L^∞-ERROR ESTIMATES FOR A CLASS OF SEMILINEAR ELLIPTIC VARIATIONAL INEQUALITIES AND QUASI-VARIATIONAL INEQUALITIES

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ABSTRACT. This paper deals with the finite element approximation of a class of variational inequalities (VI) and quasi-variational inequalities (QVI) with the right-hand side depending upon the solution. We prove that the approximation is optimally accurate in L^∞ combining the Banach fixed point theorem with the standard uniform error estimates in linear VIs and QVIs. We also prove that this approach extends successfully to the corresponding noncoercive problems.

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1. Introduction. In this paper, we study the finite element approximation of elliptic variational inequalities (VI) and quasi-variational inequalities (QVI) with nonlinear source terms. Let $\mathscr{A}$ be an elliptic second-order differential operator defined on a bounded smooth open domain Ω in $\mathbb{R}^N$, $N \geq 1$. These problems appear in the following formal framework:

- *The variational inequality*: find u such that

$$\mathscr{A}u \leq f(u), \quad u - \psi \leq 0, \quad \text{in } \Omega,$$
$$(\mathscr{A}u - f(u))(u - \psi) = 0, \quad \text{in } \Omega. \tag{1.1}$$

- *The quasi-variational inequality*: find u such that

$$\mathscr{A}u \leq f(u), \quad u - Mu \leq 0, \quad \text{in } \Omega,$$
$$(\mathscr{A}u - f(u))(u - Mu) = 0, \quad \text{in } \Omega, \tag{1.2}$$
$$u \geq 0 \quad \text{in } \Omega$$

with the addition of suitable boundary conditions.

The above VIs and QVIs may be of a great interest in stochastic control and decision sciences. The condition $u \geq 0$ in (1.2) is added because this appears natural in impulse control problems (cf. [1, 2]).

In problem (1.1) ψ is an obstacle in $W^{2,\infty}(\Omega)$ such that $\psi \geq 0$ on $\partial\Omega$ (in case of Dirichlet boundary conditions) and $(\partial\psi/\partial\nu) \leq 0$ on $\partial\Omega$ (in case of Neumann boundary condition). The nonlinearity $f(\cdot)$ is assumed to be nondecreasing and Lipschitz continuous, that is,

$$|f(x) - f(y)| \leq c|x - y| \quad \forall x, y \in \mathbb{R}, \tag{1.3}$$

where c is a positive constant.

Let $\mathbb{V}$ denote the Sobolev space $H_0^1(\Omega)$ (or $H^1(\Omega)$) and $a(u,v)$ the associated bilinear form with operator $\mathcal{A}$. Consider also the following convex sets:

$$\begin{aligned}
\mathbb{K} &= \{v \in \mathbb{V} \text{ such that } v \leq \psi \text{ a.e.}\}, \\
\mathbb{K}(u) &= \{v \in \mathbb{V} \text{ such that } v \leq M(u) \text{ a.e.}\}.
\end{aligned} \tag{1.4}$$

Then problems (1.1) and (1.2) stated in their weak forms read, respectively, as follows: find $u \in \mathbb{K}$ such that

$$a(u, v - u) \geqq (f(u), v - u) \quad \forall v \in \mathbb{K} \tag{1.5}$$

and find $u \in \mathbb{K}(u)$ such that

$$a(u, v - u) \geqq (f(u), v - u) \quad \forall v \in \mathbb{K}(u). \tag{1.6}$$

Very few papers in relation with uniform error estimates for semilinear variational inequalities exist in the literature (cf. [4, 6]). Also, as far as we know this paper contains the first L^∞-error estimate for semilinear QVIs.

Indeed, we show that, under realistic assumption on the nonlinearity, problems (1.5) and (1.6) can be properly approximated by a finite element method which turns out to be optimal in $L^\infty(\Omega)$. For this purpose, we will characterize the continuous solution (resp., the discrete solution) as the unique fixed point of a contraction in the continuous case (resp., the unique fixed point of a contraction in the discrete case). Also, beside its simplicity, this approach extends successfully to the corresponding noncoercive problems as well.

An outline of the paper is as follows: in Section 2 we associate with the VI (1.5) a fixed point mapping and prove its contraction property. Also, using the standard finite element method and a discrete maximum principle, a contraction mapping is associated with the corresponding discrete VI and optimal L^∞-error estimate is proved. Section 3 is devoted to the QVI problem for which a similar study is achieved in both the continuous and discrete cases and also a quasi-optimal uniform error estimate is established. Finally, in Section 4, we extend the method to the corresponding noncoercive problems.

2. The variational inequality. We start by giving some assumptions and notations that are needed throughout this paper. Let $\mathcal{A}$ be the elliptic second-order differential operator

$$\mathcal{A} = -\sum_{1 \leq i,\, j \leq N} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial}{\partial x_i} + a_0(x), \tag{2.1}$$

we assume that the coefficients a_{ij}, a_i, and a_0 are sufficiently smooth and satisfy the following conditions:

$$a_{ij}(x) = a_{ji}(x); \quad a_0(x) \geqq \beta > 0, \quad x \in \Omega, \tag{2.2}$$

$$\sum_{1 \leq i,\, j \leq N} a_{ij}(x)\xi i\xi_j \geqq \alpha |\xi|^2; \quad (x \in \bar{\Omega},\ \xi \in \mathbb{R}^N,\ \alpha > 0). \tag{2.3}$$

For any $u, v \in \mathbb{V}$, we define the variational form associated with the operator $\mathcal{A}$, by

$$a(u,v) = \int_\Omega \left(\sum_{1 \le i,j \le N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^N a_i(x) \frac{\partial u}{\partial x_i} v + a_0(x) \cdot uv \right) dx. \qquad (2.4)$$

We assume that $a(u,v)$ is coercive, that is, there exists $\gamma > 0$ such that for any $v \in \mathbb{V}$

$$a(v,v) \geqq \gamma \|v\|^2_{H^1(\Omega)}. \qquad (2.5)$$

We also assume that the constants c and β, respectively defined in (1.3) and (2.2), are such that

$$\frac{c}{\beta} < 1. \qquad (2.6)$$

Regarding the solvability of problems (1.5) and (1.6), it is standard that they both have a maximum solution; any solution of these problems is in $W^{2,p}(\Omega)$ for all $p \ge 2$. (See [1, 2].) In the sequel we will prove under assumption (2.6) that such solutions are the unique fixed points of contractions in $L^\infty(\Omega)$.

2.1. A contraction associated with VI (1.5). Consider the mapping

$$T_1 : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \qquad w \longrightarrow T_1 w = \zeta, \qquad (2.7)$$

where ζ is the solution of the following VI: find $\zeta \in \mathbb{K}$ such that

$$a(\zeta, v - \zeta) \geqq (f(w), v - \zeta), \quad v \in \mathbb{K}. \qquad (2.8)$$

Problem (2.8) being a coercive VI, thanks to [1], it has one and only one solution. Moreover, under the preceding assumptions, $\zeta \in W^{2,p}(\Omega)$, $2 \le p < \infty$.

PROPOSITION 2.1. *Let $\| \cdot \|_\infty$ denote the L^∞-norm. Then under conditions (1.3) and (2.6) the mapping T is a strict contraction in $L^\infty(\Omega)$, that is, for any $w, \tilde{w} \in L^\infty(\Omega)$,*

$$\left\| T_1 w - T_1 \tilde{w} \right\|_\infty \le \frac{c}{\beta} \|w - \tilde{w}\|_\infty. \qquad (2.9)$$

Therefore, T has a unique fixed point which coincides with the solution of VI (1.5).

PROOF. Let $w, \tilde{w}$ be in $L^\infty(\Omega)$. We denote by

$$\zeta = T_1 w = \sigma(f(w)); \qquad \tilde{\zeta} = T_1 \tilde{w} = \sigma(f(\tilde{w})). \qquad (2.10)$$

Setting

$$\Phi = \frac{1}{\beta} \left\| f(w) - f(\tilde{w}) \right\|_\infty. \qquad (2.11)$$

It follows that

$$\begin{aligned} f(w) &\le f(\tilde{w}) + \left\| f(w) - f(\tilde{w}) \right\|_\infty \\ &\le f(\tilde{w}) + \frac{a_0(x)}{\beta} \left\| f(w) - f(\tilde{w}) \right\|_\infty \\ &\le f(\tilde{w}) + a_0 \Phi \end{aligned} \qquad (2.12)$$

(because $a_0(x) \geqq \beta > 0$). Thus using standard comparison results in coercive variational inequalities, we get

$$\sigma(f(w)) \leq \sigma(f(\tilde{w}) + a_0(x) \cdot \Phi) \leq \sigma(f(\tilde{w}) + \Phi). \tag{2.13}$$

So

$$\zeta \leq \tilde{\zeta} + \Phi. \tag{2.14}$$

Interchanging the roles of w and $\tilde{w}$, we similarly get

$$\tilde{\zeta} \leq \zeta + \Phi. \tag{2.15}$$

Consequently,

$$\left\|T_1 w - T_1 \tilde{w}\right\|_\infty \leq \frac{1}{\beta} \left\|f(w) - f(\tilde{w})\right\|_\infty \tag{2.16}$$

and hence combining (1.3) and (2.6) the contraction property of T follows. □

2.2. The discrete variational inequality. Let Ω be decomposed into triangles and let τ_h denote the set of those elements; $h > 0$ is the mesh-size.

We assume that the triangulation τ_h is regular and quasi-uniform. Let $\mathbb{V}_h$ denote the standard piecewise linear finite element space and ϕ_i, $i = 1, 2, \ldots, m(h)$, the nodal basis functions. Let r_h denote the usual restriction operator.

The discrete maximum principle assumption (dmp): we assume that the matrix with generic coefficient $a(\varphi_i, \varphi_j)$, $1 \leq i, j \leq m(h)$, is an M-matrix (cf. [5]). Such an assumption (see [8]) is needed for the existence of discrete solutions.

Consider $\mathbb{K}_h = \{v \in \mathbb{V}_h \text{ such that } v \leq r_h \psi\}$. The discrete VI consists of seeking u_h solution to

$$a(u_h, v - u_h) \geqq (f(u_h), v - u_h) \quad \forall v \in \mathbb{K}_h. \tag{2.17}$$

Similar to the continuous problem, using the dmp, we will see that the solution of (2.17) can be characterized as the unique fixed point of an appropriate contraction in $L^\infty(\Omega)$.

2.3. A fixed point mapping associated with the VI (2.17). Consider the discrete mapping

$$T_{1h} : L^\infty(\Omega) \longrightarrow \mathbb{V}_h, \qquad w \longrightarrow T_{1h} w = \zeta_h, \tag{2.18}$$

where $\zeta_h \in \mathbb{K}_h$ solves the following coercive VI:

$$a(\zeta_h, v - \zeta_h) \geqq (f(w), v - \zeta_h) \quad \forall v \in \mathbb{K}_h. \tag{2.19}$$

PROPOSITION 2.2. *Under the dmp and assumptions (1.3) and (2.6), the mapping T_h is a contraction in $L^\infty(\Omega)$, that is,*

$$\left\|T_{1h} w - T_{1h} \tilde{w}\right\|_\infty \leq \frac{c}{\beta} \|w - \tilde{w}\|_\infty. \tag{2.20}$$

Therefore, there exists a unique fixed point which coincides with the solution of VI (2.17).

PROOF. The proof of this proposition is exactly the same as that of Proposition 2.1.

□

2.4. The finite element error analysis

REMARK 2.3. From now on, C will denote a constant independent of h.

The following lemma is very important in obtaining the convergence order of the approximation.

LEMMA 2.4. *The following inequality holds:*

$$||T_1 w - T_{1h} w||_{\infty} \le Ch^2 |\log h|^2 ||f(w)||_{\infty}. \tag{2.21}$$

PROOF. We know that

$$\zeta_h = T_{1h} w, \qquad \zeta = Tw. \tag{2.22}$$

It is also clear that ζ_h is the finite element approximation of ζ. Then due to [7], it follows that

$$||T_1 w - T_{1h} w||_{\infty} = ||\zeta - \zeta_h||_{\infty} \le Ch^2 |\log h|^2 ||f(w)||_{\infty} \tag{2.23}$$

which completes the proof. $\qquad \square$

2.5. L^{∞}-error estimate to the variational inequality (1.5)

THEOREM 2.5. *The following inequality holds:*

$$||u - u_h||_{\infty} \le Ch^2 |\log h|^2 ||f(u)||_{\infty}. \tag{2.24}$$

PROOF. It is easy to see that

$$||u - u_h||_{\infty} \le ||u - T_{1h} u|| + ||T_{1h} u - u_h||_{\infty}. \tag{2.25}$$

Also, by Propositions 2.1 and 2.2, u and u_h are the fixed points of T and T_h, respectively, that is,

$$u = T_1 u, \qquad u_h = T_{1h} u. \tag{2.26}$$

Then, applying Lemma 2.4 with $w = u$, it follows that

$$\begin{aligned}||u - u_h||_{\infty} &\le ||T_1 u - T_{1h} u||_{\infty} + ||T_{1h} u - T_{1h} u_h||_{\infty} \\ &\le Ch^2 |\log h|^2 ||f(u)||_{\infty} + \frac{c}{\beta} ||u - u_h||_{\infty}\end{aligned} \tag{2.27}$$

thus,

$$||u - u_h||_{\infty} \le \frac{Ch^2 |\log h|^2 ||f(u)||_{\infty}}{1 - c/\beta} \tag{2.28}$$

which is the desired error estimate. $\qquad \square$

REMARK 2.6. It is well known that if $\psi = +\infty$ the VI problem (1.5) reduces to the equation. Therefore, all the analysis developed in the preceding section remains valid in the unconstrained case. This leads to the same convergence order as that of the linear equation (cf. [9])

$$||u - u_h||_{\infty} \le Ch^2 |\log h| ||f(u)||_{\infty}. \tag{2.29}$$

3. The semilinear quasi-variational inequality problems. The QVI (1.6) is encountered in stochastic impulsive control problems (cf. [2]). Here, the cost function Mu represents the obstacle of impulse control defined by

$$M\varphi(x) = k + \inf \varphi(x + \xi), \quad \xi \geq 0, \, x + \xi \in \bar{\Omega}, \tag{3.1}$$

where k is a positive number.

The operator M possesses some important properties (cf. [2]): it maps $C(\bar{\Omega})$ into itself and is nondecreasing, that is,

$$M\varphi(x) \leq M\tilde{\varphi}(x) \quad \text{whenever } \varphi(x) \leq \tilde{\varphi}(x). \tag{3.2}$$

In a similar way to that of Section 2, we are able to characterize the solution of QVI (1.6) as the unique fixed point of a contraction.

3.1. A contraction associated with QVI (1.6). Let $L_+^\infty(\Omega)$ be the positive cone of $L^\infty(\Omega)$. We consider the following mapping:

$$T_2 : L_+^\infty(\Omega) \longrightarrow L_+^\infty(\Omega), \qquad w \longrightarrow T_2 w = \zeta, \tag{3.3}$$

where ζ is the solution of the following coercive QVI:

$$a(\zeta, v - \zeta) \geq (f(w), v - \zeta) \quad \forall v \in \mathbb{V}, \, \zeta \leq M\zeta, \, v \leq M\zeta. \tag{3.4}$$

Thanks to [2], the QVI (3.4) has one and only one solution which belongs to $W^{2,p}(\Omega)$, $2 \leq p < \infty$.

NOTATION 3.1. Let $F \in L_+^\infty(\Omega)$ and M the nonlinear operator defined in (3.1). We denote by $\zeta = \sigma(F, M\zeta)$ the solution of the QVI

$$a(\zeta, v - \zeta) \geq (F, v - \zeta) \quad \forall v \in V, \, \zeta \leq M\zeta; \, v \leq M\zeta. \tag{3.5}$$

Let $\zeta = \sigma(F, M\zeta)$ and $\tilde{\zeta} = \sigma(\tilde{F}, M\tilde{\zeta})$ be the solutions to QVI (3.5) with right-hand sides F and $\tilde{F}$, respectively. Then, the following comparison result holds.

LEMMA 3.2. *If $F \geq \tilde{F}$ then $\sigma(F, M\zeta) \geq \sigma(\tilde{F}, M\tilde{\zeta})$.*

PROOF. Starting from ζ^0 and $\tilde{\zeta}^0$, respectively, solutions of the equations

$$\begin{aligned}
b(\zeta^0, v) &= (F, v) \quad \forall v \in \mathbb{V}, \\
b(\tilde{\zeta}^0, v) &= (\tilde{F}, v) \quad \forall v \in \mathbb{V},
\end{aligned} \tag{3.6}$$

we consider the following iterative schemes:

$$\zeta^n = \sigma(F, M\zeta^{n-1}), \qquad \tilde{\zeta}^n = \sigma(\tilde{F}, M\tilde{\zeta}^{n-1}). \tag{3.7}$$

Clearly, for each $n \geq 1$, ζ^n and $\tilde{\zeta}^n$ are, respectively, the solutions to the following variational inequalities:

$$\begin{aligned}
b(\zeta^n, v - \zeta^n) &\geq (F, v - \zeta^n), \quad v \in \mathbb{V}, \, \zeta \leq M\zeta^{n-1}, \, v \leq M\zeta^{n-1}, \\
b(\tilde{\zeta}^n, v - \tilde{\zeta}^n) &\geq (\tilde{F}, v - \tilde{\zeta}^n), \quad v \in \mathbb{V}, \, \tilde{\zeta} \leq M\tilde{\zeta}^{n-1}, \, v \leq M\tilde{\zeta}^{n-1}.
\end{aligned} \tag{3.8}$$

So, we inductively have

For $n = 1$, $\zeta^1 = \sigma(F, M\zeta^0)$ and $\tilde{\zeta}^1 = \sigma(\tilde{F}, M\tilde{\zeta}^0)$.

Since $F \geqq \tilde{F}$ and $M\zeta^0 \geqq M\tilde{\zeta}^0$ (because M is nondecreasing), applying standard comparison result in coercive variational inequalities, we get $\zeta^1 \geqq \tilde{\zeta}^1$.

Assume now that $\zeta^{n-1} \geqq \tilde{\zeta}^{n-1}$. Since $F \geqq \tilde{F}$ and $M\zeta^{n-1} \geqq M\tilde{\zeta}^{n-1}$, applying again comparison result in coercive VI, we get $\zeta^n \geqq \tilde{\zeta}^n$.

Finally, passing to the limit, as n tends to ∞ (cf. [2, pages 343–353]) we obtain that $\zeta \geqq \tilde{\zeta}$. This completes the proof. $\qquad\square$

REMARK 3.3. The above lemma remains valid in the discrete case provided the dmp is satisfied.

PROPOSITION 3.4. *Let the conditions of Lemma 3.2 hold. Then the mapping T_2 is a contraction in $L^\infty(\Omega)$, that is,*

$$\|T_2 w - T_2 \tilde{w}\|_\infty \leq \frac{c}{\beta} \|w - \tilde{w}\|_\infty. \tag{3.9}$$

Therefore, there exists a unique fixed point which coincides with the solution of QVI (1.6).

PROOF. For $w, \tilde{w}$ in $L^\infty(\Omega)$, we consider $\zeta = T_2 w$ and $\tilde{\zeta} = T_2 \tilde{w}$ solutions to QVI (3.5) with respect to the right-hand sides

$$F = f(w), \qquad \tilde{F} = f(\tilde{w}). \tag{3.10}$$

Setting

$$\Phi = \frac{1}{\beta} \|F - \tilde{F}\|_\infty \tag{3.11}$$

and applying Lemma 3.2, it follows that

$$\sigma(F, M\zeta) \leq \sigma(\tilde{F} + (a_0(x) + \lambda)\Phi, M(\tilde{\zeta} + \Phi)) \leq \sigma(F) + \Phi \tag{3.12}$$

so,

$$\zeta \leq \tilde{\zeta} + \Phi. \tag{3.13}$$

Interchanging the roles of w and $\tilde{w}$, we similarly get

$$\tilde{\zeta} \leq \zeta + \Phi. \tag{3.14}$$

Finally, due to (1.3) and (2.6), we obtain

$$\|T_2 w - T_2 \tilde{w}\|_\infty \leq \frac{1}{\beta} \|F - \tilde{F}\|_\infty \leq \frac{c}{\beta} \|w - \tilde{w}\|_\infty. \tag{3.15}$$
$$\square$$

3.2. The discrete quasi-variational inequality. Let $\mathbb{K}_h(u_h) = \{v \in \mathbb{V}_h$ such that $v \leq r_h M u_h\}$. The discrete quasi-variational inequality consists of finding $u_h \in \mathbb{K}_h(u_h)$ such that

$$a(u_h, v - u_h) \geqq (f(u_h), v - u_h) \quad \forall v \in \mathbb{K}_h(u_h). \tag{3.16}$$

3.3. A contraction associated with discrete QVI ([3.16](#)). Consider the following discrete mapping:

$$T_{2h} : L_+^\infty(\Omega) \longrightarrow \mathbb{V}_h, \qquad w \longrightarrow T_h w = \zeta_h, \tag{3.17}$$

where ζ_h is the solution of the following discrete coercive QVI:

$$a(\zeta_h, v - \zeta_h) \geqq (f(w), v - \zeta_h) \quad \forall v \in \mathbb{V}_h, \ \zeta_h \leq r_h M \zeta_h, \ v \leq r_h M \zeta_h. \tag{3.18}$$

PROPOSITION 3.5. *Under the dmp assumption, the mapping T_{2h} is a contraction in $L^\infty(\Omega)$, that is,*

$$\left\| T_{2h} w - T_{2h} \tilde{w} \right\|_\infty \leq \frac{c}{\beta} \| w - \tilde{w} \|_\infty. \tag{3.19}$$

Therefore, there exists a unique fixed point, which coincides with u_h, the solution of discrete QVI ([3.16](#)).

PROOF. The proof is very similar to that of [Proposition 3.4](#). □

3.4. L^∞-error estimate for the QVI ([1.6](#)). Adapting [7], we have the following lemma.

LEMMA 3.6. *The following inequality holds:*

$$\| T_2 w - T_{2h} w \|_\infty \leq C h^2 |\log h|^3 \| f(w) \|_\infty. \tag{3.20}$$

PROOF. The proof is very similar to that of [Lemma 2.4](#). □

THEOREM 3.7. *The following inequality holds:*

$$\| u - u_h \|_\infty \leq \frac{C h^2 |\log h|^3}{(1 - c/\beta)}. \tag{3.21}$$

PROOF. The proof of this theorem is exactly the same as that of [Theorem 2.5](#). We make use of Propositions [3.4](#), [3.5](#), and [Lemma 3.6](#). □

4. The noncoercive problems. We assume that ([2.5](#)) does not hold. In this situation it is well known (cf. [1, 2]) that the question of existence for the corresponding noncoercive problems can be treated as follows (cf. [1, 2]): there exists $\lambda > 0$ large enough such that

$$a(v, v) + \lambda(v, v) \geqq \delta \| v \|^2, \quad \delta > 0. \tag{4.1}$$

As a result of this, VI ([1.5](#)), (resp., QVI ([1.6](#))) transform into

$$b(u, v - u) \geqq (f(u) + \lambda u, v - u) \quad \forall v \in \mathbb{K}, \tag{4.2}$$

$$b(u, v - u) \geqq (f(u) + \lambda u, v - u) \quad \forall v \in \mathbb{K}(u), \tag{4.3}$$

where clearly the new variational form $b(u, v) = a(u, v) + \lambda(u, v)$ satisfies the strong coercivity assumption.

Now, in order to approximate the continuous solutions, we shall proceed as in the previous sections. Indeed, we construct the respective fixed point mappings

$$T_{1\lambda} : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \qquad w \longrightarrow T w = \zeta_\lambda, \tag{4.4}$$

where ζ_λ is the solution of the coercive variational inequality below

$$b(\zeta_\lambda, v - \zeta_\lambda) \geqq (f(w) + \lambda w, v - \zeta_\lambda) \quad \forall v \in \mathbb{K},$$
$$T_{2\lambda} : L_+^\infty(\Omega) \longrightarrow L_+^\infty(\Omega), \tag{4.5}$$
$$w \longrightarrow T_{2\lambda} = \zeta_\lambda,$$

where $\zeta_{2\lambda}$ is the solution of the coercive variational inequality below

$$b(\zeta_\lambda, v - \zeta_\lambda) \geqq (f(w) + \lambda w, v - \zeta_\lambda) \quad \forall v \in \mathbb{K}(\zeta_\lambda). \tag{4.6}$$

PROPOSITION 4.1. *The mappings $T_{1\lambda}$ and $T_{2\lambda}$ are contractions in $L^\infty(\Omega)$ with the constant of contraction equal to $(c + \lambda)/(\lambda + \beta)$. Therefore, their unique fixed points coincide with the solutions of the VI (4.2) and QVI (4.3), respectively.*

PROOF. We sketch the proof for the VI. The QVI's case being very similar. It suffices to set

$$\zeta_\lambda = \sigma(F, \psi); \qquad \tilde{\zeta}_\lambda = \sigma(\tilde{F}, \psi), \qquad \Phi = \frac{1}{\lambda + \beta} \|F - \tilde{F}\|_\infty, \tag{4.7}$$

where $F = f(w) + \lambda w$; $\tilde{F} = f(\tilde{w}) + \lambda \tilde{w}$, and next use the same arguments as those developed in the proof of Proposition 2.1. □

4.1. The discrete noncoercive problems. The discrete noncoercive VI and QVI are defined, respectively, as follows:

$$b(u_h, v - u_h) \geqq (f(u_h) + \lambda u_h, v - u_h) \quad \forall v \in \mathbb{K}_h,$$
$$b(u_h, v - u_h) \geqq (f(u_h) + \lambda u_h, v - u_h) \quad \forall v \in \mathbb{K}_h(u_h). \tag{4.8}$$

Their associated fixed point mappings are, respectively,

$$T_{1\lambda,h} : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \qquad w \longrightarrow T_{1\lambda,h} w = \zeta_{\lambda,h}, \tag{4.9}$$

where $\zeta_{1\lambda,h}$ is the unique solution of the following coercive VI:

$$b(\zeta_{\lambda,h}, v - \zeta_{\lambda,h}) \geqq (f(w) + \lambda w, v - \zeta_{\lambda,h}) \quad \forall v \in \mathbb{K}_h,$$
$$T_{2\lambda,h} : L^\infty(\Omega) \longrightarrow L^\infty(\Omega), \qquad w \longrightarrow T_{2\lambda,h} w = \zeta_{\lambda,h}, \tag{4.10}$$

where $\zeta_{1\lambda,h}$ is the unique solution of the following coercive QVI:

$$b(\zeta_{\lambda,h}, v - \zeta_{\lambda,h}) \geqq (f(w) + \lambda w, v - \zeta_{\lambda,h}) \quad \forall v \in \mathbb{K}_h(\zeta_{\lambda,h}). \tag{4.11}$$

Similar to the continuous case, one can easily prove under the "dmp" that $T_{1\lambda,h}$ and $T_{2\lambda,h}$ are contractions in $L^\infty(\Omega)$ with as constant of contraction: $(c + \lambda)/(\lambda + \beta)$.

4.2. L^∞-error estimates for the noncoercive problems

4.2.1. The noncoercive VI. Adapting [7], we have the following lemma.

LEMMA 4.2. *The following inequality holds:*

$$||T_{1\lambda}w - T_{1\lambda,h}w||_\infty \le Ch^2|\log h|^2||f(w) + \lambda w||_\infty, \tag{4.12}$$

and then the error estimate follows.

THEOREM 4.3. *The following inequality holds:*

$$||u - u_h||_\infty \le \frac{Ch^2|\log h|^2}{1 - (c + \lambda)/(\lambda + \beta)}||f(u) + \lambda u||_\infty. \tag{4.13}$$

PROOF. The proof is very similar to that of Theorem 2.5.   □

4.2.2. The noncoercive QVI

LEMMA 4.4. *Adapting [7], then*

$$||T_{2\lambda}w - T_{2\lambda,h}w||_\infty \le Ch^2|\log h|^3||f(w) + \lambda w||_\infty. \tag{4.14}$$

Consequently, using the fact that $T_{2\lambda}$ and $T_{2\lambda,h}$ are contractions in $L^\infty(\Omega)$, one can easily get the following theorem.

THEOREM 4.5. *The following inequality holds:*

$$||u - u_h||_\infty \le \frac{Ch^2|\log h|^3}{1 - (c + \lambda)/(\lambda + \beta)}||f(u) + \lambda u||_\infty. \tag{4.15}$$

COROLLARY 4.6. *If the right-hand side is independent of u, problem (4.2) reduces to the well-known linear noncoercive variational inequalities of stochastic control [8], while problem (4.3) reduces to the linear noncoercive quasi-variational inequality of impulse control [2, 3, 4]. In this situation the approximation convergence orders (4.13) and (4.15) transforms, respectively, into:*
For the VI of stochastic control (cf. [8])

$$||u - u_h||_\infty \le \frac{Ch^2|\log h|^2}{1 - \lambda/(\lambda + \beta)}||u||_\infty. \tag{4.16}$$

For the QVI of impulse control (cf. [3])

$$||u - u_h||_\infty \le \frac{Ch^2|\log h|^3}{1 - \lambda/(\lambda + \beta)}||u||_\infty. \tag{4.17}$$

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