and if $NM^{-1} \geq 0$, then $NA^{-1} \geq NM^{-1}$,*
(4) $\varrho(M^{-1}N) = \varrho(A^{-1}N)/(1+\varrho(A^{-1}N)) = \varrho(NA^{-1})/(1+\varrho(NA^{-1})) < 1$,
(5) *Conversely, if $\varrho(M^{-1}N) < 1$, then $A^{-1} \geq 0$.*

The relation in item (4), similar to the result of Theorem 2.1, follows from Theorem 1.11 and Remark 1.12. From Theorem 3.1 we have the following result.

COROLLARY 3.2. *Each weak nonnegative (as well as by Corollary 1.14 nonnegative and regular) splitting of $A = M - N$ is convergent if and only if $A^{-1} \geq 0$. In other words, if A is not a monotone matrix, it is impossible to construct a convergent weak nonnegative splitting.*

It is obvious by Theorem 3.1 that when both weak nonnegative splittings of a monotone matrix $A = M_1 - N_1 = M_2 - N_2$ are of the same type, the inequality

$$N_2 \geq N_1 \tag{3.1}$$

implies either

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0 \tag{3.2}$$

or

$$N_2A^{-1} \geq N_1A^{-1} \geq 0. \tag{3.3}$$

When both weak nonnegative splittings are of different types, one of the matrices $A^{-1}N_2$ and $A^{-1}N_1$ or N_2A^{-1} and N_1A^{-1} may have negative entries, which does not allow us to conclude that the inequality $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$ is satisfied. Let us assume that $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ which implies by Theorem 3.1 that $A^{-1}N_1 \geq 0$ and $N_2A^{-1} \geq 0$, then from (3.1) we have

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0, \qquad N_2A^{-1} \geq N_1A^{-1} \ngeq 0, \tag{3.4}$$

which leads to the conclusion that the second splitting should be a nonnegative splitting. In the case when $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, implying that $N_1A^{-1} \geq 0$ and $A^{-1}N_2 \geq 0$, then from (3.1) one obtains,

$$N_2A^{-1} \geq N_1A^{-1} \geq 0, \qquad A^{-1}N_2 \geq A^{-1}N_1 \ngeq 0, \tag{3.5}$$

hence, it can be concluded again that the second splitting should be a nonnegative splitting.

Thus, the above considerations allow us to prove the following theorem.

THEOREM 3.3. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same type, where $A^{-1} \geq 0$. If $N_2 \geq N_1$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.6}$$

The result of this theorem, proven originally by Varga [14] for regular splittings, carries over to the case when both weak nonnegative splittings are of the same type.

Some results (see [20, Theorems 3.3 and 3.16]) are summarized in the following lemma.

LEMMA 3.4 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of A, where $A^{-1} \geq 0$. If one of the following inequalities:*

(a) $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$ *(or $M_2^{-1}N_2 \geq M_1^{-1}N_1 \geq 0$),*

(b) $A^{-1}N_2 \geq N_1A^{-1} \geq 0$ *(or* $M_2^{-1}N_2 \geq N_1M_1^{-1} \geq 0$*),*
(c) $N_2A^{-1} \geq N_1A^{-1} \geq 0$ *(or* $N_2M_2^{-1} \geq N_1M_1^{-1} \geq 0$*),*
(d) $N_2A^{-1} \geq A^{-1}N_1 \geq 0$ *(or* $N_2M_2^{-1} \geq M_1^{-1}N_1 \geq 0$*),*
(e) $A^{-1}N_2 \geq (A^{-1}N_1)^T \geq 0$ *(or* $M_2^{-1}N_2 \geq (M_1^{-1}N_1)^T \geq 0$*),*
(f) $A^{-1}N_2 \geq (N_1A^{-1})^T \geq 0$ *(or* $M_2^{-1}N_2 \geq (N_1M_1^{-1})^T \geq 0$*),*
(g) $N_2A^{-1} \geq (N_1A^{-1})^T \geq 0$ *(or* $N_2M_2^{-1} \geq (N_1M_1^{-1})^T \geq 0$*),*
(h) $N_2A^{-1} \geq (A^{-1}N_1)^T \geq 0$ *(or* $N_2M_2^{-1} \geq (M_1^{-1}N_1)^T \geq 0$*)*

is satisfied, then

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.7}$$

In the case of the weaker condition $M_1^{-1} \geq M_2^{-1}$, the contrary behaviour is observed. As is demonstrated in the examples in [20], when both weak nonnegative splittings of a monotone matrix A are of the same type, with $M_1^{-1} \geq M_2^{-1}$ (or even $M_1^{-1} > M_2^{-1}$), it may occur that $\varrho(M_1^{-1}N_1) > \varrho(M_2^{-1}N_2)$. In the case of nonnegative splittings we have the following result.

THEOREM 3.5 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two nonnegative splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.8}$$

In particular, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{3.9}$$

But for different types of weak nonnegative splittings there is a similar result.

THEOREM 3.6 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different types, that is, either $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ or $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.10}$$

In particular, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{3.11}$$

REMARK 3.7. Obviously, the case of two mixed splittings of $A = M_1 - N_1 = M_2 - N_2$ (i.e., when one of them is nonnegative and the second is weak nonnegative) is fulfilled by the assumptions of Theorem 3.6.

When both splittings are of the same type, there is not a general recipe for the choice of additional conditions to the assumption $M_1^{-1} \geq M_2^{-1}$ in order to ensure inequality (2.16). However, some additional natural conditions, appearing in many applications, are illustrated by the following result.

THEOREM 3.8 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of a symmetric matrix A, where $A^{-1} \geq (>)0$. If $M_1^{-1} \geq (>)M_2^{-1}$ and at least one of M_1 and M_2 is a symmetric matrix, then*

$$\varrho(M_1^{-1}N_1) \leq (<)\varrho(M_2^{-1}N_2). \tag{3.12}$$

In the case of the Berman-Plemmons weak regular splitting, corresponding to the weak nonnegative splitting of the first type, Elsner [5] showed that the assumption $M_1^{-1} \geq M_2^{-1} \geq 0$ may not be a sufficient hypothesis for ensuring inequality (2.16), and he stated the result of Theorem 3.6 for the case when one of the splittings is a regular one. This means that Elsner restored the need of condition (2.35) sticking originally in the Ortega-Rheinboldt's definition of weak regular splitting. It is evident that the Elsner's result is a particular case of Theorem 3.6. This topic is discussed in detail in [25].

The Ortega-Rheinboldt's definition of weak regular splitting is used by Lanzkron, Rose, and Szyld [7], but Szyld in his earlier paper of Marek and Szyld [9] uses the Berman-Plemmons' definition of weak regular splitting and he just refers it to the Ortega-Rheinboldt's paper [11]. Lanzkron, Rose, and Szyld [7] have proven the following theorem for Ortega-Rheinboldt's weak regular splittings.

Theorem 3.9 (see [7, Theorem 3.1]). *Let $A = M_1 - N_1 = M_2 - N_2$ be convergent weak regular splittings (i.e., nonnegative splittings) such that*

$$M_1^{-1} \geq M_2^{-1}, \tag{3.13}$$

and let x and z be the nonnegative Frobenius eigenvectors of $M_1^{-1}N_1$ and $M_2^{-1}N_2$, respectively. If $N_2 z \geq 0$ or if $N_1 x \geq 0$ with $x > 0$, then

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.14}$$

As can be deduced from Theorem 3.1, the term "convergent" is equivalent to the assumption that $A^{-1} \geq 0$. Since $M^{-1} \geq 0$, $M^{-1}N \geq 0$, and $\varrho(M^{-1}N) < 1$ by the assumption, then

$$A^{-1} = [I - M^{-1}N]^{-1}M^{-1} = [I + M^{-1}N + (M^{-1}N)^2 + (M^{-1}N)^3 + \cdots]M^{-1} \geq 0 \tag{3.15}$$

and conversely, if $A^{-1} \geq 0$, then $\varrho(M^{-1}N) < 1$.

As follows from Theorem 3.5, the hypothesis $M_1^{-1} \geq M_2^{-1}$ is a sufficient condition in this theorem, and the assumptions $N_2 z \geq 0$ or $N_1 x \geq 0$ with $x > 0$ are superfluous because they follow from the properties of nonnegative splittings reported in [6] as well. For each nonnegative splitting of $A = M - N$, where $A^{-1} \geq 0$ and $1 > \lambda = \varrho(M^{-1}N) \geq 0$, one can write

$$M^{-1}Nx = \lambda x, \quad \text{where } x \geq 0 \tag{3.16}$$

or equivalently

$$Nx = \lambda Mx, \qquad NM^{-1}Mx = \lambda Mx, \qquad NM^{-1}y = \lambda y. \tag{3.17}$$

Since NM^{-1} is also a nonnegative matrix, then its eigenvector $y = Mx \geq 0$, hence

$$Nx = \lambda y \geq 0. \tag{3.18}$$

Thus, this theorem supplied with additional but completely superfluous conditions, used frequently as a reference in Marek-Szyld [9] and other papers, is equivalent to Theorem 3.5.

In the case of condition (G) $A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1} \geq 0$, equivalent to condition (F) $A^{-1}M_2A^{-1} \geq A^{-1}M_1A^{-1} \geq 0$, we have the following results.

THEOREM 3.10 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two nonnegative splittings, or two weak nonnegative splittings of different types, that is, either $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ or $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, where $A^{-1} \geq 0$. If $A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1} \geq 0$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.19}$$

In particular, if $A^{-1} > 0$ and $A^{-1}N_2A^{-1} > A^{-1}N_1A^{-1} \geq 0$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{3.20}$$

Another class of conditions with transpose matrices have been considered by the author in [20] and the results are summarized below.

THEOREM 3.11 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of A but of the same type, that is, either $M_1^{-1}N_1 \geq 0$ and $M_2^{-1}N_2 \geq 0$ or $N_1M_1^{-1} \geq 0$ and $N_2M_2^{-1} \geq 0$, where $A^{-1} \geq 0$. If $N_2^T \geq N_1$ or $(M_1^{-1})^T \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.21}$$

Climent and Perea [3] showed in simple examples of regular splittings that this theorem as well as its counterparts for weak and weaker splittings (Theorems 6.8, 6.9, and 6.10 given in [20]) fail. In the first example (see [3, Example 3]) they consider the condition $N_2^T \geq N_1$, and in the second case (see [3, Example 4]) the condition $(M_1^{-1})^T \geq M_2^{-1}$. The example of regular splittings given below shows that when both these conditions are simultaneously fulfilled, Theorem 3.11 is also not true. For the monotone matrix

$$A = \begin{bmatrix} 4 & -2 & -1 \\ -4 & 8 & -2 \\ -8 & -4 & 12 \end{bmatrix}, \tag{3.22}$$

we consider regular splittings $A = M_1 - N_1 = M_2 - N_2$, where

$$M_1 = \begin{bmatrix} 4 & 0 & 0 \\ -4 & 8 & 0 \\ -8 & -4 & 12 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_1^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{5}{24} & \frac{1}{24} & \frac{1}{12} \end{bmatrix}, \tag{3.23}$$

$$M_2 = \begin{bmatrix} 4 & -2 & -1 \\ 0 & 8 & -2 \\ 0 & 0 & 12 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 8 & 4 & 0 \end{bmatrix}, \quad M_2^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{16} & \frac{1}{32} \\ 0 & \frac{1}{8} & \frac{1}{48} \\ 0 & 0 & \frac{1}{12} \end{bmatrix}. \tag{3.24}$$

Hence

$$M_1^{-1}N_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{8} \\ 0 & \frac{5}{12} & \frac{7}{24} \end{bmatrix}, \qquad M_2^{-1}N_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} & 0 \\ \frac{2}{3} & \frac{1}{12} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}. \tag{3.25}$$

Evidently $N_2^T \geq N_1$ and $(M_1^{-1})^T \geq M_2^{-1}$, but

$$\varrho(M_1^{-1}N_1) = \frac{2}{3} \approx 0.6667 > \varrho(M_2^{-1}N_2) = \frac{7+\sqrt{73}}{24} \approx 0.6477. \tag{3.26}$$

The above splittings of matrix (3.22) show not only that Theorem 3.11 fails, but it also illustrates the behavior of the Gauss-Seidel method used in the algorithms of the SOR method which its performance is studied in [29].

Usually a diagonally dominant matrix A is defined by the following decomposition:

$$A = D - L - U, \tag{3.27}$$

where D, L, and U are nonsingular diagonal, strictly lower triangular and strictly upper triangular parts of A respectively, and the standard iterative schemes are defined as follows.

The Jacobi method.

$$M_J = D, \qquad N_J = L + U, \qquad \mathcal{B}_1 = M_J^{-1}N_J = D^{-1}(L+U). \tag{3.28}$$

The forward Gauss-Seidel method.

$$M_{fG} = D - L, \qquad N_{fG} = U, \qquad \mathcal{L}_1^f = M_{fG}^{-1}N_{fG} = (D-L)^{-1}U. \tag{3.29}$$

The backward Gauss-Seidel method.

$$M_{bG} = D - U, \qquad N_{bG} = L \qquad \mathcal{L}_1^b = M_{bG}^{-1}N_{bG} = (D-U)^{-1}L. \tag{3.30}$$

As can be seen, unlike the Jacobi iteration, the Gauss-Seidel iteration depends on the ordering of the unknowns. Forward Gauss-Seidel begins the update of x with the first component, whereas for backward Gauss-Seidel with the last component. Usually, when A is a nonsymmetric matrix, the spectral radii $\varrho(\mathcal{L}_1^f)$ and $\varrho(\mathcal{L}_1^b)$ may have different values as is just illustrated by the splittings of the matrix (3.22); the first splitting represents the forward Gauss-Seidel method and the second one represents the backward Gauss-Seidel method.

However, for the symmetric case we have the following result.

Theorem 3.12 (see [29]). *Let $A = D - L - U$ be a symmetric matrix with the nonsingular matrix D, then*

$$\varrho(\mathcal{L}_1^f) = \varrho(\mathcal{L}_1^b). \tag{3.31}$$

PROOF. We can write

$$\mathcal{L}_1^f = (D-L)^{-1}U = (D-U^T)^{-1}U, \tag{3.32}$$

$$\mathcal{L}_1^b = (D-U)^{-1}L = (D-U)^{-1}U^T. \tag{3.33}$$

Then we have

$$\begin{aligned}\varrho(\mathcal{L}_1^b) &= \varrho((D-U)^{-1}U^T) = \varrho(U^T(D-U)^{-1}) \\ &= \varrho\left([U^T(D-U)^{-1}]^T\right) = \varrho\left([(D-U)^{-1}]^T U\right) \\ &= \varrho((D-U^T)^{-1}U) = \varrho(\mathcal{L}_1^f)\end{aligned} \tag{3.34}$$

which completes the proof. □

COMMENTS ON CLIMENT-PEREA'S RESULTS (see [3]). The authors of [3] try not only to extend the author's results given in [20] to bounded operators in a general Banach space and in some cases to a Hilbert space, but they want to apply the Perron-Frobenius theory of nonnegative matrices for proving results with matrices that may not be nonnegative, as it appears in the proof of [3, Theorem 5].

The trivial Corollary 1.14 of this paper, summarizing results of Corollaries 3.1 and 6.1 given in [20], became [3, Theorem 1]. [3, Theorems 2, 3, and 4] contain only the well-known results.

In [3, Theorem 5] Climent and Perea want to prove the result of Theorem 3.3 for the assumption $N_2 \geq N_1$ when both weak nonnegative splittings of $A = M_1 - N_1 = M_2 - N_2$ are just of different types for $A^{-1} \geq 0$. Since their proof is based on an "interesting" methodology; therefore, it is worth to present their approach.

If $A = M_1 - N_1$ is a splitting of the first type, then from the assumption $N_2 \geq N_1$

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0. \tag{3.35}$$

For $\varrho(A^{-1}N_1)$ there exists an eigenvector $x \geq 0$ such that $A^{-1}N_1x = \varrho(A^{-1}N_1)x$ (as can be concluded from Theorem 3.1), then by (3.35), one obtains,

$$A^{-1}N_2x - \varrho(A^{-1}N_1)x \geq 0 \tag{3.36}$$

and from Lemma 1 given in [3], it follows

$$\varrho(A^{-1}N_1) \leq \varrho(A^{-1}N_2), \tag{3.37}$$

and by Theorem 3.1, the inequality $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$ can be concluded.

On the other hand, if $A = M_2 - N_2$ is a weak nonnegative splitting of the second type for which $N_2A^{-1} \geq 0$ and $A^{-1}N_2 \not\geq 0$, then there is a contradiction to inequality (3.35), but this does not disturb to conclude by Climent and Perea that inequality (3.37) is equivalent to

$$\varrho(A^{-1}N_1) \leq \varrho(N_2A^{-1}). \tag{3.38}$$

How Climent and Perea conclude the result (3.38), is not shown in [3]. Inequality (3.37) is deduced from inequality (3.36) by means of the results of the Perron-Frobenius theory of nonnegative matrices. As an illustration, consider the following

matrix examples:

$$A^{-1}N_1=\begin{bmatrix}1&0\\1&0\end{bmatrix},\qquad A^{-1}N_2=\begin{bmatrix}0&-1\\0&2\end{bmatrix},\qquad N_2A^{-1}=\begin{bmatrix}0&0\\1&2\end{bmatrix}. \tag{3.39}$$

Obviously, $\varrho(A^{-1}N_1)\le\varrho(A^{-1}N_2)$, but this inequality cannot be deduced from (3.36) because inequality (3.36) is not satisfied for both of the above matrices $A^{-1}N_2$ and N_2A^{-1}.

From inequality (3.35), it follows that the second splitting should be a weak nonnegative splitting of the first type. Since for each weak nonnegative splitting we have $A^{-1}N\ge M^{-1}N$ or $NA^{-1}\ge NM^{-1}$, it may occur that, for instance, $NA^{-1}\ge 0$ with $NM^{-1}\not\ge 0$. Just such an example of splitting, considered in [6], has been submitted to the author by Daniel Szyld for the following matrix

$$A=\begin{bmatrix}1&-2&1\\0&2&-2\\-1&0&2\end{bmatrix}=M_1-N_1=M_2-N_2,\quad \text{where } A^{-1}=\begin{bmatrix}2&2&1\\1&\frac{3}{2}&1\\1&1&1\end{bmatrix}>0, \tag{3.40}$$

$$M_1=\begin{bmatrix}1&-\frac{3}{2}&\frac{3}{4}\\0&2&-\frac{201}{100}\\-1&-\frac{1}{2}&3\end{bmatrix},\qquad N_1=\begin{bmatrix}0&\frac{1}{2}&-\frac{1}{4}\\0&0&-\frac{1}{100}\\0&-\frac{1}{2}&1\end{bmatrix},$$

$$M_1^{-1}=\frac{1}{696}\begin{bmatrix}999&825&303\\402&750&402\\400&400&400\end{bmatrix},\qquad M_2=\begin{bmatrix}1&-\frac{3}{2}&\frac{3}{4}\\0&2&-2\\0&-\frac{1}{2}&3\end{bmatrix},$$

$$N_2=\begin{bmatrix}0&\frac{1}{2}&-\frac{1}{4}\\0&0&0\\1&-\frac{1}{2}&1\end{bmatrix},\qquad M_2^{-1}=\frac{1}{40}\begin{bmatrix}40&33&12\\0&24&16\\0&4&16\end{bmatrix},$$

$$M_1^{-1}N_1=\frac{1}{696}\begin{bmatrix}0&348&45\\0&0&309\\0&0&296\end{bmatrix},\qquad N_1M_1^{-1}=\frac{1}{696}\begin{bmatrix}101&275&101\\-4&-4&-4\\199&25&199\end{bmatrix},$$

$$M_2^{-1}N_2=\frac{1}{20}\begin{bmatrix}6&17&1\\8&-4&8\\8&-4&8\end{bmatrix},\qquad N_2M_2^{-1}=\frac{1}{40}\begin{bmatrix}0&11&4\\0&0&0\\40&17&20\end{bmatrix},$$

$$A^{-1}N_2=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{2}\\1&0&\frac{3}{4}\\1&0&\frac{3}{4}\end{bmatrix}\ge A^{-1}N_1=\begin{bmatrix}0&\frac{1}{2}&\frac{23}{50}\\0&0&\frac{147}{200}\\0&0&\frac{74}{100}\end{bmatrix}\ge 0,$$

$$N_2A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 \\ \frac{5}{2} & \frac{9}{4} & \frac{3}{2} \end{bmatrix} \geq N_1A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{100} & -\frac{1}{100} & -\frac{1}{100} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \not\geq 0. \tag{3.41}$$

In this example the first splitting is weak nonnegative of the first type with $N_1A^{-1} \not\geq 0$, but the second splitting is weak nonnegative of the second type with $A^{-1}N_2 \geq 0$ and $N_2A^{-1} \geq 0$. Since in this example the assumption $N_2 \geq N_1$ is satisfied and moreover, $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$, one can conclude that

$$0.4253 = \varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2) = 0.6531. \tag{3.42}$$

However, in this case some contradiction appears with the Perron-Frobenius theory of nonnegative matrices again. Namely, the relation

$$\varrho(M^{-1}N) = \frac{\varrho(A^{-1}N)}{1+\varrho(A^{-1}N)} < 1 \tag{3.43}$$

have been derived with the assumption that $M^{-1}N$ and $A^{-1}N$ are nonnegative matrices. In the above example we have $M_2^{-1}N_2 \not\geq 0$ and $A^{-1}N_2 \geq 0$, but this does not allow us to derive a conclusion from the above relation. Since we have $\varrho(A^{-1}N) = \varrho(NA^{-1})$ as well as $\varrho(M^{-1}N) = \varrho(NM^{-1})$, and the second splitting is a weak nonnegative splitting of the second type, the result (3.42) can be concluded by the relation

$$\varrho(NM^{-1}) = \frac{\varrho(NA^{-1})}{1+\varrho(NA^{-1})} < 1. \tag{3.44}$$

The above result indicates that there exists a subclass of weak nonnegative splittings with stronger conditions $A^{-1}N \geq 0$ and $NA^{-1} \geq 0$ which allows us to prove inequality (3.38) when both splittings of A are weak nonnegative splittings of different types [6]. But this requires proving that $A^{-1}N$ and NA^{-1} are nonnegative matrices at least for one of these splittings, which seems to be a difficult or even impossible task. Thus, we see that the above approach does not allow us to prove Theorem 3.3 for the case when both weak nonnegative splittings of $A = M_1 - N_1 = M_2 - N_2$ are of different types and accompanied by $N_1A^{-1} \not\geq 0$ and $A^{-1}N_2 \not\geq 0$ or $A^{-1}N_1 \not\geq 0$ and $N_2A^{-1} \not\geq 0$. However, in such a case Theorem 3.3 is valid and this fact can be proved in a simple way as can be seen below.

Theorem 3.13. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same or different type, where $A^{-1} \geq 0$. If $N_2 \geq N_1$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{3.45}$$

Proof. The case when both splittings are of the same type has been proven in Theorem 3.3. Assume that both splittings of A are of different types. The condition $N_2 \geq N_1$ implies that

$$M_2 \geq M_1. \tag{3.46}$$

Since $M_1^{-1} \geq 0$ and $M_2^{-1} \geq 0$ by the assumption, then from (3.46) we have

$$M_1^{-1} \geq M_2^{-1} \geq 0 \tag{3.47}$$

which is the hypothesis of Theorem 3.6, and the result of this theorem follows immediately from Theorem 3.6 valid for weak nonnegative splittings of different types. □

As can be easily verified, condition (3.47) is satisfied in the examples of splittings of matrix (3.40) with $M_1^{-1} > M_2^{-1} \geq 0$ which implies the strict inequality in (3.42).

Referring to the remaining Climent-Perea's results, it should be mentioned that Theorems 6 and 7 in [3] are trivial extensions of Theorems 3.4 and 3.7 (given in [20] as Theorems 3.7, 3.8, 5.3, and 5.4), respectively.

Theorems 8, 9, and 10 in [3] are an attempt for improving the result of Theorem 3.11 and Climent and Perea claim that A must be a symmetric matrix as a necessary condition.

First, such an assumption is completely useless in the convergence analysis of nonsymmetric problems.

Second, there are examples of splittings of nonsymmetric matrices A showing that Theorem 3.11 holds, for instance, for the splittings of the Gauss-Seidel method, derived from both Climent-Perea's examples (see [3, Examples 3 and 4]), Theorem 3.11 holds, which is a contradiction to the Climent-Perea's necessary condition that A must be a symmetric matrix. It is worth to think about additional conditions ensuring that with the hypotheses

$$N_2^T \geq N_1 \tag{3.48}$$

or

$$\left(M_1^{-1}\right)^T \geq M_2^{-1}. \tag{3.49}$$

Theorem 3.11 will be held, of course, when A is a nonsymmetric monotone matrix.

Third, assuming that $M_2^{-1}N_2 \geq 0$ and let $A = M_3 - N_3$ be such a splitting for which $M_3 = M_2^T$ and $N_3 = N_2^T$, then

$$\left(M_2^{-1}N_2\right)^T = N_2^T\left(M_2^{-1}\right)^T = N_3M_3^{-1} \geq 0 \tag{3.50}$$

hence, it can be concluded that $\varrho(M_2^{-1}N_2) = \varrho(N_3M_3^{-1}) = \varrho(M_3^{-1}N_3)$. Thus, the second and third splittings are equivalent and we see that when A is a symmetric monotone matrix, the use of conditions (3.48) and (3.49) does not provide new results in comparison to the classic conditions $N_3 \geq N_1$ and $M_1^{-1} \geq M_3^{-1}$ used as hypotheses in Theorem 3.8.

Fourth, using the Climent-Perea's language, one can say that they assumed again a false hypothesis in Theorem 8 in [3]. As follows from (3.48), when $A = M_1 - N_1$ is a weak nonnegative splitting of the first type ($M_1^{-1}N_1 \geq 0$), one obtains

$$A^{-1}N_2^T \geq A^{-1}N_1 \geq 0, \tag{3.51}$$

since $A = A^T$, hence

$$A^{-1}N_2^T = \left(N_2A^{-1}\right)^T \geq A^{-1}N_1 \geq 0 \qquad \text{or} \qquad N_2A^{-1} \geq \left(A^{-1}N_1\right)^T \geq 0. \tag{3.52}$$

When $A = M_1 - N_1$ is a weak nonnegative splitting of the second type ($N_1 M_1^{-1} \geq 0$), one obtains

$$N_2^T A^{-1} \geq N_1 A^{-1} \geq 0, \tag{3.53}$$

hence

$$N_2^T A^{-1} = \left(A^{-1} N_2\right)^T \geq N_1 A^{-1} \geq 0 \qquad \text{or} \qquad A^{-1} N_2 \geq \left(N_1 A^{-1}\right)^T \geq 0. \tag{3.54}$$

Then, by item (f) or (h) of Lemma 3.4, the result of [3, Theorem 8] follows immediately. Thus, as can be concluded from the above considerations, this result of Climent-Perea, being a particular case of the more general Lemma 3.4 (or [20, Theorem 3.16]) holds if both weak nonnegative splittings of a symmetric monotone matrix A are of different types. When both splittings are of the same type, then condition (3.48) may implies difficulties in the proof, similar to those of [3, Theorem 5] and discussed above.

The equivalence of [3, Theorems 9 and 10] with Theorems 3.6 and 3.11 can be shown in a similar way, where passing to conditions (3.28) and (3.29) is accompanied with the change of splitting types.

[3, Theorems 11 and 12] are again particular cases of Lemma 4.4, and [3, Theorem 3] is an example of manipulation of the author's results [20] because all assumptions considered in this theorem were just analyzed in detail in [20]. In the last six theorems of Section 4 in [3], Climent and Perea collect conditions and duplicate the results known already.

The author finds in Climent-Perea's paper [3] only one valuable result showing that Theorem 3.11 and its versions for weak and weaker splittings given in [20] fail. The inspection of theorems in [3] shows that there are no new results or ideas useful for matrix splitting applications, except the trivial extension of known results to bounded operators. Theorems 8 and 10 in [3] are an attempt in saving Theorem 3.11 for symmetric monotone matrices; however, as was already shown these theorems are particular cases of known results. Thus, it seems that a generalization of existing results was the main intention of Climent and Perea and all examples in [3] are given only for matrix splittings for which the original results are satisfied. It is interesting that in a rich collection of splitting examples given in [3], which seems to be a challenge for making exercises with matrix operations and finding inverse matrices by potential readers, there is no any example illustrating Theorem 5 in [3] for the assumption $N_2 \geq N_1$ when both weak nonnegative splittings are of different types.

Finally we have the following corollary.

Corollary 3.14. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings, where $A^{-1} \geq 0$, then*

(a) *the assumption $N_2 \geq N_1$ allows us to prove that $\varrho(M_1^{-1} N_1) \leq \varrho(M_2^{-1} N_2)$ for an arbitrary type of splittings,*

(b) *if both splittings are of the same type, then the assumption $M_1^{-1} \geq M_2^{-1} \geq 0$ is not a sufficient condition for proving that $\varrho(M_1^{-1} N_1) \leq \varrho(M_2^{-1} N_2)$.*

4. Weak and weaker splittings. As was stated in the previous section, weak nonnegative splittings, determined by conditions (2.31) and either (2.33) or (2.35), are

convergent if and only if $A^{-1} \geq 0$, which means that both conditions $A^{-1} \geq 0$ and $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$ are equivalent. In the case of weak and weaker splittings (based on conditions (2.33) and/or (2.35)), the assumption $A^{-1} \geq 0$ is not a sufficient condition in order to ensure the convergence of a given splitting of A; it is also possible to construct a convergent weak or weaker splitting when $A^{-1} \ngeq 0$. Moreover, as can be shown in examples, the conditions $A^{-1}N \geq 0$ or $NA^{-1} \geq 0$ may not ensure that a given splitting of A will be a weak or weaker splitting.

As a result of [20, Theorem 6.1] we have

Theorem 4.1. *Let $A = M - N$ be a weaker splitting of A. If $A^{-1}N \geq 0$ or $NA^{-1} \geq 0$, then*

(1) *if $M^{-1}N \geq 0$, then $A^{-1}N \geq M^{-1}N$ and if $NM^{-1} \geq 0$, then $NA^{-1} \geq NM^{-1}$,*

(2) $\varrho(M^{-1}N) = \varrho(A^{-1}N)/(1+\varrho(A^{-1}N)) = \varrho(NA^{-1})/(1+\varrho(NA^{-1})) < 1$.

Thus, in this case of a convergent weaker splitting there are three conditions $M^{-1}N \geq 0$ (or $NM^{-1} \geq 0$), $A^{-1}N \geq 0$ (or $NA^{-1} \geq 0$) and $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$, and any two conditions imply the third. However, the two last conditions may also imply a convergent splitting for which $M^{-1}N \ngeq 0$ and $NM^{-1} \ngeq 0$.

Theorem 4.2. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weaker splittings of A of the same type, that is, either $M_1^{-1}N_1 \geq 0$ and $M_2^{-1}N_2 \geq 0$ or $N_1M_1^{-1} \geq 0$ and $N_2M_2^{-1} \geq 0$. If $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$ or $N_2A^{-1} \geq N_1A^{-1} \geq 0$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{4.1}$$

Remark 4.3. Obviously, when $A^{-1} \geq 0$, the condition $N_2 \geq N_1$ is included in the hypotheses of Theorem 4.2

The first part of the relation in item (2) of Theorem 4.1 has been proven in 1970 by Marek [8] using the name "splitting of a positive type" and corresponding to the weaker splitting of the first type, and he stated the result of Theorem 4.2 for the hypothesis $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$.

It is interesting to ask if Theorem 4.2 is valid for the condition $N_2 \geq N_1$ when both weaker splittings of a monotone matrix A (i.e., $A^{-1} \geq 0$) are of different types. As was discussed in Section 3 for weak nonnegative splittings, considering the inequalities such as the assumptions of Theorem 4.2 does not lead for proving the inequality $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$. The proof of Theorem 3.13 is based on considering the condition $M_2 \geq M_1$ implied by $N_2 \geq N_1$. In the case of weaker splittings we may have that $M_1^{-1} \ngeq 0$ and $M_2^{-1} \ngeq 0$ and the condition (3.47) may not be valid. Thus, Theorem 4.2 may not hold when both weaker splittings are of different types.

Weak splittings are analyzed in [1, 8, 9, 10, 12, 13, 20]. In [20, Section 6], some comparison theorems for convergent weak splittings of a monotone matrix A are proven under the conditions considered previously for weak nonnegative splittings as well as with more composed hypotheses. It is evident that Lemma 3.4, Theorems 3.5, and 3.6, Remark 3.7, Corollary 3.14, and Theorem 3.10 have their counterparts for weak and weaker splittings, as is given below.

Lemma 4.4 (see [20])**.** *Let $A = M_1 - N_1 = M_2 - N_2$ be two weaker splittings of A. If one of the following inequalities:*

(a) $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$,

(b) $A^{-1}N_2 \geq N_1A^{-1} \geq 0$,
(c) $N_2A^{-1} \geq N_1A^{-1} \geq 0$,
(d) $N_2A^{-1} \geq A^{-1}N_1 \geq 0$,
(e) $A^{-1}N_2 \geq (A^{-1}N_1)^T \geq 0$,
(f) $A^{-1}N_2 \geq (N_1A^{-1})^T \geq 0$,
(g) $N_2A^{-1} \geq (N_1A^{-1})^T \geq 0$,
(h) $N_2A^{-1} \geq (A^{-1}N_1)^T \geq 0$,

is satisfied, then

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{4.2}$$

THEOREM 4.5 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{4.3}$$

In particular, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{4.4}$$

THEOREM 4.6 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weaker splittings of different types, that is, either $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ or $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{4.5}$$

In particular, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{4.6}$$

REMARK 4.7. Obviously, the case of two mixed splittings of $A = M_1 - N_1 = M_2 - N_2$ (i.e., when one of them is weak and the second is weaker) is fulfilled by the assumptions of Theorem 4.6.

COROLLARY 4.8. *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weaker splittings, where $A^{-1} \geq 0$, then*

(a) *the assumption $N_2 \geq N_1$ allows us to prove that $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$ if both splittings are of the same type, this assumption may not be valid when both splittings are of different types,*
(b) *if both splittings are of the same type, then the assumption $M_1^{-1} \geq M_2^{-1} \geq 0$ is not a sufficient condition for proving that $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$.*

THEOREM 4.9 (see [20]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak splittings, or two convergent weaker splittings of different types, that is, either $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ or $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, where $A^{-1} \geq 0$. If $A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1} \geq 0$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{4.7}$$

In particular, if $A^{-1} > 0$ and $A^{-1}N_2A^{-1} > A^{-1}N_1A^{-1} \geq 0$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{4.8}$$

Some conditions and comparison theorems for convergent weaker splittings were considered by Song [12], where he uses the terminology "nonnegative splitting" for the weaker splitting of the first type. His result is given in the following lemma.

Lemma 4.10 (see [12]). *Let $A = M - N$ be a weaker splitting of a nonsingular matrix A. If $A^{-1}M \geq 0$, then*

$$\varrho(M^{-1}N) = \frac{\varrho(A^{-1}M) - 1}{\varrho(A^{-1}M)} < 1. \tag{4.9}$$

Conversely, if $\varrho(M^{-1}N) < 1$, then $A^{-1}M \geq 0$.

The condition $A^{-1}M \geq 0$, ensuring by Lemma 4.10 that a given weaker splitting is convergent suggests some generality of results presented in [12]. However, as can be easily shown, this condition is equivalent to the conditions $A^{-1}M \geq I$ and $A^{-1}N \geq 0$. In reality,

$$A^{-1}M = A^{-1}[A+N] = I + A^{-1}N \geq 0, \tag{4.10}$$

which means that some diagonal entries of $A^{-1}N$ may be negative with values ≥ -1. But on the other hand for a convergent weak splitting of A we have

$$A^{-1}N = [I + M^{-1}N]^{-1}M^{-1}N = \left[\sum_{i=0}^{\infty} (M^{-1}N)^i\right] M^{-1}N \geq M^{-1}N \geq 0, \tag{4.11}$$

which implies that $A^{-1}N \geq 0$, hence $A^{-1}M = I + A^{-1}N \geq I$ as follows from (4.10), which gives us $\varrho(A^{-1}M) = 1 + \varrho(A^{-1}N)$. Thus, Lemma 4.10 is completely equivalent to the following result.

Lemma 4.11. *Let $A = M - N$ be a weaker splitting of a nonsingular matrix A. If $A^{-1}M \geq 0$ or $MA^{-1} \geq 0$, then $A^{-1}N \geq 0$ and $A^{-1}M = I + A^{-1}N$ or $NA^{-1} \geq 0$ and $MA^{-1} = I + NA^{-1}$, and*

$$\varrho(M^{-1}N) = \frac{\varrho(A^{-1}M) - 1}{\varrho(A^{-1}M)} = \frac{\varrho(A^{-1}N)}{\varrho(A^{-1}N) + 1} < 1. \tag{4.12}$$

Conversely, if $\varrho(M^{-1}N) < 1$, then $A^{-1}N \geq 0$ or $NA^{-1} \geq 0$.

As follows from the scheme of condition implications shown in Figure 2.1, both conditions (D) and (E) are equivalent and comparison theorems based on the hypothesis $A^{-1}M_2 \geq A^{-1}M_1 \geq 0$ (or $M_2A^{-1} \geq M_1A^{-1} \geq 0$) are equivalent to Theorem 4.2.

Both equivalent conditions $A^{-1}M \geq 0$ and $A^{-1}N \geq 0$ imply by Lemma 4.11 that a weaker splitting of A is convergent but, as will be shown in the examples, these conditions do not ensure that each splitting will be a weak or weaker splitting.

The verification of both mentioned conditions requires the explicit form of A^{-1} which may be cumbersome or impracticable. However, in this case of a monotone matrix A, we have the following useful results based on simply verifiable conditions.

Lemma 4.12. *Let $A = M - N$ be a weaker splitting A, where $A^{-1} \geq 0$. If $M \geq A$ or equivalently $N \geq 0$, then the splitting is a convergent weak splitting of A, characterized by $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$.*

PROOF. Since $A^{-1} \geq 0$, then the simple condition

$$M \geq A \tag{4.13}$$

or $M = A + N \geq A$; giving equivalently

$$N \geq 0 \tag{4.14}$$

implies that

$$A^{-1}M \geq I \geq 0, \qquad MA^{-1} \geq I \geq 0, \tag{4.15}$$

$$A^{-1}N \geq 0, \qquad NA^{-1} \geq 0. \tag{4.16}$$

Hence, by Lemma 4.11, it is evident that the splitting of A is convergent, and by Theorem 4.1, is a weak splitting. □

THEOREM 4.13. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak splittings of A, where $A^{-1} \geq 0$. If $M_2 \geq M_1 \geq A$ or equivalently $N_2 \geq N_1 \geq 0$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2). \tag{4.17}$$

Moreover, if $A^{-1} > 0$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2). \tag{4.18}$$

In order to illustrate the above results, consider some splittings for the following example of the monotone matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = M_i - N_i, \quad \text{where } A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad M_1^{-1}N_1 = \frac{1}{15}\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}, \quad \varrho(M_1^{-1}N_1) = \frac{4}{5},$$

$$M_2 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \quad M_2^{-1}N_2 = \frac{1}{6}\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}, \quad \varrho(M_2^{-1}N_2) = \frac{5}{6},$$

$$M_3 = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad M_3^{-1}N_3 = \frac{1}{7}\begin{bmatrix} -4 & 10 \\ 10 & -4 \end{bmatrix}, \quad \varrho(M_3^{-1}N_3) = \frac{2\sqrt{29}}{7},$$

$$M_4 = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}, \quad N_4 = \begin{bmatrix} -4 & 3 \\ 2 & -4 \end{bmatrix}, \quad M_4^{-1}N_4 = \begin{bmatrix} 2 & 1 \\ 0 & \frac{5}{2} \end{bmatrix}, \quad \varrho(M_4^{-1}N_4) = \frac{5}{2},$$

$$M_5 = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}, \quad N_5 = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}, \quad M_5^{-1}N_5 = \frac{1}{14}\begin{bmatrix} 8 & 0 \\ 5 & 7 \end{bmatrix}, \quad \varrho(M_5^{-1}N_5) = \frac{4}{7}. \tag{4.19}$$

As can be easily noticed, conditions (4.13) and (4.14) are satisfied only in the first three splittings. The first two splittings are convergent weak splittings, while the third

splitting is not either weaker or convergent. The fourth splitting is a disconvergent weak splitting. The fifth is a convergent weaker splitting of the first type because $N_5M_5^{-1} \not\geq 0$. The first two splittings illustrate the results of Theorem 4.13, where for $M_2 \geq M_1 > 0$ (or equivalently $N_2 \geq N_1 > 0$) we have $\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2)$. Lemma 4.4 is illustrated by the first, second, and fifth splittings for which $A^{-1}N_2 \geq A^{-1}N_1 \geq A^{-1}N_5 \geq 0$ and $\varrho(M_5^{-1}N_5) < \varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2)$.

As can be seen in the above examples of splittings, the inequalities $M \geq A$ or $N \geq 0$ are not either sufficient or necessary conditions for existing weak and weaker or convergent weak and weaker splittings of monotone matrices. It is evident that the inequalities $A^{-1}N \geq 0$ or $NA^{-1} \geq 0$, implying $A^{-1}M = I + A^{-1}N \geq 0$ or $MA^{-1} = I + NA^{-1} \geq 0$, are necessary conditions by Lemma 4.11, but as can be seen in the third splitting these inequalities are not sufficient conditions for existing convergent weak and weaker splittings. Thus, the criteria for the construction of convergent weak and weaker splittings of even monotone matrices remain an open question.

Comparison theorems for convergent weaker splittings of the first type, using the eigenvectors of $M_1^{-1}N_1$ and/or $M_2^{-1}N_2$ in hypotheses, are considered by Marek and Szyld [9], where in the case of matrix splittings their theorem has the following form.

Theorem 4.14 (see [9, Theorem 3.11]). *Let $A = M_1 - N_1 = M_2 - N_2$ be weaker splittings with $T_1 = M_1^{-1}N_1$, $T_2 = M_2^{-1}N_2$, and $\varrho(T_1) < 1$, $\varrho(T_2) < 1$. Let $x \geq 0$ and $z \geq 0$ be such that $T_1x = \varrho(T_1)x$, $T_2z = \varrho(T_2)z$. If either $N_1x \geq 0$ or $N_2z \geq 0$ with $z > 0$, and if*

$$M_1^{-1} \geq M_2^{-1}, \tag{4.20}$$

then

$$\varrho(T_1) \leq \varrho(T_2). \tag{4.21}$$

Moreover, if $M_1^{-1} > M_2^{-1}$ and if $N_1 \neq N_2$, then

$$\varrho(T_1) < \varrho(T_2). \tag{4.22}$$

In the proof for the assumption $N_1x \geq 0$, Marek and Szyld obtained the following relation

$$M_1x = \frac{1}{\varrho(T_1)}N_1x \geq 0. \tag{4.23}$$

Consider the following example of regular splitting

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix} = M_1 - N_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.24}$$

for which

$$T_1 = M_1^{-1}N_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad \varrho(T_1) = 0. \tag{4.25}$$

Thus, for the above example of regular splitting, relation (4.23) fails.

Theorem 4.14 is illustrated in the example of nonnegative splittings of a monotone matrix A (see [9, Example 3.10]) for which Marek and Szyld notice that the Lanzkron-Rose-Szyld Theorem 3.9 (accompanied by superfluous assumptions as was shown in the previous section and equivalent to Theorem 3.5) holds. Similarly as in the case of Theorem 3.9, it can be easily shown that for $A^{-1} \geq 0$ Theorem 4.14 is equivalent to Theorem 4.5 when both splittings of A are weak splittings or to Theorem 4.6 when both splittings are weaker splittings of different types. But in this case, additional hypotheses of Theorem 4.14 are evidently superfluous.

Moreover, Marek and Szyld comment (see [9, Remark 3.12]) that the author's result Theorem 2.3 does not hold for weak regular splittings without additional hypothesis. As was already mentioned Marek and Szyld use the weaker Berman-Plemons' definition of weak regular splitting [2] but they refer it to the Ortega-Rheinboldt's paper [11], where there is the original definition of weak regular splitting for which Theorem 2.3 holds. Thus, this comment of Marek and Szyld is a next example of the confusion following from unjustified removing the condition $NM^{-1} \geq 0$ existing in the Ortega-Rheinboldt's definition of weak regular splitting.

In the recent study of weak and weaker splittings for the case when $A^{-1} \leq 0$, the author obtained the following results [27, 28] being counterparts to Theorems 3.3, 4.5, and 4.6.

Theorem 4.15 (see [27]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weaker splittings of the same type, where $A^{-1} \leq 0$. If $N_2 \geq N_1$, then*

$$\varrho(M_1^{-1}N_1) \geq \varrho(M_2^{-1}N_2). \tag{4.26}$$

Theorem 4.16 (see [27]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak splittings of A, where $A^{-1} \leq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \geq \varrho(M_2^{-1}N_2). \tag{4.27}$$

In particular, if $A^{-1} < 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) > \varrho(M_2^{-1}N_2). \tag{4.28}$$

Theorem 4.17 (see [27]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weaker splittings of different types, that is, either $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ or $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, where $A^{-1} \leq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \geq \varrho(M_2^{-1}N_2). \tag{4.29}$$

In particular, if $A^{-1} < 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) > \varrho(M_2^{-1}N_2). \tag{4.30}$$

Thus, we see that passing from the assumption $A^{-1} \geq 0$ to the assumption $A^{-1} \leq 0$ changes the inequality sign in the inequalities for spectral radii.

Consider the case of conditions (H) of Figure 2.1. As can be seen, conditions (H1) and (H2) are implied indirectly by condition (C), on the other hand all conditions (H1), (H2),

(H3), and (H4) are implied directly by condition (E). Both conditions (H1) and (H2) allow us to prove Theorem 2.3 for regular splittings as is shown by relations (2.9) to (2.16). Now it seems to be interesting to ask, is it possible to use the separation of conditions (H1) and (H2) as well as (H3) and (H4) in order to prove the inequality $\varrho(M_1^{-1}N_1) \le \varrho(M_2^{-1}N_2)$? Beauwens [1] assumed all these conditions as separate hypotheses and he stated for weaker splittings of the first type the following results.

Theorem 4.18 (see [1, page 342]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings of A such that M_1 and M_2 are nonsingular, $M_1^{-1}N_1$ and $M_2^{-1}N_2$ are nonnegative and convergent. Then, any of the four assumptions*

(a) $(A^{-1}N_2 - A^{-1}N_1)A^{-1}N_1 \ge 0$, *(H1)*
(b) $(A^{-1}N_2 - A^{-1}N_1)A^{-1}N_2 \ge 0$, *(H2)*
(c) $A^{-1}N_1(A^{-1}N_2 - A^{-1}N_1) \ge 0$, *(H3)*
(d) $A^{-1}N_2(A^{-1}N_2 - A^{-1}N_1) \ge 0$. *(H4)*

implies

$$\varrho(M_1^{-1}N_1) \le \varrho(M_2^{-1}N_2). \tag{4.31}$$

Corollary 4.19 (see [1, page 342]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings of A such that M_1 and M_2 are nonsingular, $M_1^{-1}N_1$ and $M_2^{-1}N_2$ are nonnegative and convergent. Then, any of the two assumptions*

(a) $(M_1^{-1} - M_2^{-1})N_1 \ge 0$,
(b) $(M_1^{-1} - M_2^{-1})N_2 \ge 0$ *implies*

$$\varrho(M_1^{-1}N_1) \le \varrho(M_2^{-1}N_2). \tag{4.32}$$

It appears that Beauwens succeeded only with assumptions (a) and (c) of Theorem 4.18, and (a) of Corollary 4.19, and in 1986 he has corrected his results [1, (Erratum)]. Later in 1991 Song [12] has shown in the example of regular splittings derived from a 3×3 diagonal matrix A that these results fail for the remaining assumptions. This allows us to conclude that only conditions (H1) and (H3) can be considered as separate hypotheses but in the case of conditions (H2) and (H4), it is necessary to use additional assumptions.

This topic is discussed in detail in [25] and it was shown that when $A^{-1}N_2$ is a nonsingular matrix, which corresponds to the nonsingularity of N_2, conditions (H2) or (H4) are sufficient hypotheses. Song [12] showed that if at least one of $A^{-1}N_1$ and $A^{-1}N_2$ is an irreducible matrix, then Theorem 4.18 is valid with items (b) and (d) as well.

As can be seen in the scheme of condition implications, condition (D) implies further conditions (K1), (K2), (K3) and (K4) equivalent to (L1), (L2), (L3), and (L4), respectively. Since $A^{-1}M_1$ and $A^{-1}M_2$ are nonsingular matrices, conditions (K) can be used as separate hypotheses of weak splittings in comparison theorems proven by means of the methodology presented in [25]. In the case of conditions (L) the nonsingular matrices $I + A^{-1}N_1$ and $I + A^{-1}N_2$ can be extracted in both hand-sides of the inequality, which allows us to prove comparison theorems under the separate hypotheses (L) using again the same methodology [25].

Further generalizations for weaker splittings of $A = M_1 - N_1 = M_2 - N_2$ with $A^{-1}N_1 \ge 0$ and $A^{-1}N_2 \ge 0$ may be proven under composed hypotheses derived from some power combinations of conditions (H) and (K) or (L).

Comparison theorems for weaker splittings of the first type are considered in Miller and Neumann [10] with the following hypothesis

$$(A^{-1}N_1)^{i+j} \le (A^{-1}N_1)^i (A^{-1}N_2)^j, \tag{4.33}$$

and in Song [12] for such hypotheses as, for instance

$$(A^{-1}M_1)^{i+j} \le (A^{-1}M_1)^i (A^{-1}M_2)^j, \tag{4.34}$$

where $i \ge 1$ and $j \ge 1$. In the case of condition (4.34), Song [12] assumed that nonsingular matrices $A^{-1}M_1 \ge 0$ and $A^{-1}M_2 \ge 0$ are irreducible. However, as is shown in [25] only the nonsingularity of these matrices is a sufficient condition for proving the inequality $\varrho(M_1^{-1}N_1) \le \varrho(M_2^{-1}N_2)$.

5. Conclusion. The splittings defined in the successive items of Definition 1.13 extend successively a class of splittings of $A = M - N$ for which the matrices N and M^{-1} may lose the properties of nonnegativity. Distinguishing both types of weak nonnegative and weaker splittings leads to further extensions allowing us to analyze cases when $M^{-1}N$ may have negative entries even if NM^{-1} is a nonnegative matrix.

Conditions ensuring that a splitting of $A = M - N$ will be convergent are discussed in Section 1 for a general case. The main result of this section, Theorem 1.1, shows that for an arbitrary splitting of $A = M - N$ with the nonsingular matrices A and M, the matrices $M^{-1}N$ and $A^{-1}N$ (as well as NM^{-1} and NA^{-1}) commute. These commutative properties of $M^{-1}N$ and $A^{-1}N$ allowed us to determine the dependence of the eigenvalue spectra of both matrices, as is shown in the results of Section 1.

As follows from Theorem 3.1, the splittings defined in first three items of Definition 1.13 are convergent if and only if $A^{-1} \ge 0$, which means that both conditions $A^{-1} \ge 0$ and $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$ are equivalent. In the case of weak and weaker splittings, the assumption $A^{-1} \ge 0$ is not a sufficient condition in order to ensure the convergence of a given splitting of A; it is also possible to construct a convergent weak or weaker splitting when $A^{-1} \not\ge 0$. Moreover, the conditions $A^{-1}N \ge 0$ or $NA^{-1} \ge 0$ may not ensure that a given splitting of A will be a weak or weaker splitting. As can be seen from Theorem 4.1, for a convergent weaker splitting there are three conditions $M^{-1}N \ge 0$ (or $NM^{-1} \ge 0$), $A^{-1}N \ge 0$ (or $NA^{-1} \ge 0$), and $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$, and any two conditions imply the third; however, as was already mentioned in Section 4, two last conditions may imply, not necessarily a weak splitting. Thus, the criteria for the construction of convergent weak or weaker splittings of even monotone matrices remain an open question.

Comparison theorems, proven under the progressively weakening conditions presented in the scheme of condition implications shown in Figure 2.1, provide successive generalizations, but it is accompanied with an increased complexity in the verification of hypotheses.

The conditions $N_2 \ge N_1$ and progressively weaker $M_1^{-1} \ge M_2^{-1}$ may be considered as natural conditions appearing in many applications. It should be emphasized that for verifying the last condition, it is not always necessary to compute inverses because the validity of this inequality can be very often deduced from the structure of the matrices M_1 and M_2 (cf. [20, Section 4], which justifies a natural character of this condition.

For the verification of condition (E) $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$ (and similar conditions of Lemmas 3.4 and 4.4) equivalent to (D) $A^{-1}M_2 \geq A^{-1}M_1 \geq 0$, the Csordas-Varga's condition (G) $A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1} \geq 0$ equivalent to Song's condition (F) $A^{-1}M_2A^{-1} \geq A^{-1}M_1A^{-1} \geq 0$ as well as conditions (H1) to (H4) used by Beauwens [1] in Theorem 4.15, it is necessary to know explicitly the matrix A^{-1} which can be a cumbersome or impractical task when the matrix A has a large order. On the other hand, when A^{-1} is known, the solution can be obtained directly from the equation $x = A^{-1}b$, and the convergence analysis based on the above conditions becomes an aimless task.

However, condition (E) $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$ (as well as similar conditions of Lemmas 3.4 and 4.4) has an important theoretical meaning. Just, showing the validity of this condition allows us to prove many comparison theorems.

Marek and Szyld [9] introduced conditions with verification based on the knowledge of eigenvectors of iterative matrices, and similar conditions are also used as hypotheses of comparison theorems in [20]. However, the arithmetical effort for verification of such hypotheses may be comparable to the arithmetical effort required for the iterative or direct solution of the equation $Ax = b$. The hypotheses of Theorem 4.14 are based on the knowledge of the eigenvectors $x \geq 0$ and/or $z \geq 0$ such that $M_1^{-1}N_1x = \varrho(M_1^{-1}N_1)x$ and $M_2^{-1}N_2z = \varrho(M_2^{-1}N_2)z$. If the condition $N_1x \geq 0$ is not satisfied, then it is necessary to find the second eigenvector $z \geq 0$, that is, both spectral radii $\varrho(M_1^{-1}N_1)$ and $\varrho(M_2^{-1}N_2)$ are known now, and their direct comparison provides the sought result, not necessarily satisfying the theorem thesis, but this weakens the meaning of such theorems in applications. As was already pointed out, in the case of monotone matrices A, the Marek-Szyld's Theorem 4.14 reduces to simpler Theorems 4.5 and/or 4.6.

Csordas and Varga [4], introducing condition (2.21), that is,

$$(A^{-1}N_2)^jA^{-1} \geq (A^{-1}N_1)^jA^{-1} \geq 0, \quad j \geq 1 \tag{5.1}$$

for regular splittings, in some sense opened a new category of composed conditions, and their work was continued by Miller and Neumann [10], who considered hypothesis (4.14), and by Song [12, 13] in the case of hypothesis (4.15), where $i \geq 1$ and $j \geq 1$. However, the authors of [10, 12, 13] did not give any simple examples of splittings showing that conditions (4.14) and (4.15) are not satisfied with $i = 1$ and $j = 1$ but they hold for $i > 1$ and/or $j > 1$. Song [12, 13] illustrates his results only in examples of regular splittings, with $j = 1$ and $l = 1$, for which Varga's Theorem 2.2 is satisfied. Thus, the results obtained under these hypotheses have only a theoretical meaning.

Finally, it is worth to mention that only the comparison theorems based on the conditions $N_2 \geq N_1$ and $M_1^{-1} \geq M_2^{-1}$ found applications in actual practice [14, 15, 16, 20, 30].

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