

MOTION OF TWO POINT VORTICES IN A STEADY, LINEAR, AND ELLIPTICAL FLOW

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ABSTRACT. For a pair of point vortices in an inviscid, incompressible fluid in the plane, the relative and absolute motion are determined when the vortices move under the influence of (1) each other, and (2) a steady, linear, and elliptical background flow.

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1. Introduction. The *point vortex model* [1, 4, 5] is an idealization of the motion of a collection of vortices in an inviscid, incompressible fluid in the plane. Each vortex is assumed to be a point, and to induce in the surrounding fluid a velocity field, namely that of a Rankine vortex whose core has shrunk to a point. Each such point P moves with a velocity equal to the sum of the velocities induced by the other points, and the velocity field induced by P moves, without change of form, with the same velocity as P itself.

We investigate the absolute and relative motion in the plane of a pair of point vortices that are embedded in a steady flow whose velocity field has the form

$$\langle -\alpha y, \beta x \rangle, \quad (1.1)$$

where α and β are constants such that $\alpha \geq \beta > 0$. The flow (1.1) carries fluid particles counterclockwise around the origin, in elliptical trajectories. Kimura and Hasimoto [3] have analyzed a similar problem in which two vortices move in a simple shear flow $\langle \alpha y, 0 \rangle$. They require their vortices to be identical; here that requirement is dropped.

Here are the basic equations and notation needed for our analysis.

First, we need some information about the flow (1.1) (henceforth called the “background flow”). The position $\langle x, y \rangle$ of a given fluid particle in the background flow satisfies the equations

$$\frac{dx}{dt} = -\alpha y, \quad \frac{dy}{dt} = \beta x, \quad (1.2)$$

which have a general solution

$$x = x_0 \cos \omega t - D y_0 \sin \omega t, \quad y = D^{-1} x_0 \sin \omega t + y_0 \cos \omega t, \quad (1.3)$$

where $x_0 = x(0)$, $y_0 = y(0)$, and

$$D = \sqrt{\frac{\alpha}{\beta}}, \quad \omega = \sqrt{\alpha\beta}. \quad (1.4)$$

Thus, a fluid particle that begins at (x_0, y_0) will complete one counterclockwise revolution around the ellipse $x^2/\alpha + y^2/\beta = x_0^2/\alpha + y_0^2/\beta$ in time $2\pi/\sqrt{\alpha\beta}$.

It follows from (1.3) that the linear transformation $L_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by

$$L_t \left(\begin{bmatrix} X \\ Y \end{bmatrix} \right) = \begin{bmatrix} \cos \omega t & -D \sin \omega t \\ D^{-1} \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \tag{1.5}$$

takes as input the location of a given fluid particle in the background flow at time 0, and gives as output the particle's location at time t . The inverse transformation

$$L_t^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \cos \omega t & D \sin \omega t \\ -D^{-1} \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{1.6}$$

takes as input the location of a given fluid particle in the background flow at time t , and gives as output the particle's location at time 0.

Next, we introduce the equations of motion of the vortices. Denote by $\langle x_j, y_j \rangle$ ($j = 1, 2$) the position of the j th vortex, and put

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1.7}$$

Then, because the velocity of each vortex is the sum of the background flow's velocity and the velocity induced by the other vortex, the vortices' positions satisfy the following differential equations:

$$\frac{dx_1}{dt} = \kappa_2 \frac{y_2 - y_1}{r^2} - \alpha y_1; \tag{1.8}$$

$$\frac{dy_1}{dt} = -\kappa_2 \frac{x_2 - x_1}{r^2} + \beta x_1; \tag{1.9}$$

$$\frac{dx_2}{dt} = -\kappa_1 \frac{y_2 - y_1}{r^2} - \alpha y_2; \tag{1.10}$$

$$\frac{dy_2}{dt} = \kappa_1 \frac{x_2 - x_1}{r^2} + \beta x_2; \tag{1.11}$$

here κ_1 and κ_2 are nonzero constants.

Finally, to obtain differential equations for the vortices' *relative* position, we first define

$$\xi = x_2 - x_1, \quad \eta = y_2 - y_1, \quad \kappa = \kappa_1 + \kappa_2; \tag{1.12}$$

then, by subtracting (1.8) from (1.10) and (1.9) from (1.11), we get

$$\frac{d\xi}{dt} = -\left(\frac{\kappa}{r^2} + \alpha\right)\eta, \tag{1.13a}$$

$$\frac{d\eta}{dt} = \left(\frac{\kappa}{r^2} + \beta\right)\xi. \tag{1.13b}$$

The system (1.13) has a Hamiltonian

$$H = -\frac{1}{2}[\kappa \log(\xi^2 + \eta^2) + \beta\xi^2 + \alpha\eta^2]; \tag{1.14}$$

that is, $\partial H/\partial \eta$ equals the right-hand side of (1.13a) and $-\partial H/\partial \xi$ equals the right-hand side of (1.13b). Each solution curve of (1.13) is contained in a level curve of H . (Cf. [6, pages 43–45] for an introduction to Hamiltonians.)

In polar coordinates r and θ defined by

$$\xi = r \cos \theta, \quad \eta = r \sin \theta, \tag{1.15}$$

where r satisfies (1.7), equations (1.13) and (1.14) take the form

$$\frac{dr}{dt} = -2^{-1}(\alpha - \beta)r \sin 2\theta, \tag{1.16a}$$

$$\frac{d\theta}{dt} = \frac{\kappa}{r^2} + \alpha \sin^2 \theta + \beta \cos^2 \theta, \tag{1.16b}$$

$$H = -2^{-1}[2\kappa \log r + \beta r^2 \cos^2 \theta + \alpha r^2 \sin^2 \theta]. \tag{1.17}$$

We are now ready to begin our analysis. In Section 2, we consider absolute motion; we consider relative motion in Sections 3.1, 3.2, 3.3, and 3.4. The character of the relative motion depends on whether $\alpha = \beta$ (when the background flow is solid-body rotation) or $\alpha > \beta$ (when the background flow is elliptical but not circular); in the latter case the behavior depends on the sign of κ .

2. Absolute motion. Theorems 2.1 and 2.2 below describe the absolute motion in the cases $\kappa \neq 0$ and $\kappa = 0$, respectively.

For $\kappa \neq 0$, the center of vorticity of the vortices $\langle x_j, y_j \rangle$ ($j = 1, 2$) is defined to be $\langle x_c, y_c \rangle$, where

$$x_c = \kappa^{-1}(\kappa_1 x_1 + \kappa_2 x_2), \quad y_c = \kappa^{-1}(\kappa_1 y_1 + \kappa_2 y_2). \tag{2.1}$$

THEOREM 2.1. Fix α and β , where $\alpha \geq \beta > 0$, and let $\langle x_j, y_j \rangle$ ($j = 1, 2$) be a solution of the system (1.8), (1.9), (1.10), and (1.11). If κ defined by (1.12) is nonzero, then the center of vorticity moves with the background flow.

PROOF. By computing $\kappa^{-1}\{\kappa_1[(1.8)] + \kappa_2[(1.10)]\}$, we find that $dx_c/dt = -\alpha y_c$. Similarly, $dy_c/dt = \beta y_c$. Thus, since the background flow is given by (1.1), the proof is complete. \square

THEOREM 2.2. Fix α and β , with $\alpha \geq \beta > 0$, and pick real numbers κ_1 and κ_2 such that $\kappa = \kappa_1 + \kappa_2 = 0$. Define D , ω , and L_t by (1.4) and (1.5). Finally, choose real numbers X_1, Y_1, X_2 , and Y_2 , with $(X_2 - X_1)^2 + (Y_2 - Y_1)^2 \neq 0$, and set $\xi_0 = X_2 - X_1$ and $\eta_0 = Y_2 - Y_1$.

Then the system (1.7), (1.8), (1.9), (1.10), and (1.11) has a unique solution satisfying $x_j(0) = X_j$ and $y_j(0) = Y_j$ ($j = 1, 2$); that solution is

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = L_t \left(\begin{bmatrix} X_j \\ Y_j \end{bmatrix} \right) + \frac{\kappa_1}{D^{-1}\xi_0^2 + D\eta_0^2} \left\{ tL_t \left(\begin{bmatrix} -D\eta_0 \\ D^{-1}\xi_0 \end{bmatrix} \right) + G(t)L_t \left(\begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} \right) \right\}, \tag{2.2}$$

where

$$G(t) = -\frac{D - D^{-1}}{2(\alpha - \beta)} \log \left\{ (\xi_0^2 + \eta_0^2)^{-1} \cdot [(\xi_0^2 + \eta_0^2) \cos^2 \omega t - (D - D^{-1})\xi_0 \eta_0 \sin 2\omega t + (D^{-2}\xi_0^2 + D^2\eta_0^2) \sin^2 \omega t] \right\}. \tag{2.3}$$

PROOF. We will rewrite the system (1.8), (1.9), (1.10), and (1.11) in terms of new variables \hat{x}_j and \hat{y}_j defined by

$$\begin{bmatrix} \hat{x}_j \\ \hat{y}_j \end{bmatrix} = L_t^{-1} \left(\begin{bmatrix} x_j \\ y_j \end{bmatrix} \right). \tag{2.4}$$

We hope in this way to simplify the system by eliminating (or at least reducing) the effect of the background flow.

To convert (1.8), (1.9), (1.10), and (1.11) to the new variables, we first rewrite (2.4) as

$$\hat{x}_j = x_j \cos \omega t + D y_j \sin \omega t, \quad \hat{y}_j = -D^{-1} x_j \sin \omega t + y_j \cos \omega t. \tag{2.5}$$

We then differentiate the four equations in (2.5) with respect to t , use (1.8), (1.9), (1.10), and (1.11) to eliminate the derivatives of x_j and y_j , and apply (1.12), (1.4), and the condition $\kappa = 0$; the result is

$$\frac{d\hat{x}_j}{dt} = \kappa_1 \frac{(-\eta \cos \omega t + D\xi \sin \omega t)}{r^2}, \quad \frac{d\hat{y}_j}{dt} = \kappa_1 \frac{(D^{-1}\eta \sin \omega t + \xi \cos \omega t)}{r^2}. \tag{2.6}$$

Now by (1.13),

$$\frac{d\xi}{dt} = -\alpha\eta, \quad \frac{d\eta}{dt} = \beta\xi. \tag{2.7}$$

This last system is just (1.2) with x and y replaced by ξ and η ; thus, by (1.3), the definitions of ξ_0 and η_0 , and (1.12), the general solution of (2.7) is

$$\xi = \xi_0 \cos \omega t - D\eta_0 \sin \omega t, \quad \eta = D^{-1}\xi_0 \sin \omega t + \eta_0 \cos \omega t. \tag{2.8}$$

After solving (2.8) for $\cos \omega t$ and $\sin \omega t$ and substituting the result into (2.6), we obtain

$$\begin{aligned} \frac{d\hat{x}_j}{dt} &= \kappa_1 (D^{-1}\xi_0^2 + D\eta_0^2)^{-1} \left[-D\eta_0 + (D - D^{-1})\xi_0 \frac{\xi\eta}{(\xi^2 + \eta^2)} \right], \\ \frac{d\hat{y}_j}{dt} &= \kappa_1 (D^{-1}\xi_0^2 + D\eta_0^2)^{-1} \left[D^{-1}\xi_0 + (D - D^{-1})\eta_0 \frac{\xi\eta}{(\xi^2 + \eta^2)} \right]. \end{aligned} \tag{2.9}$$

But by (2.7), $(d/dt)(\xi^2 + \eta^2) = -2(\alpha - \beta)\xi\eta$. This last equation allows us to integrate (2.9), after which, using (2.3) and (2.8), we find that

$$\begin{aligned} \hat{x}_j &= \hat{x}_j(0) + \kappa_1 (D^{-1}\xi_0^2 + D\eta_0^2)^{-1} [-D\eta_0 t + \xi_0 G(t)], \\ \hat{y}_j &= \hat{y}_j(0) + \kappa_1 (D^{-1}\xi_0^2 + D\eta_0^2)^{-1} [D^{-1}\xi_0 t + \eta_0 G(t)]. \end{aligned} \tag{2.10}$$

Finally, we put (2.10) into matrix form and apply L_t to both sides; (2.2) then follows because, by (2.5), $\hat{x}_j(0) = x_j(0)$ and $\hat{y}_j(0) = y_j(0)$. This completes the proof of Theorem 2.2. \square

COROLLARY 2.3. *Under the hypotheses of Theorem 2.2,*

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = \frac{\kappa_1 t}{D^{-1}\xi_0^2 + D\eta_0^2} L_t \left(\begin{bmatrix} -D\eta_0 \\ D^{-1}\xi_0 \end{bmatrix} \right) + O(1). \tag{2.11}$$

PROOF. This follows trivially from (2.2), (2.3), and (2.5). □

From Corollary 2.3, along with (1.4) and the interpretation of L_t given in Section 1, it follows that, when $\kappa = 0$, the two vortices move in a spiral around and away from the origin. More precisely, each vortex stays a bounded distance from a moving point which behaves as follows:

- (a) it moves counterclockwise around the origin with period $2\pi/\sqrt{\alpha\beta}$;
- (b) it lies, at time t , on the ellipse

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = (\kappa_1 t)^2 \frac{(D^2 \eta_0^2 / \alpha + D^{-2} \xi_0^2 / \beta)}{(D^{-1} \xi_0^2 + D \eta_0^2)^2}. \tag{2.12}$$

3. Relative motion

3.1. The case $\alpha = \beta$. The following theorem is a direct consequence of (1.16).

THEOREM 3.1. Fix real numbers α, β , and κ such that $\alpha = \beta > 0$, and consider a pair of vortices whose positions satisfy equations (1.7), (1.8), (1.9), (1.10), and (1.11). The line segment joining the two vortices has constant length and rotates with constant, possibly zero, angular velocity $\kappa r_0^{-2} + \alpha$, where r_0 is the segment's length.

3.2. The case $\kappa = 0$. From the proof of Theorem 2.2 (see (2.7) and (2.8)) we have the following result.

THEOREM 3.2. Fix real numbers α, β, κ_1 , and κ_2 such that $\alpha \geq \beta > 0$ and $\kappa = \kappa_1 + \kappa_2 = 0$, and consider a pair of vortices whose positions satisfy equations (1.7), (1.8), (1.9), (1.10), and (1.11). In (ξ, η) -coordinates (1.12), the second vortex moves around the first, with period $2\pi/\sqrt{\alpha\beta}$, on the ellipse $\xi^2/\alpha + \eta^2/\beta = \xi_0^2/\alpha + \eta_0^2/\beta$.

Theorems 3.1 and 3.2 agree in the case where $\alpha = \beta$ and $\kappa = 0$.

3.3. The case $\alpha > \beta, \kappa > 0$. Our investigation of the motion when $\alpha > \beta$ and $\kappa \neq 0$ depends on understanding the level curves of the Hamiltonian H in (1.17), which in turn requires us to analyze the function

$$g_w(r) = (\alpha - \beta)^{-1} [4r^{-2}(\kappa \log r + w) + \alpha + \beta]. \tag{3.1}$$

For $\kappa > 0$, the following lemma gives the information we need.

LEMMA 3.3. Pick α, β , and κ , with $\alpha > \beta > 0$ and $\kappa > 0$, and define $g_w(r)$ by (3.1). Set

$$r^* = r^*(w) = e^{1/2-w/\kappa}. \tag{3.2}$$

Then,

- (a) g_w is increasing on $(0, r^*]$; $\lim_{r \rightarrow 0^+} g_w(r) = -\infty$; $g_w(r) > 1$ for $r \geq r^*$;
- (b) given a number u in $[-1, 1]$, the equation $g_w(r) = u$ has exactly one solution r in $(0, \infty]$, namely, $r = f_w(u)$, where f_w is the inverse function of the restriction of $g_w(r)$ to the interval $(0, r^*]$;
- (c) $0 < f_w(u) \leq r^*$ for all real w and all u in $[-1, 1]$;
- (d) for each real w , and each fixed u in $[-1, 1]$, $f_w(u)$ is a decreasing function of w ;
- (e) $\lim_{w \rightarrow -\infty} f_w(u) = 0$ and $\lim_{w \rightarrow -\infty} f_w(u) = +\infty$, uniformly for u in $[-1, 1]$.

PROOF. The first two statements in (a) are obvious; the third holds because (i) g_w is decreasing for $r \geq r^*$, while (ii) $\lim_{r \rightarrow \infty} g_w(r) > 1$. Part (b) follows immediately from (a), and (c) from (b).

To prove (d), we fix u in $[-1, 1]$ and pick real numbers z and w such that $z < w$. Then $f_z(u) > f_w(u)$; otherwise, since $g_w(r)$ is an increasing function of $r \leq r^*$ for fixed w , and an increasing function of w for fixed r , we would have

$$u = g_z(f_z(u)) \leq g_z(f_w(u)) < g_w(f_w(u)) = u. \tag{3.3}$$

The first limit in (e) is a consequence of (c) and (3.2). To establish the second limit, we first calculate, using (3.1), that $g_w(\sqrt[3]{-w}) \rightarrow -\infty$ as $w \rightarrow -\infty$; thus, when w is a sufficiently large negative, $u > g_w(\sqrt[3]{-w})$ for all u in $[-1, 1]$. The second limit then follows when we apply f_w to this last inequality. This completes the proof of (e) and of Lemma 3.3. \square

THEOREM 3.4. Fix real numbers α, β, κ_1 , and κ_2 such that $\alpha > \beta > 0$ and $\kappa = \kappa_1 + \kappa_2 > 0$, and consider a pair of vortices whose positions satisfy (1.7), (1.8), (1.9), (1.10), and (1.11). In (ξ, η) -coordinates, the second vortex moves around the first counterclockwise in a simple closed curve, with period

$$T = 8 \int_0^{\pi/2} \frac{d\theta}{2\kappa/[f_w(\cos 2\theta)]^2 + (\alpha + \beta) - (\alpha - \beta)\cos 2\theta}. \tag{3.4}$$

The period T is a decreasing function of w such that $T \rightarrow 2\pi/\sqrt{\alpha\beta}$ as $w \rightarrow -\infty$, and $T \rightarrow 0$ as $w \rightarrow \infty$. The maximum separation r of the vortices occurs when $\theta = 0, \pi$, and the minimum when $\theta = \pi/2, 3\pi/2$.

PROOF. The identities $\sin^2 \theta = (1 - \cos 2\theta)/2$ and $\cos^2 \theta = (1 + \cos 2\theta)/2$ allow us to rewrite (1.16b) and (1.17) as

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{\kappa}{r^2} + \frac{(\alpha + \beta)}{2} - \frac{[(\alpha - \beta)\cos 2\theta]}{2}, \\ H &= -\kappa \log r - \frac{(\alpha + \beta)r^2}{4} + \frac{[(\alpha - \beta)r^2 \cos 2\theta]}{4}. \end{aligned} \tag{3.5}$$

Using (3.1), the equation $H(r, \theta) = w$ can be rewritten as $g_w(r) = \cos 2\theta$, or, by Lemma 3.3(b), as

$$r = f_w(\cos 2\theta). \tag{3.6}$$

The latter is a simple closed curve, symmetric with respect to the ξ - and η -axes, and enclosing the origin. Each trajectory of (1.16) lies on a curve (3.6) for some w . By (3.5),

$$\frac{d\theta}{dt} > \frac{(\alpha + \beta)}{2} - \frac{(\alpha - \beta)}{2} > 0, \tag{3.7}$$

so the motion is counterclockwise. By (3.5) and (3.6), the period T is given by (3.4). By (3.4), along with Lemma 3.3(d), (e), T is a decreasing function of w such that $T \rightarrow 2\pi/\sqrt{\alpha\beta}$ as $w \rightarrow -\infty$, and $T \rightarrow 0$ as $w \rightarrow \infty$. Finally, the statements about the separation of the vortices (which, by symmetry, need only be verified for θ in $[0, \pi/2]$), follow from (1.16a) since the motion is counterclockwise. This completes the proof of Theorem 3.4. \square

Because f_w is a decreasing function of w , smaller values of w correspond to larger curves; that is, if $z < w$, then the curve $r = f_z(\cos 2\theta)$ encloses the curve $r = f_w(\cos 2\theta)$. Thus a consequence of [Theorem 3.4](#) is that, if two vortices are close to each other, then their period of rotation around each other is what it would be if there were no background flow, while, if the vortices are far apart, then that period is approximately what it would be if the vortices did not affect each others' motion.

3.4. The case $\alpha > \beta, \kappa < 0$. The Hamiltonian H defined by (1.14) has maxima at the points $\pm P$, where, in (ξ, η) coordinates, $P = (\sqrt{-\kappa/\beta}, 0)$. Also, H has saddle points at $\pm Q$, where $Q = (0, \sqrt{-\kappa/\alpha})$. The points $\pm P$ and $\pm Q$ are the only stationary points of the system (1.13). If the pair of vortices begin with relative position given by $\pm P$ or $\pm Q$, then they maintain that relative position while their center of vorticity revolves about the origin. The values of H at those points are

$$M \equiv H(\pm P) = \frac{\kappa[1 - \log(-\kappa/\beta)]}{2} > \frac{\kappa[1 - \log(-\kappa/\alpha)]}{2} = H(\pm Q) \equiv S, \tag{3.8}$$

and the behavior of a trajectory lying on a level curve $H = w$ depends on where w lies in relation to M and S . As in [Section 3.3](#), we use the function g_w of (3.1) to explore that behavior. The following lemma gives the information we need; I omit the proof, which is similar to that of [Lemma 3.3](#).

LEMMA 3.5. *Pick α, β , and κ , with $\alpha > \beta > 0$ and $\kappa < 0$; define $g_w(r), r^*, M$, and S by (3.1), (3.2), and (3.8). Then,*

- (a) g_w is decreasing on $(0, r^*]$ and increasing on $[r^*, \infty)$; $\lim_{r \rightarrow 0^+} g_w(r) = +\infty$; $\lim_{r \rightarrow \infty} g_w(r) > 1$;
- (b) $g_w(r^*(w))$ is an increasing function of w such that (i) $g_M(r^*(M)) = 1$; (ii) $-1 < g_w(r^*(w)) < 1$ if $S < w < M$; (iii) $g_S(r^*(S)) = -1$; and (iv) $g_w(r^*(w)) < -1$ if $w < S$;
- (c) given $w < M$ and u in $(g(r^*), 1]$, the equation $g_w(r) = u$ has exactly two solutions r in $(0, \infty]$, namely, $r_1 = f_w(u)$ and $r_2 = h_w(u)$, where f_w and h_w are the inverse functions of the restrictions of $g_w(r)$ to the intervals $(0, r^*]$ and $[r^*, \infty)$; if $u = g(r^*)$ then the equation has exactly one solution, namely $f_w(u) = h_w(u) = r^*$;
- (d) $0 < f_w(u) < r^*$ and $h_w(u) > r^*$ for all $w < M$ and all u in $(g(r^*), 1]$;
- (e) for each fixed u in $[-1, 1]$, $f_w(u)$ is an increasing function of w and $h_w(u)$ is a decreasing function of w ;
- (f) $\lim_{w \rightarrow -\infty} f_w(u) = 0$ and $\lim_{w \rightarrow -\infty} h_w(u) = \infty$.

The following definitions are helpful in describing the level curves of H . With ξ and η given by (1.12), and polar coordinates r, θ given by (1.7) and (1.15), we define four curves in the $\xi\eta$ -plane (see [Figure 3.1](#)):

$$\begin{aligned} C_1 : r = f_S(\cos 2\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}; & \quad C_2 : r = h_S(\cos 2\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}; \\ C_3 : r = f_S(\cos 2\theta), \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}; & \quad C_4 : r = h_S(\cos 2\theta), \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}. \end{aligned} \tag{3.9}$$

We also define four open, connected sets: R_1 is the inside of $C_1 \cup C_2 \cup \{Q, -Q\}$, excluding P ; R_2 is the inside of $C_3 \cup C_4 \cup \{Q, -Q\}$, excluding $-P$; R_3 is the inside

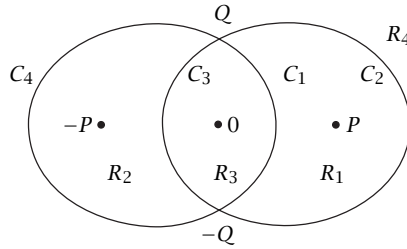


FIGURE 3.1

of $C_1 \cup C_3 \cup \{Q, -Q\}$, excluding the origin; and R_4 is the outside of $C_2 \cup C_4 \cup \{Q, -Q\}$. (The curve $C_1 \cup C_2 \cup \{Q, -Q\}$ encloses P because $f_5(1) < \sqrt{-\kappa/\beta} < h_5(1)$ by (3.8) and Lemma 3.5(a).) Then $H(R_1) = H(R_2) = (S, M)$ and $H(R_2) = H(R_3) = (-\infty, S)$; this results from (3.8) along with (i) $\lim_{(\xi, \eta) \rightarrow -\infty} H(\xi, \eta) = -\infty$, (ii) $H \equiv S$ on $\{Q, -Q\} \cup \bigcup_{i=1}^4 C_i$, and (iii) H has no critical points in $\bigcup_{i=1}^4 R_i$.

By the Poincaré-Bendixson theorem and a corollary [2, Theorem, page 248 and Theorem 3, page 252], each region R_i is a union of periodic orbits of (1.13). The following theorem gives more detail.

THEOREM 3.6. Fix real numbers α, β, κ_1 , and κ_2 such that $\alpha > \beta > 0$ and $\kappa = \kappa_1 + \kappa_2 < 0$, define M and S by (3.8), let H be given by (1.14), (1.15), (1.16), and (1.17), and consider a pair of vortices whose positions satisfy (1.7), (1.8), (1.9), (1.10), and (1.11). For those vortices, define ξ and η by (1.12), and put $\xi_0 = \xi(0)$ and $\eta_0 = \eta(0)$. Then:

(a) If $(\xi_0, \eta_0) \in R_1 \cup R_2$, then the line segment joining the vortices periodically rocks from side to side in such a way that its maximum and minimum angles with the positive ξ -direction are $\pm\theta^*$, where

$$\theta^* = 2^{-1} \cos^{-1} g_w(r^*) = 2^{-1} \cos^{-1} [(\alpha - \beta)^{-1} (2\kappa e^{2w/\kappa-1} + \alpha + \beta)] \tag{3.10}$$

and $w = H(\xi_0, \eta_0)$. The period is

$$T = 4(\alpha - \beta)^{-1} \int_{f_w(1)}^{h_w(1)} r^{-1} \{1 - [g_w(r)]^2\}^{-1/2} dr. \tag{3.11}$$

The maximum and minimum length of the segment occur at the two instants in the cycle when the segment is horizontal.

(b) If $(\xi_0, \eta_0) \in \bigcup_{i=1}^4 C_i$ then, as $t \rightarrow \infty$, the line segment joining the vortices tends to a vertical position. The segment's length approaches $\sqrt{-\kappa/\alpha}$.

(c) If $(\xi_0, \eta_0) \in R_3 \cup R_4$ then, in (ξ, η) -coordinates, the second vortex moves around the first in a simple closed curve.

If $(\xi_0, \eta_0) \in R_3$, then the motion is clockwise, with period

$$T = -8 \int_0^{\pi/2} \left\{ \frac{2\kappa}{[f_w(\cos 2\theta)]^2} + (\alpha + \beta) - (\alpha - \beta) \cos 2\theta \right\} d\theta. \tag{3.12}$$

The period T is an increasing function of w such that $T \rightarrow 0$ as $w \rightarrow -\infty$. (That is, the period is small when the vortices are close to each other.) The maximum separation r of the vortices occurs when $\theta = \pi/2, 3\pi/2$, and the minimum when $\theta = 0, \pi$.

If $(\xi_0, \eta_0) \in R_4$, then the motion is counterclockwise, and the period is

$$T = 8 \int_0^{\pi/2} \left\{ \frac{2\kappa}{[h_w(\cos 2\theta)]^2} + (\alpha + \beta) - (\alpha - \beta) \cos 2\theta \right\} d\theta. \tag{3.13}$$

The period T is an increasing function of w such that $T \rightarrow 2\pi/\sqrt{\alpha\beta}$ as $w \rightarrow -\infty$. (That is, the period is close to the background flow period when the vortices are far apart.) The maximum separation r of the vortices occurs when $\theta = 0, \pi$, and the minimum when $\theta = \pi/2, 3\pi/2$.

PROOF. In proving (a), we can assume that $(\xi_0, \eta_0) \in R_1$; this is because (1.13) is unchanged when ξ and η are replaced by $-\xi$ and $-\eta$. Then $(\xi(t), \eta(t)) \in R_1$ for all t . The trajectory is contained in a level set $H = w$ such that $S < w < M$. As in the proof of Theorem 3.4, the equation $H = w$ can be written in the form $g_w(r) = \cos 2\theta$. By Lemma 3.5(a), (b), and (c), this last equation has solutions r if and only if

$$\cos 2\theta \geq g_w(r^*). \tag{3.14}$$

Since $R_1 \subset \{-\pi/2 < \theta < \pi/2\}$, the solutions are

$$r = f_w(\cos 2\theta), \quad r = h_w(\cos 2\theta), \quad \text{where } \theta \in [-\theta^*, \theta^*]. \tag{3.15}$$

Equations (3.15) together represent a simple closed curve; this is a consequence of Lemma 3.5(d) and the equation (from (3.10)) $f_w(\cos 2\theta^*) = h_w(\cos 2\theta^*)$. Therefore the motion is periodic, with the maximum and minimum values of θ stated in part (a) of Theorem 3.6. To verify the formula (3.11) for the period, we first rewrite (1.16a), for θ in $[0, \theta^*]$, as

$$\frac{dr}{dt} = -2^{-1}(\alpha - \beta)r\sqrt{1 - [g_w(r)]^2}. \tag{3.16}$$

We then define T_f and T_h to be the amounts of time spent by the second vortex in the parts of the upper half-plane $\{\eta > 0\}$ where $r < r^*$ and $r > r^*$, respectively. After separating variables in (3.16), we find that

$$T_f = \frac{2}{\alpha - \beta} \int_{f_w(1)}^{r^*} \frac{dr}{r\sqrt{1 - [g_w(r)]^2}}, \quad T_h = \frac{2}{\alpha - \beta} \int_{r^*}^{h_w(1)} \frac{dr}{r\sqrt{1 - [g_w(r)]^2}}, \tag{3.17}$$

which yields (3.11). Finally, by (3.15) and Lemma 3.5(a), the smallest and largest values of r are, respectively, $f_w(1)$ and $h_w(1)$; these occur when $\theta = 0$. Thus the minimum and maximum separations of the vortices occur when the segment joining them is horizontal, and the proof of (a) is complete.

Part (b) is clear since the boundary of each curve C_i is $\{Q, -Q\}$.

We prove (c) only in the case where $(\xi_0, \eta_0) \in R_3$; the proof for (ξ_0, η_0) in R_4 is similar. Put $w = H(\xi_0, \eta_0)$. Then, since $w < S$, it follows from Lemma 3.5(b), (c) that the level set $H = w$ consists of two disjoint simple closed curves $r = f_w(\cos 2\theta)$ and $r = h_w(\cos 2\theta)$. By Lemma 3.5(d), the former is the one that lies in R_3 . By Lemma 3.5(a), $dr/d\theta > 0$ on the part of that curve in the first quadrant. But $dr/dt < 0$ there by (1.16a), so the motion is clockwise. The statements about the vortices' separation, and

the formula (3.12) for the period, are established as in the proofs of the corresponding facts in Theorem 3.4. The period T is an increasing function of w such that $T \rightarrow 0$ as $w \rightarrow -\infty$ by (3.12) and Lemma 3.5(e), (f). This completes the proof of Theorem 3.6. \square

Under the hypotheses of Theorem 3.6, the solutions of the linearization of (1.13) about $P = (\sqrt{-\kappa/\beta}, 0)$ have period $2\pi/\sqrt{2\beta(\alpha-\beta)}$. The following statements are probably true, but we have been unable to prove them: (i) if $w \in (S, M)$, then the period T is a decreasing function of w such that $\lim_{w \rightarrow M^-} T = 2\pi/\sqrt{2\beta(\alpha-\beta)}$; (ii) $\lim_{w \rightarrow S} T = \infty$.

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