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ON n-FOLD FUZZY IMPLICATIVE/COMMUTATIVE IDEALS OF BCK-ALGEBRAS

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Abstract. We consider the fuzzification of the notion of an n-fold implicative ideal, an n-fold (weak) commutative ideal. We give characterizations of an n-fold fuzzy implicative ideal. We establish an extension property for n-fold fuzzy commutative ideals.

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1. Introduction. Huang and Chen [1] introduced the notion of n-fold implicative ideals and n-fold (weak) commutative ideals. The aim of this paper is to discuss the fuzzification of n-fold implicative ideals, n-fold commutative ideals and n-fold weak commutative ideals. We show that every n-fold fuzzy implicative ideal is an n-fold fuzzy positive implicative ideal, and so a fuzzy ideal, and give a condition for a fuzzy ideal to be an n-fold fuzzy implicative ideal. Using the level set, we provide a characterization of an n-fold fuzzy implicative ideal. We also give a condition for a fuzzy ideal to be an n-fold fuzzy (weak) commutative ideal. We show that every n-fold fuzzy positive implicative ideal which is an n-fold fuzzy weak commutative ideal is an n-fold fuzzy implicative ideal. Finally, we establish an extension property for n-fold fuzzy commutative ideals.

2. Preliminaries. We include some elementary aspects of BCK-algebras that are necessary for this paper, and for more details we refer to [1, 2, 4, 5]. By a *BCK-algebra* we mean an algebra $(X;*,0)$ of type $(2,0)$ satisfying the axioms:

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply $x=y$, for all $x,y,z\in X$.

We can define a partial ordering $\le$ on X by $x\le y$ if and only if $x*y=0$. In any BCK-algebra X, the following hold:

(P1) $x*0=x$,
(P2) $x*y\le x$,
(P3) $(x*y)*z=(x*z)*y$,
(P4) $(x*z)*(y*z)\le x*y$,
(P5) $x\le y$ implies $x*z\le y*z$ and $z*y\le z*x$.

Throughout, X will always mean a BCK-algebra unless otherwise specified. A nonempty subset I of X is called an *ideal* of X if it satisfies:

(I1) $0\in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A nonempty subset I of X is said to be an *implicative ideal* of X if it satisfies:

(I1) $0 \in I$,

(I3) $(x * (y * x)) * z \in I$ and $z \in I$ imply $x \in I$.

A nonempty subset I of X is said to be a *commutative ideal* of X if it satisfies:

(I1) $0 \in I$,

(I4) $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \to [0,1]$. For a fuzzy set μ in X and $t \in [0,1]$ define $U(\mu;t)$ to be the set $U(\mu;t) := \{x \in X \mid \mu(x) \geq t\}$.

A fuzzy set μ in X is said to be a *fuzzy ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Note that every fuzzy ideal μ of X is order reversing, that is, if $x \leq y$ then $\mu(x) \geq \mu(y)$.

A fuzzy set μ in X is called a *fuzzy implicative ideal* of X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F3) $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$ for all $x, y, z \in X$.

A fuzzy set μ in X is called a *fuzzy commutative ideal* of X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F4) $\mu(x * (y * (y * x))) \geq \min\{\mu((x * y) * z), \mu(z)\}$ for all $x, y, z \in X$.

3. n-fold fuzzy implicative ideals.

For any elements x and y of a BCK-algebra X, $x * y^n$ denotes

$$(\cdots((x * y) * y) * \cdots) * y \tag{3.1}$$

in which y occurs n times. Huang and Chen [1] introduced the concept of n-fold implicative ideals as follows.

Definition 3.1 (see [1]). A subset A of X is called an *n-fold implicative ideal* of X if

(I1) $0 \in A$,

(I5) $(x * (y * x^n)) * z \in A$ and $z \in A$ imply $x \in A$ for every $x, y, z \in X$.

We consider the fuzzification of the concept of n-fold implicative ideal.

Definition 3.2. A fuzzy set μ in X is called an *n-fold fuzzy implicative ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F5) $\mu(x) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\}$ for every $x, y, z \in X$.

Notice that the 1-fold fuzzy implicative ideal is a fuzzy implicative ideal.

Theorem 3.3. *Every n-fold fuzzy implicative ideal is a fuzzy ideal.*

Proof. The condition (F2) follows from taking $y = 0$ in (F5). □

The following example shows that the converse of Theorem 3.3 may not be true.

Example 3.4. Let $X = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, in which the operation $*$ is defined by $x * y = \max\{0, x - y\}$ for all $x, y \in X$. Then X is a BCK-algebra (see [1, Example 1.3]). Let μ be a fuzzy set in X given by $\mu(0) = t_0 > t_1 = \mu(x)$ for all $x(\neq 0) \in X$. Then μ is a fuzzy ideal of X. But μ is not a 2-fold fuzzy implicative ideal of X because

$$\mu(3) = t_1 < t_0 = \mu(0) = \min\{\mu((3 * (14 * 3^2)) * 0), \mu(0)\}. \tag{3.2}$$

We give a condition for a fuzzy ideal to be an n-fold fuzzy implicative ideal.

Theorem 3.5. *A fuzzy ideal μ of X is n-fold fuzzy implicative if and only if $\mu(x) \geq \mu(x * (y * x^n))$ for all $x, y \in X$.*

Proof. Necessity is by taking $z = 0$ in (F5). Suppose that a fuzzy ideal μ satisfies the inequality $\mu(x) \geq \mu(x * (y * x^n))$ for all $x, y \in X$. Then

$$\mu(x) \geq \mu(x * (y * x^n)) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\}. \tag{3.3}$$

Hence μ is an n-fold fuzzy implicative ideal of X. □

Theorem 3.6. *A fuzzy set μ in X is an n-fold fuzzy implicative ideal of X if and only if the nonempty level set $U(\mu; t)$ of μ is an n-fold implicative ideal of X for every $t \in [0, 1]$.*

Proof. Assume that μ is an n-fold fuzzy implicative ideal of X and $U(\mu; t) \neq \emptyset$ for every $t \in [0, 1]$. Then there exists $x \in U(\mu; t)$. It follows from (F1) that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in U(\mu; t)$. Let $x, y, z \in X$ be such that $(x * (y * x^n)) * z \in U(\mu; t)$ and $z \in U(\mu; t)$. Then $\mu((x * (y * x^n)) * z) \geq t$ and $\mu(z) \geq t$, which imply from (F5) that

$$\mu(x) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\} \geq t \tag{3.4}$$

so that $x \in U(\mu; t)$. Therefore $U(\mu; t)$ is an n-fold implicative ideal of X. Conversely, suppose that $U(\mu; t)(\neq \emptyset)$ is an n-fold implicative ideal of X for every $t \in [0, 1]$. For any $x \in X$, let $\mu(x) = t$. Then $x \in U(\mu; t)$. Since $0 \in U(\mu; t)$, we get $\mu(0) \geq t = \mu(x)$ and so $\mu(0) \geq \mu(x)$ for all $x \in X$. Now assume that there exist $a, b, c \in X$ such that

$$\mu(a) < \min\{\mu((a * (b * a^n)) * c), \mu(c)\}. \tag{3.5}$$

Selecting $s_0 = (1/2)(\mu(a) + \min\{\mu((a * (b * a^n)) * c), \mu(c)\})$, then

$$\mu(a) < s_0 < \min\{\mu((a * (b * a^n)) * c), \mu(c)\}. \tag{3.6}$$

It follows that $(a * (b * a^n)) * c \in U(\mu; s_0)$, $c \in U(\mu; s_0)$, and $a \notin U(\mu; s_0)$. This is a contradiction. Hence μ is an n-fold fuzzy implicative ideal of X. □

Definition 3.7 (see [3]). A fuzzy set μ in X is called an *n-fold fuzzy positive implicative ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F6) $\mu(x * y^n) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\}$ for all $x, y, z \in X$.

Lemma 3.8 (see [3, Theorem 3.13]). *Let μ be a fuzzy set in X. Then μ is an n-fold fuzzy positive implicative ideal of X if and only if the nonempty level set $U(\mu;t)$ of μ is an n-fold positive implicative ideal of X for every $t \in [0,1]$.*

Lemma 3.9 (see [1, Theorem 2.5]). *Every n-fold implicative ideal is an n-fold positive implicative ideal.*

Using Theorem 3.6 and Lemmas 3.8 and 3.9, we have the following theorem.

Theorem 3.10. *Every n-fold fuzzy implicative ideal is an n-fold fuzzy positive implicative ideal.*

4. n-fold fuzzy commutative ideals

Definition 4.1 (see [1]). A subset A of X is called an *n-fold commutative ideal* of X if

(I1) $0 \in A$,

(I6) $(x*y)*z \in A$ and $z \in A$ imply $x*(y*(y*x^n)) \in A$ for all $x,y,z \in X$.

A subset A of X is called an *n-fold weak commutative ideal* of X if

(I1) $0 \in A$,

(I7) $(x*(x*y^n))*z \in A$ and $z \in A$ imply $y*(y*x) \in A$ for all $x,y,z \in X$.

We consider the fuzzification of n-fold (weak) commutative ideals as follows.

Definition 4.2. A fuzzy set μ in X is called an *n-fold fuzzy commutative ideal* of X if

(F1) $\mu(0) \ge \mu(x)$ for all $x \in X$,

(F7) $\mu(x*(y*(y*x^n))) \ge \min\{\mu((x*y)*z),\mu(z)\}$ for all $x,y,z \in X$.

A fuzzy set μ in X is called an *n-fold fuzzy weak commutative ideal* of X if

(F1) $\mu(0) \ge \mu(x)$ for all $x \in X$,

(F8) $\mu(y*(y*x)) \ge \min\{\mu((x*(x*y^n))*z),\mu(z)\}$ for all $x,y,z \in X$.

Note that the 1-fold fuzzy commutative ideal is a fuzzy commutative ideal. Putting $y = 0$ and $y = x$ in (F7) and (F8), respectively, we know that every n-fold fuzzy commutative (or fuzzy weak commutative) ideal is a fuzzy ideal.

Theorem 4.3. *Let μ be a fuzzy ideal of X. Then*

(i) *μ is an n-fold fuzzy commutative ideal of X if and only if*

$$\mu(x*(y*(y*x^n))) \ge \mu(x*y) \quad \forall x,y \in X. \tag{4.1}$$

(ii) *μ is an n-fold fuzzy weak commutative ideal of X if and only if*

$$\mu(y*(y*x)) \ge \mu(x*(x*y^n)) \quad \forall x,y \in X. \tag{4.2}$$

Proof. The proof is straightforward. □

Lemma 4.4 (see [3, Theorem 3.12]). *A fuzzy set μ in X is an n-fold fuzzy positive implicative ideal of X if and only if μ is a fuzzy ideal of X in which the following inequality holds:*

(F9) $\mu((x*z^n)*(y*z^n)) \ge \mu((x*y)*z^n)\ \forall x,y,z \in X.$

Theorem 4.5. *If μ is both an n-fold fuzzy positive implicative ideal and an n-fold fuzzy weak commutative ideal of X, then it is an n-fold fuzzy implicative ideal of X.*

Proof. Let $x, y \in X$. Using Theorem 4.3(ii), Lemma 4.4, (P3), and (III), we have

$$\begin{aligned} \mu(x*(x*(y*x^n))) &\geq \mu((y*x^n)*((y*x^n)*x^n)) \\ &\geq \mu((y*(y*x^n))*x^n) \\ &= \mu((y*x^n)*(y*x^n)) \\ &= \mu(0). \end{aligned} \tag{4.3}$$

It follows from (F1) and (F2) that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x*(x*(y*x^n))), \mu(x*(y*x^n))\} \\ &\geq \min\{\mu(0), \mu(x*(y*x^n))\} \\ &= \mu(x*(y*x^n)) \end{aligned} \tag{4.4}$$

so from Theorem 3.5, μ is an n-fold fuzzy implicative ideal of X. □

Theorem 4.6 (extension property for n-fold fuzzy commutative ideals)**.** *Let μ and ν be fuzzy ideals of X such that $\mu(0) = \nu(0)$ and $\mu \subseteq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in X$. If μ is an n-fold fuzzy commutative ideal of X, then so is ν.*

Proof. Let $x, y \in X$. Taking $u = x*(x*y)$, we have

$$\begin{aligned} \nu(0) = \mu(0) &= \mu(u*y) \\ &\leq \mu(u*(y*(y*u^n))) \\ &\leq \nu(u*(y*(y*u^n))) \\ &= \nu((x*(x*y))*(y*(y*u^n))) \\ &= \nu((x*(y*(y*u^n)))*(x*y)). \end{aligned} \tag{4.5}$$

Since $x*(y*(y*x^n)) \leq x*(y*(y*u^n))$ and since ν is order reversing, it follows that

$$\begin{aligned} \nu(x*(y*(y*x^n))) &\geq \nu(x*(y*(y*u^n))) \\ &\geq \min\{\nu((x*(y*(y*u^n)))*(x*y)), \nu(x*y)\} \\ &\geq \min\{\nu(0), \nu(x*y)\} \\ &= \nu(x*y). \end{aligned} \tag{4.6}$$

Hence, by Theorem 4.3(i), ν is an n-fold fuzzy commutative ideal of X. □

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