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SOME EXAMPLES OF NONTRIVIAL HOMOTOPY GROUPS OF MODULES

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ABSTRACT. The concept of the homotopy theory of modules was discovered by Peter Hilton as a result of his trip in 1955 to Warsaw, Poland, to work with Karol Borsuk, and to Zurich, Switzerland, to work with Beno Eckmann. The idea was to produce an analog of homotopy theory in topology. Yet, unlike homotopy theory in topology, there are two homotopy theories of modules, the injective theory, $\overline{\pi}_n(A,B)$, and the projective theory, $\underline{\pi}_n(A,B)$. They are dual, but not isomorphic. In this paper, we deliver and carry out the precise calculation of the first known nontrivial examples of absolute homotopy groups of modules, namely, $\overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, $\overline{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, and $\underline{\pi}_n(\mathbb{Z},\mathbb{Z})$, where $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}$ are regarded as $\mathbb{Z}C_k$-modules with trivial action. One interesting phenomenon of the results is the periodicity of these homotopy groups, just as for the Ext groups.

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1. Introduction. It is well known that $\mathrm{Ext}^n_\Lambda(A,B)$ may be regarded either as the value of the left nth derived functor of $\mathrm{Hom}_\Lambda(-,B)$ on the Λ-module A or as the value of the right nth derived functor of $\mathrm{Hom}_\Lambda(A,-)$ on the Λ-module B. In this paper, we study the *injective homotopy groups* $\overline{\pi}_n(A,B)$ and the *projective homotopy groups* $\underline{\pi}_n(A,B)$ which are, respectively, the value of the right nth derived functor of $\mathrm{Hom}_\Lambda(-,B)$ on the Λ-module A and the value of the left nth derived functor of $\mathrm{Hom}_\Lambda(A,-)$ on the Λ-module B. We refer to these abelian groups as *homotopy groups* because of their strong analogy with the homotopy groups encountered in the homotopy theory of pointed topological spaces (see [2, Chapter 13] and [1, Chapter 2]). Yet, unlike homotopy theory in topology, there are two homotopy theories in module theory, the injective theory and the projective theory. They are dual, but not isomorphic.

Our principal concern in this paper is to carry out some calculations to demonstrate the nontriviality of these groups. We would regard the groups $\overline{\pi}_n(A,B)$ as trivial if either $\overline{\pi}_n(A,B)=0$ or $\overline{\pi}_n(A,B)=\mathrm{Hom}_\Lambda(\Sigma^n A,B)$, where ΣA is the cokernel of $A\subseteq CA$, with CA an injective container of A; a similar remark applies to $\underline{\pi}_n(A,B)$. Now if Λ is a principal ideal domain, then $\overline{\pi}_n(A,B)=0$ for $n\geqq 1$, so that the homotopy theory of modules for such rings Λ is not very interesting. We therefore turn our attention to a case of particular interest in the calculation of $\mathrm{Ext}^n_\Lambda(A,B)$, namely, that in which Λ is the integral group ring of the finite cyclic group C_k. Our main thrust will be to calculate certain injective homotopy groups.

It has been conjectured that, if the abelian group $\mathbb{Q}/\mathbb{Z}$ is regarded as a trivial C_k-module, then $\overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$ should, in general, be nonzero (actually, this was shown by Bea Bleile in her Master's thesis at the University of New England); we verify this conjecture, and, in fact, carry out the precise calculation of these groups. Furthermore, mainly using the *long exact* $(\overline{\pi},\operatorname{Ext}_\Lambda)$*-sequence in the first variable* (see Theorem 2.7), we obtain another nontrivial example, namely, $\overline{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is also regarded as a trivial C_k-module. In addition, we produce an example of nontrivial projective homotopy groups, namely, $\underline{\pi}_n(\mathbb{Z},\mathbb{Z})$. One feature of the nontrivial examples found is the periodicity of the homotopy groups in question, just as for the Ext groups.

2. Examples of nontrivial injective homotopy groups of modules. As mentioned in the introduction, we will concentrate our attention on the case that Λ is the integral group ring of the finite cyclic group C_k with generator τ. First we produce a theorem, namely, Theorem 2.2, which suggests that $\overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$ may indeed become our successful candidate for nontriviality, where $\mathbb{Q}/\mathbb{Z}$ is regarded as a trivial C_k-module. The theorem requires an elementary lemma.

Lemma 2.1. *Let D, B be abelian groups.*
(i) *If D is divisible, then* $\operatorname{Hom}(D,B)$ *is torsionfree.*
(ii) *If D is divisible and torsionfree, then* $\operatorname{Hom}(D,B)$ *is torsionfree and divisible.*

The proof is left to the reader. We now state the theorem.

Theorem 2.2. *Let the abelian groups D and B be regarded as trivial C_k-modules. If D is divisible, then*

$$\overline{\pi}_n(D,B)=\begin{cases}\operatorname{Hom}(D,B)_k\equiv\operatorname{Hom}(D,B)/k\operatorname{Hom}(D,B) & \textit{for } n \textit{ even},\\ 0 & \textit{for } n \textit{ odd}.\end{cases}\tag{2.1}$$

Proof. Using the fact that $D\cong\operatorname{Hom}(\mathbb{Z},D)$, we produce an injective resolution of D as a trivial C_k-module by applying $\operatorname{Hom}(-,D)$ to a well-known projective resolution of $\mathbb{Z}$, where $\mathbb{Z}$ is regarded as a trivial C_k-module: since $\mathbb{Z}C_k$ is a free, hence projective, C_k-module, the exact sequence

$$\cdots \xrightarrow{\rho} \mathbb{Z}C_k \xrightarrow{\sigma} \mathbb{Z}C_k \xrightarrow{\rho} \mathbb{Z}C_k \xrightarrow{\epsilon} \mathbb{Z}\tag{2.2}$$

forms a projective resolution of $\mathbb{Z}$, where the maps ϵ, ρ, and σ are the augmentation of $\mathbb{Z}C_k$, multiplication by $\tau-1$, and multiplication by $\tau^{k-1}+\cdots+\tau+1$, respectively. Applying $\operatorname{Hom}(-,D)$ to (2.2), we obtain a cochain complex of abelian groups, thus,

$$\operatorname{Hom}(\mathbb{Z},D) \xrightarrow{\epsilon^*} \operatorname{Hom}(\mathbb{Z}C_k,D) \xrightarrow{\rho^*} \operatorname{Hom}(\mathbb{Z}C_k,D) \xrightarrow{\sigma^*} \operatorname{Hom}(\mathbb{Z}C_k,D) \xrightarrow{\rho^*} \cdots.\tag{2.3}$$

Note that (2.3) is exact because D is divisible, hence injective. In addition, since D is divisible, $\operatorname{Hom}(\mathbb{Z}C_k,D)$ is an injective C_k-module (see [3, Theorem I.8.2]). One can easily check that the maps ϵ^*, ρ^*, and σ^* are module-homomorphisms with respect to the module structures assigned to $\operatorname{Hom}(\mathbb{Z},D)$ and $\operatorname{Hom}(\mathbb{Z}C_k,D)$. Hence, (2.3) forms an injective resolution of $D(\cong\operatorname{Hom}(\mathbb{Z},D))$ as C_k-module.

We next give (2.3) a simpler appearance: since $\operatorname{Hom}(\mathbb{Z}C_k,D)\cong D^k$ (k copies of D) as abelian groups, we assign to D^k the C_k-module structure inherited from this isomorphism, and notice that the C_k-action on D^k is essentially the cyclic permutation of the summands, given by $\tau\cdot(0,\dots,0,d_{(i)},0,\dots,0) = (0,\dots,0,d_{(i+1)},0,\dots,0)$. Thus, $\mathrm{Hom}(\mathbb{Z}C_k,D)\cong D^k$ as C_k-modules, and the injective resolution (2.3) becomes

$$D \rightarrowtail^{\Delta} D^k \xrightarrow{\rho^*} D^k \xrightarrow{\sigma^*} D^k \xrightarrow{\rho^*} \cdots, \tag{2.4}$$

where $\epsilon^* = \Delta =$ the diagonal map; $\Delta(d) = (d,d,\dots,d)$.

We then apply $\mathrm{Hom}_{C_k}(-,B)$ to (2.4) and obtain a chain complex of C_k-modules, thus,

$$\cdots \xrightarrow{\rho^{**}} \mathrm{Hom}_{C_k}(D^k,B) \xrightarrow{\sigma^{**}} \mathrm{Hom}_{C_k}(D^k,B) \xrightarrow{\rho^{**}} \mathrm{Hom}_{C_k}(D^k,B) \xrightarrow{\Delta^*} \mathrm{Hom}_{C_k}(D,B), \tag{2.5}$$

whose homology groups are the homotopy groups $\overline{\pi}_n(D,B)$.

In order to calculate the homology groups of (2.5), we simplify (2.5) to a chain complex of abelian groups, that is,

$$\cdots \longrightarrow \mathrm{Hom}(D,B) \xrightarrow{k} \mathrm{Hom}(D,B) \xrightarrow{0} \mathrm{Hom}(D,B) \xrightarrow{k} \mathrm{Hom}(D,B), \tag{2.6}$$

by first showing $\mathrm{Hom}_{C_k}(D^k,B)\cong\mathrm{Hom}(D,B)$ as abelian groups: let $\nabla : D^k \twoheadrightarrow D$ be the *folding map*, given by $\nabla(d_1,\dots,d_k) = \sum_{j=1}^k d_j$; since ∇ is surjective, the induced map $\nabla^* : \mathrm{Hom}(D,B)\to\mathrm{Hom}(D^k,B)$ is monomorphic. Thus, $\mathrm{Hom}(D,B)\cong$ image $\nabla^* = \{\phi\nabla \mid \phi\in\mathrm{Hom}(D,B)\}$, where the last set is indeed $\mathrm{Hom}_{C_k}(D^k,B)$. Under this isomorphism, the map $\rho^{**} : \mathrm{Hom}_{C_k}(D^k,B)\to\mathrm{Hom}_{C_k}(D^k,B)$ is seen to correspond to the zero map $0 : \mathrm{Hom}(D,B)\to\mathrm{Hom}(D,B)$, whereas $\sigma^{**} : \mathrm{Hom}_{C_k}(D^k,B)\to\mathrm{Hom}_{C_k}(D^k,B)$ corresponds to $k : \mathrm{Hom}(D,B)\to\mathrm{Hom}(D,B)$ (multiplication by k). Moreover, $\Delta^* : \mathrm{Hom}_{C_k}(D^k,B)\to\mathrm{Hom}_{C_k}(D,B)$ also corresponds to $k : \mathrm{Hom}(D,B)\to\mathrm{Hom}(D,B)$. Finally, the assumption that D is divisible, together with Lemma 2.1(i), implies that the map k in (2.6) is monomorphic, so that calculating the homology groups of (2.6) tells us that, as claimed,

$$\overline{\pi}_n(D,B) = \begin{cases} \mathrm{Hom}(D,B)/k\mathrm{Hom}(D,B) & \text{for } n \text{ even},\\ 0 & \text{for } n \text{ odd}.\end{cases} \tag{2.7}$$

□

Corollary 2.3. *If, in addition, D is torsionfree, then $\mathrm{Hom}(D,B)_k = 0$ so that all the homotopy groups $\overline{\pi}_n(D,B)$ vanish in this case.*

Proof. If, in addition, D is torsionfree, $\mathrm{Hom}(D,B)$ is not only torsionfree but divisible, by Lemma 2.1(ii). Therefore, the map k is epimorphic, which yields $\mathrm{Hom}(D,B)/k\mathrm{Hom}(D,B) = 0$. Thus, in this case, all the homotopy groups are zero. □

Corollary 2.4. *If $\mathbb{Q}$ is regarded as a trivial C_k-module, then, for any trivial C_k-module B, $\overline{\pi}_n(\mathbb{Q},B) = 0$ for all n.*

Corollary 2.4 is just a special case of Corollary 2.3. Further, if $\mathbb{Q}/\mathbb{Z}$ is regarded as a trivial C_k-module, then $\overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) = 0$ for all n, because $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}) = 0$, so that (2.6) is, in this case, just the zero sequence.

We now display our first example of nontrivial injective homotopy groups of modules, namely, $\overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, by first stating a simple lemma.

Lemma 2.5. *Let $A' \overset{\mu}{\rightarrowtail} A \overset{\epsilon}{\twoheadrightarrow} A''$ be a short exact sequence of abelian groups. If A'' is torsionfree, then the induced sequence (reduction* $\operatorname{mod} k$*) $A'_k \overset{\mu_k}{\rightarrowtail} A_k \overset{\epsilon_k}{\twoheadrightarrow} A''_k$, where $A_k = A/kA$, is also short exact, for all k.*

The proof is left to the reader; note that the lemma asserts that A' is pure in A.

Theorem 2.6. *If $\mathbb{Q}/\mathbb{Z}$ is regarded as a trivial C_k-module, then (we adopt the notational device of distinguishing between the group C_k and the abelian group $\mathbb{Z}/k$)*

$$\overline{\pi}_n(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{2.8}$$

Proof. By Theorem 2.2, we only need to calculate $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})_k$: consider the short exact sequence $\mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ of abelian groups; the associated six-term Hom-Ext sequence in the first variable

$$\begin{aligned} &\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, -) \rightarrowtail \operatorname{Hom}(\mathbb{Q}, -) \longrightarrow \operatorname{Hom}(\mathbb{Z}, -) \\ &\qquad \longrightarrow \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, -) \longrightarrow \operatorname{Ext}(\mathbb{Q}, -) \longrightarrow \operatorname{Ext}(\mathbb{Z}, -) \end{aligned} \tag{2.9}$$

yields a short exact sequence

$$\operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \rightarrowtail \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \twoheadrightarrow \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}). \tag{2.10}$$

Moreover, $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{R}$ as abelian groups (see [3, Exercise III.6.2]). Therefore, (2.10) is essentially

$$\mathbb{Z} \rightarrowtail \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \twoheadrightarrow \mathbb{R}. \tag{2.11}$$

Also, the associated six-term Hom-Ext sequence in the second variable

$$\begin{aligned} &\operatorname{Hom}(-, \mathbb{Z}) \rightarrowtail \operatorname{Hom}(-, \mathbb{Q}) \longrightarrow \operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z}) \\ &\qquad \longrightarrow \operatorname{Ext}(-, \mathbb{Z}) \longrightarrow \operatorname{Ext}(-, \mathbb{Q}) \longrightarrow \operatorname{Ext}(-, \mathbb{Q}/\mathbb{Z}) \end{aligned} \tag{2.12}$$

yields $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$, so that (2.11) is equivalent to the short exact sequence $\mathbb{Z} \rightarrowtail \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \mathbb{R}$. Finally, since $\mathbb{R}$ is torsionfree and divisible, by Lemma 2.5, the isomorphism $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})_k \cong \mathbb{Z}/k$ is guaranteed by the following diagram:

$$\begin{tikzcd} k\mathbb{Z} \arrow[r, tail] \arrow[d, hook] & k\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \arrow[r, two heads] \arrow[d, hook] & \mathbb{R} \arrow[d, equal] \\ \mathbb{Z} \arrow[r, tail] \arrow[d, two heads] & \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \arrow[r, two heads] \arrow[d, two heads] & \mathbb{R} \arrow[d] \\ \mathbb{Z}/k \arrow[r, tail] & \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})_k \arrow[r, two heads] & 0 \end{tikzcd} \tag{2.13}$$

□

In the short exact sequence $\mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ of trivial C_k-modules, we already know $\overline{\pi}_n(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ (see Theorem 2.6) and $\overline{\pi}_n(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ (see Corollary 2.4). By applying the *long exact $(\overline{\pi}, \operatorname{Ext}_\Lambda)$-sequence in the first variable*, the homotopy groups $\overline{\pi}_n(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

naturally become our next candidates for nontriviality. We first quote the long exact sequence (see [2, Theorem 13.16]).

Theorem 2.7 (see [2]). *Given a short exact sequence $A' \overset{\lambda}{\rightarrowtail} A \overset{\kappa}{\twoheadrightarrow} A''$, there exists, for each B, a long exact sequence*

$$\begin{aligned}
\cdots &\xrightarrow{\partial} \overline{\pi}_n(A'',B) \xrightarrow{\kappa^*} \overline{\pi}_n(A,B) \xrightarrow{\lambda^*} \overline{\pi}_n(A',B) \xrightarrow{\partial} \overline{\pi}_{n-1}(A'',B) \xrightarrow{\kappa^*} \cdots \\
&\xrightarrow{\lambda^*} \overline{\pi}_1(A',B) \xrightarrow{\partial} \overline{\pi}(A'',B) \xrightarrow{\kappa^*} \overline{\pi}(A,B) \xrightarrow{\lambda^*} \overline{\pi}(A',B) \\
&\xrightarrow{\delta^*} \operatorname{Ext}^1_\Lambda(A'',B) \xrightarrow{\kappa^*} \operatorname{Ext}^1_\Lambda(A,B) \xrightarrow{\lambda^*} \operatorname{Ext}^1_\Lambda(A',B) \xrightarrow{\delta} \operatorname{Ext}^2_\Lambda(A'',B) \xrightarrow{\kappa^*} \cdots \\
&\xrightarrow{\delta} \operatorname{Ext}^n_\Lambda(A'',B) \xrightarrow{\kappa^*} \operatorname{Ext}^n_\Lambda(A,B) \xrightarrow{\lambda^*} \operatorname{Ext}^n_\Lambda(A',B) \xrightarrow{\delta} \operatorname{Ext}^{n+1}(A'',B) \xrightarrow{\kappa^*} \cdots .
\end{aligned} \tag{2.14}$$

We call (2.14) the *long exact* $(\overline{\pi},\operatorname{Ext}_\Lambda)$*-sequence in the first variable*.

Theorem 2.8. *If $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ are regarded as trivial C_k-modules, then*

$$\overline{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \cong \begin{cases} 0 & \text{for } n \text{ even}, \\ \mathbb{Z}/k & \text{for } n \text{ odd}. \end{cases} \tag{2.15}$$

Proof. As mentioned earlier, Theorem 2.7 provides us with a long exact sequence of abelian groups, that is,

$$\begin{aligned}
\cdots &\longrightarrow \overline{\pi}_n(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \longrightarrow \overline{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \longrightarrow \overline{\pi}_{n-1}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \\
&\longrightarrow \overline{\pi}_{n-1}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \longrightarrow \cdots .
\end{aligned} \tag{2.16}$$

Since $\overline{\pi}_n(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) = 0$ for all n,

$$\overline{\pi}_n(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \cong \overline{\pi}_{n-1}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \cong \begin{cases} 0 & \text{for } n \text{ even and } n > 0, \\ \mathbb{Z}/k & \text{for } n \text{ odd}. \end{cases} \tag{2.17}$$

Therefore, it only remains to show that $\overline{\pi}(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) = 0$.

To find $\overline{\pi}(\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, we first embed $\mathbb{Z}$ into $\mathbb{Q}$ in the canonical way, and embed $\mathbb{Q}$ into the injective C_k-module $\mathbb{Q}^k$ as in (2.4), so that we have a commutative diagram of C_k-modules, that is,

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, tail, "\Delta\iota"] \arrow[dr, hook, "\iota"'] & & \mathbb{Q}^k \text{ (injective } C_k\text{-module)} \\
& \mathbb{Q} \arrow[ur, tail, "\Delta"'] &
\end{tikzcd} \tag{2.18}$$

When applying $\operatorname{Hom}_{C_k}(-,\mathbb{Q}/\mathbb{Z})$ to (2.18), we obtain a diagram of C_k-modules, thus,

$$\begin{tikzcd}
\operatorname{Hom}_{C_k}(\mathbb{Q}^k,\mathbb{Q}/\mathbb{Z}) \arrow[rr, "(\Delta\iota)^*"] \arrow[dr, "\Delta^*"'] & & \operatorname{Hom}_{C_k}(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \\
& \operatorname{Hom}_{C_k}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \arrow[ur, "\iota^*"'] &
\end{tikzcd} \tag{2.19}$$

which is equivalent to the following diagram of abelian groups:

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \arrow[rr, "\iota^* k"] \arrow[dr, "k"'] & & \mathrm{Hom}(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \\
& \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \arrow[ur, "\iota^*"'] &
\end{tikzcd} \tag{2.20}$$

Next, since $\mathbb{Q}$ is divisible and torsionfree, by Lemma 2.1(ii), $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})$ is torsionfree and divisible. Therefore, the map k is surjective. Finally, since $\iota^* : \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{Q}/\mathbb{Z})$ is induced from the short exact sequence $\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ of abelian groups with the target group $\mathbb{Q}/\mathbb{Z}$ injective, the map ι^* is epimorphic. Hence, the map $\iota^* k$ in (2.20) is epimorphic, so $\overline{\pi}(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) = 0$. $\square$

We remark that we would like to prove that $\overline{\pi}(\mathbb{Z},\mathbb{Q}/\mathbb{Z}) = 0$, using the full force of the doubly-infinite long exact $(\overline{\pi},\mathrm{Ext}_\Lambda)$-sequence in the first variable. Furthermore, we would like to find some examples of nontrivial injective (or projective) homotopy groups $\overline{\pi}_n(A,B)$ (or $\underline{\pi}_n(A,B)$), where A or B is not a trivial C_k-module. One more note is that we conjecture the following: "let A and B be regarded as trivial C_k-modules. If the injective homotopy groups $\overline{\pi}_n(A,B)$ are nontrivial, then, as abelian groups, B contains $\mathbb{Q}/\mathbb{Z}$ as a direct summand."

3. An example of nontrivial projective homotopy groups of modules. Dualizing the procedure for the injective theory, we calculate the projective homotopy groups $\underline{\pi}_n(A,B)$ by constructing a projective resolution of B, applying $\mathrm{Hom}_\Lambda(A,-)$ to this resolution, and computing the homology groups of the resulting chain complex. Having described two examples of nontrivial injective homotopy groups, we now produce an example of nontrivial projective homotopy groups.

Theorem 3.1. *If $\mathbb{Z}$ is regarded as a trivial C_k-module, then*

$$\underline{\pi}_n(\mathbb{Z},\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}. \end{cases} \tag{3.1}$$

Proof. We start with the well-known projective resolution (2.2) of $\mathbb{Z}$, namely,

$$\cdots \xrightarrow{\rho} \mathbb{Z}C_k \xrightarrow{\sigma} \mathbb{Z}C_k \xrightarrow{\rho} \mathbb{Z}C_k \overset{\epsilon}{\twoheadrightarrow} \mathbb{Z}. \tag{3.2}$$

Applying $\mathrm{Hom}_{C_k}(\mathbb{Z},-)$, we have a chain complex of C_k-modules, thus,

$$\cdots \xrightarrow{\rho_*} \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k) \xrightarrow{\sigma_*} \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k) \xrightarrow{\rho_*} \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k) \xrightarrow{\epsilon_*} \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}). \tag{3.3}$$

The structure of the maps in $\mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k)$ quickly shows us that $\mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k) \cong \mathbb{Z}$ as abelian groups. Under this isomorphism, the map $\rho_* : \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k) \to \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k)$ is seen to correspond to the zero map $0 : \mathbb{Z} \to \mathbb{Z}$, $\sigma_* : \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k) \to \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k)$ to $k : \mathbb{Z} \to \mathbb{Z}$ (multiplication by k), and $\epsilon_* : \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z}C_k) \to \mathrm{Hom}_{C_k}(\mathbb{Z},\mathbb{Z})$ also to $k : \mathbb{Z} \to \mathbb{Z}$. Therefore, (3.3) is equivalent to a chain complex of abelian groups, namely,

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{k} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{k} \mathbb{Z}, \tag{3.4}$$

whose homology groups are the homotopy groups $\underline{\pi}_n(\mathbb{Z},\mathbb{Z})$. The proof is thus complete since the map $k:\mathbb{Z}\to\mathbb{Z}$ is monomorphic with cokernel $\mathbb{Z}/k$. $\square$

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