IJMMS 27:2 (2001) 91–98
PII. S0161171201010262
http://ijmms.hindawi.com

GLOBAL ATTRACTIVITY WITHOUT STABILITY FOR LIÉNARD TYPE SYSTEMS

MARIAN MUREŞAN

(Received 14 March 2000)

Abstract. We are concerned with some conditions such as the trivial solution of a planar system of differential equations (including the Liénard system) that is globally attractive but not stable. We emphasize the connection with some nonoscillatory conditions. The results are related to the previous ones obtained by Hara in 1993.

2000 Mathematics Subject Classification. 34D45.

1. Introduction. The present paper is concerned with some conditions such as the trivial solution of the following planar system of differential equations

$$x'(t) = \varphi(x(t), y(t)), \qquad y'(t) = -g(x(t)), \tag{1.1}$$

which is globally attractive but not stable. The above system is a generalization of the Liénard system

$$x'(t) = y(t) - F(x(t)), \qquad y'(t) = -g(x(t)). \tag{1.2}$$

We assume that

$$F, g \in C(\mathbb{R}, \mathbb{R}), \qquad F(0) = 0, \qquad xg(x) > 0, \quad \text{for } x \neq 0. \tag{1.3}$$

From (1.3), system (1.2) has a unique solution for each initial value problem associated with it, the origin is the unique critical point of (1.2) and all the trajectories are oriented clockwise. Similar to [5, 6] we assume that

$$\varphi \in C^1(\mathbb{R}^2, \mathbb{R}), \qquad \varphi(0,0) = 0, \qquad xg(x) > 0, \quad \text{for } x \neq 0, \tag{1.4}$$

and that there exist constants k and h such that

$$k \le \varphi'_y(x, y) \le h, \quad \text{for } (x, y) \in \mathbb{R}^2. \tag{1.5}$$

It is obvious that under conditions (1.4) and (1.5), the origin of the Euclidean plane is the unique critical point of system (1.1) and the trajectories are oriented clockwise. Hence, system (1.1), with (1.4) and (1.5), includes system (1.2) with (1.3).

From [7], we know that the first example of a planar system such that the zero solution is attractive but not stable was introduced in [4]. This example was followed by others (see [8] and [1, page 191]).

More recently this problem has been studied by Hara [2], who used the Liénard system (1.2) under assumptions (1.3). See also [3].

For definitions and relations between the notions of stability and attractivity as well as a phase-portrait of a planar system which is globally attractive but not stable, see [7, Section I.2].

The aim of this paper is to extend Hara's results in [2] to system (1.1), namely, to give general conditions such that the zero solution of system (1.1) is globally attractive but not stable.

2. The result.

The *characteristic curve* of system (1.1) is given by $\{(x,y) \mid \varphi(x,y) = 0\}$. We write $D_1 = \{(x,y) \mid x > 0,\ \varphi(x,y) > 0\}$, $D_2 = \{(x,y) \mid x > 0,\ \varphi(x,y) < 0\}$, $D_3 = \{(x,y) \mid x < 0,\ \varphi(x,y) < 0\}$, and $D_4 = \{(x,y) \mid x < 0,\ \varphi(x,y) > 0\}$. The behaviour of the trajectories of system (1.1) under assumptions (1.4) and (1.5) was introduced in [6, Lemma 2.1]. For the sake of completeness we recall it now.

Lemma 2.1 (see [6]). *Consider system (1.1) under assumptions (1.4) and (1.5). Every trajectory of system (1.1) passing through a point $B(x_0,y_0)$ $(x_0 \neq 0)$, which belongs to the characteristic curve, intersects the vertical axis at two points $A(0,y_A)$ $(y_A \geq 0)$ and $C(0,y_C)$ $(y_C \leq 0)$. More precisely, if $x_0 > 0$, the solution of (1.1) leaving the point B at $t = 0$ either traverses the positive y-axis at some finite $-t_A > 0$ as t decreases or tends to the origin as $t \to \underline{t}$ $(\underline{t} \geq -\infty)$, remaining in the region D_1, and either traverses the negative y-axis at some finite $t_C > 0$ as t increases or tends to the origin as $t \to \overline{t}$ $(\overline{t} \leq +\infty)$, remaining in the region D_2.*

If $x_0 < 0$, the solution of (1.1) leaving the point B at $t = 0$ either traverses the positive y-axis at some finite $t_A > 0$ as t increases or tends to the origin as $t \to \overline{t}$ $(\overline{t} \leq +\infty)$, remaining in the region D_4, and either traverses the negative y-axis at some finite $-t_C > 0$ as t decreases or tends to the origin as $t \to \underline{t}$ $(\underline{t} \geq -\infty)$, remaining in the region D_3.

Proof. Suppose that $x_0 > 0$ and let $t = 0$ be the moment at which the trajectory meets the characteristic curve at the point $B(x_0,y_0)$. First we consider the case when $t \geq 0$. Since

$$\frac{d}{dt}\varphi(x(t),y(t))\bigg|_{t=0} = -\varphi'_y(x_0,y_0)g(x_0) < 0, \tag{2.1}$$

the solution $(x(t),y(t))$ enters in the region D_2 and does not intersect anymore the characteristic curve as long as $x(t) > 0$. Then $x' \leq 0$, $y' \leq 0$. Suppose that this trajectory starting in B does not meet the y-axis. Then we can find $\overline{x} \in [0,x_0)$ such that $x(t) \to \overline{x}$ and $y(t) \to -\infty$ for $t \to \overline{t}$ or $x(t) \to 0$, $y(t) \to 0$ for $t \to \overline{t}$ and $(x(t),y(t)) \in D_2$ for $t \in [0,\overline{t})$.

Suppose that $y(t) \to -\infty$ for $t \to \overline{t}$. We consider the auxiliary system

$$x'(t) = h[y(t) - y_0], \qquad y'(t) = -g(x(t)) \tag{2.2}$$

and the following Liapunov function:

$$V(x,y) = \frac{1}{2}(y-y_0)^2 + \frac{1}{h}\int_0^x g(\xi)\,d\xi. \tag{2.3}$$

For $c > 0$, $V(x,y) = c$ is a closed curve. Since

$$\frac{d}{dt}V(x,y)\Big|_{22(2.2)} = 0, \tag{2.4}$$

any trajectory of system (2.2) lies on $V(x,y) = c$. If we consider the trajectory passing through the point B, then $V(x_0,y_0) =: c_0$ and $\int_0^x g(\xi)d\xi = hc_0$. Since $xg(x) > 0$ for $x \neq 0$, the equation $\int_0^x g(\xi)d\xi = hc_0$ has two values for x, one positive and one negative satisfying it. For $(x(t),y(t)) \in D_2$ we have

$$x'(t)\big|_{2(1.1)} = \varphi(x(t),y(t)) \geq h[y(t)-y_0] = x'(t)\big|_{2(2.2)}. \tag{2.5}$$

So

$$y'(x)\big|_{2(1.1)} = \frac{y'(t)\big|_{2(1.1)}}{x'(t)\big|_{2(1.1)}} \geq \frac{y'(t)\big|_{2(2.2)}}{x'(t)\big|_{2(2.2)}} = y'(x)\big|_{2(2.2)}. \tag{2.6}$$

Thus the trajectory of system (1.1) which starts from B and enters D_2 is above the trajectory of system (2.2) which starts from B. Since the last trajectory meets the negative y-axis, so will the trajectory of system (1.1). Hence we get a contradiction to the assumption that $y(t) \to -\infty$ for $t \to \bar{t}$.

Consequently, the trajectory of system (1.1) passing through the point B either approaches the origin of the Euclidean plane for $t \to \bar{t}$ or crosses the y-axis at a finite distance, say, $C(0,y_C)$ $(y_C < 0)$.

In a similar way, if $t < 0$, we conclude that the trajectory of system (1.1) passing through the point B either approaches the origin of the Euclidean plane for $t \to \underline{t}$ or crosses the y-axis at a finite distance, say, $A(0,y_A)$ $(y_A > 0)$.

The case $x_0 < 0$ runs similarly. □

We write $\varphi_\pm(x,0) = \max\{0, \pm\varphi(x,0)\}$ and

$$\Gamma_\pm(x) = \int_0^x \frac{g(s)}{1+\varphi_\pm(s,0)}ds. \tag{2.7}$$

Lemma 2.2 (see [6]). *Under conditions (1.4), (1.5), and*

$$\liminf_{x\to+\infty}\varphi(x,0) < +\infty, \qquad \limsup_{x\to-\infty}\varphi(x,0) > -\infty, \tag{2.8}$$

for each $(x_0,y_0) \in D_1$ with $x_0 > 0$, the trajectory of system (1.1) which passes through (x_0,y_0) crosses the characteristic curve at $x > x_0$ if and only if

$$\limsup_{x\to+\infty}[\Gamma_+(x) - \varphi(x,0)] = +\infty. \tag{2.9}$$

For each (x_0,y_0) with $x_0 < 0$ and $\varphi(x_0,y_0) < 0$, the trajectory of system (1.1) which passes through (x_0,y_0) crosses the characteristic curve at $x < x_0$ if and only if

$$\limsup_{x\to-\infty}[\Gamma_-(x) + \varphi(x,0)] = +\infty. \tag{2.10}$$

Now consider a new assumption

$$\varphi(x,0) < 0, \quad x > 0. \tag{2.11}$$

In the following lemma we assume that all the assumptions of Lemma 2.1 and, in addition, (2.11) hold.

Lemma 2.3. *Assume that* $x > 0$, *then* $y_A > 0$.

Proof. If $x > 0$, the characteristic curve traverses the first quadrant. In this case $y_A \geq y_0 > 0$, where $B(x_0, y_0)$ is on the characteristic curve with $x_0 > 0$. □

Theorem 2.4 (see [5]). *Let* (x,y) *be a solution of system (1.1) defined on the interval* $J = [t_0, \bar{t})$ *such that*

(i) *the assumptions (1.4), (1.5), and (2.11) are satisfied;*
(ii) *the point* $B = (x_0, y_0)$, *with* $x_0 > 0$, *belongs to the characteristic curve;*
(iii) *for any* $x > 0$

$$\frac{1}{\varphi(x,0)} \int_{0^+}^{x} \frac{g(s)}{\varphi(s,0)} ds \leq \frac{1}{4h}. \tag{2.12}$$

Then the solution of system (1.1) which passes through the point B remains in the region D_2 *and converges to the origin for* $t \to \bar{t}$.

Now we introduce a version of Theorem 2.3 in [5].

Theorem 2.5. *Suppose that there exists* $a > 0$ *such that*

$$\varphi(x,0) < 0, \quad \frac{1}{\varphi(x,0)} \int_{0^+}^{x} \frac{g(s)}{\varphi(s,0)} ds \leq \frac{1}{4h}, \quad x \in (-a, 0], \tag{2.13}$$

then, for each point $B = (x_0, y_0)$ *with* $x_0 \in (-a, 0)$, $\varphi(x_0, y_0) = 0$, *the trajectory of the system (1.1) passing through the point B approaches the origin as* $t \to \underline{t}$ *through the region bounded by the characteristic curve, the horizontal axis, and the line* $x = x_0$.

Proof. Suppose that there exists a trajectory of (1.1) passing through the point $B = (x_0, y_0)$, with $x_0 \in (-a, 0)$ and $\varphi(x_0, y_0) = 0$, which does not tend to the origin through the region mentioned above. Then by Lemma 2.1, this trajectory meets the negative y-axis at $C = (0, y_C)$ $(y_C < 0)$ at a finite time $-t_C > 0$. Consequently, the trajectory meets the negative x-axis at some point $(b, 0)$, $x_0 < b < 0$. It will be shown that this leads to a contradiction. Therefore we construct a sequence $(y_n(\cdot))_n$ defined on the interval $[x_0, b]$ by the following recurrence relation

$$y_{n+1}(x) = \int_b^x -\frac{g(s)}{\varphi(s, y_n(s))} ds, \quad y_1(x) = 0, \ x \in [x_0, b], \ n \in \mathbb{N}, \tag{2.14}$$

having the properties

$$\begin{gathered} 0 \leq y_n(x) \leq y_{n+1}(x) < -c_n \varphi(x,0), \quad n \in \mathbb{N}, \\ c_{n+1} = \frac{1}{4h} \frac{1}{1 - hc_n}, \quad c_1 = \frac{1}{4h}, \ n \in \mathbb{N}, \\ y_n(b) = 0, \quad n \in \mathbb{N}. \end{gathered} \tag{2.15}$$

For $n = 1$, from (2.14) and using (2.13), we have

$$
\begin{aligned}
y_2(x) &= \int_b^x -\frac{g(s)}{\varphi(s,y_1(s))}ds \\
&= \int_0^x -\frac{g(s)}{\varphi(s,0)}ds - \int_0^b -\frac{g(s)}{\varphi(s,0)}ds \\
&< \int_0^x -\frac{g(s)}{\varphi(s,0)}ds \le -\frac{1}{4h}\varphi(x,0).
\end{aligned} \tag{2.16}
$$

Hence,

$$
0 = y_1(x) < y_2(x) < -\frac{1}{4h}\varphi(x,0), \quad x \in [x_0, b]. \tag{2.17}
$$

We denote

$$
c_1 = \frac{1}{4h}. \tag{2.18}
$$

For $n = 2$, taking into account the following inequalities:

$$
\varphi(s,y_2(s)) \le \varphi(s,0) + hy_2(s) \le \frac{3}{4}\varphi(s,0), \tag{2.19}
$$

we have

$$
\begin{aligned}
y_3(x) &= \int_b^x -\frac{g(s)}{\varphi(s,y_2(s))}ds < \int_0^x -\frac{g(s)}{\varphi(s,y_2(s))}ds \le \int_0^b -\frac{g(s)}{(3/4)\varphi(s,0)}ds \\
&\le -\frac{4}{3}\frac{1}{4h}\varphi(x,0) = -\frac{1}{3h}\varphi(x,0).
\end{aligned} \tag{2.20}
$$

We denote

$$
c_2 = \frac{1}{3h}. \tag{2.21}
$$

Suppose that $y_{n+1}(x) < -c_n\varphi(x,0)$, and using the recurrence relation (2.14), we search a constant c_{n+1} such that $y_{n+2}(x) < -c_{n+1}\varphi(x,0)$.

Since

$$
\begin{aligned}
y_{n+2}(x) &= \int_b^x -\frac{g(s)}{\varphi(s,y_{n+1}(s))}ds < \int_0^x -\frac{g(s)}{\varphi(s,y_{n+1}(s))}ds \\
&\le \int_0^b -\frac{g(s)}{(1-hc_n)\varphi(s,0)}ds \le -\frac{1}{4h}\frac{1}{1-hc_n}\varphi(x,0),
\end{aligned} \tag{2.22}
$$

we take

$$
c_{n+1} = \frac{1}{4h}\frac{1}{1-hc_n}. \tag{2.23}
$$

Thus all the properties of (2.15) are satisfied. It is obvious that the sequence defined by (2.23) with (2.18) is convergent and tends to $1/2h$. The function sequence defined by (2.14) converges uniformly to a continuous function, let it be $\overline{y}(\cdot)$. From (2.15) we have

$$\begin{aligned}\overline{y}(x) &\le -\frac{1}{2h}\varphi(x,0),\\ \overline{y}(x) &= \int_b^x -\frac{g(s)}{\varphi(s,\overline{y}(s))}ds, \quad x \in [x_0,b].\end{aligned} \tag{2.24}$$

But the last relation in (2.24) shows that the function $\overline{y}(\cdot)$ is a solution of the following Cauchy problem:

$$\overline{y}'(x) = -\frac{g(x)}{\varphi(x,\overline{y}(x))}, \quad \overline{y}(b) = 0, \quad x \in [x_0,b]. \tag{2.25}$$

Then

$$y(x) = \overline{y}(x), \quad x \in [x_0,b]. \tag{2.26}$$

On the other side

$$\overline{y}(x_0) \le -\frac{1}{2h}\varphi(x_0,0) \le \frac{y_0}{2}. \tag{2.27}$$

This is a contradiction and the theorem is proved. □

Now we introduce our main result.

Theorem 2.6. *Consider system (1.1) under the assumptions (1.4), (1.5), (2.8), (2.9), (2.10), (2.11), (2.12), and (2.13). Then the origin is globally attractive but not stable.*

Proof. First of all we remark that (2.11) implies that the first part of (2.8) holds.

Consider a point $B = (x_0, y_0)$, with $-a < x_0 < 0$, on the characteristic curve. Then, by Theorem 2.5, the trajectory of system (1.1) passing through the point B approaches the origin as $t \to \underline{t}$. This shows that the origin is not stable.

Consider a point $B = (0, y_0)$, $y_0 > 0$. From the phase-portrait analysis, the trajectory of system (1.1) which passes through the point B enters the region D_1 and crosses the characteristic curve, by Lemma 2.2, as t increases.

Consider a point $B = (x_0, y_0)$, with $x_0 > 0$, on the characteristic curve. Then, by Theorem 2.4, the trajectory of system (1.1) passing through the point B approaches the origin as $t \to \overline{t}$, so that it remains in the region D_2.

Consider a point $B = (x_0, y_0) \in D_2$. Then, by the proof of Theorem 2.4, the trajectory of system (1.1) passing through the point B approaches the origin as $t \to \overline{t}$ so that it remains in the region D_2.

Consider a point $B = (x_0, y_0)$, with $x_0 > 0$, $y_0 \le 0$, and the trajectory of system (1.1) passing through B at the moment $t = 0$. Then $y'(0) = -g(x_0) < 0$, and by Lemma 2.1 this trajectory crosses the y-axis in a finite time and at a finite point $y < y_0$ as t increases.

Consider a point $B = (0, y_0)$, with $y_0 < 0$, and the trajectory of system (1.1) passing through B at the moment $t = 0$. Then, by Lemma 2.2 it has to cross the characteristic curve in a finite time.

Going further and invoking once again Lemma 2.1, we reach the positive y-semi axis. □

Remark 2.7. (a) A similar result holds if instead of (2.11) we consider

$$\varphi(x,0) < 0, \quad x < 0, \tag{2.28}$$

and instead of (2.12)

$$\frac{1}{\varphi(x,0)} \int_x^{0^-} \frac{g(s)}{\varphi(s,0)} ds \le \frac{1}{4h}, \quad \forall x < 0. \tag{2.29}$$

(b) In [6], the following repulsivity conditions, amongst other conditions, were used

(i) there exists an $a > 0$ such that $|\varphi(x,0)| \ne 0$, for $0 < x \le a$ and

$$\frac{1}{\varphi(x,0)} \int_{0^+}^{x} \frac{g(s)}{\varphi(s,0)} ds \ge \alpha > \frac{1}{4k}, \quad \forall x \in (0,a]; \tag{2.30}$$

(ii) there exists an $a > 0$ such that $|\varphi(x,0)| \ne 0$, for $-a < x < 0$ and

$$\frac{1}{\varphi(x,0)} \int_{0^-}^{x} \frac{g(s)}{\varphi(s,0)} ds \ge \alpha > \frac{1}{4k}, \quad \forall x \in [-a,0). \tag{2.31}$$

These conditions are useful in oscillation problems to exclude the case when $(x(t),y(t)) \to 0$, for $t \to \bar{t}$.

In order to have a globally attractive origin we need a nonrepulsive origin. The following assumptions seem to be natural:

- there exists an $a > 0$ such that

$$\begin{gathered} |\varphi(x,0)| \ne 0, \quad \forall x \in (0,a], \\ \frac{1}{\varphi(x,0)} \int_{0^+}^{x} \frac{g(s)}{\varphi(s,0)} ds \le \frac{1}{4h}, \quad \forall x \in (0,a]; \end{gathered} \tag{2.32}$$

- there exists an $a > 0$ such that

$$\begin{gathered} |\varphi(x,0)| \ne 0, \quad \forall\, x \in [-a,0), \\ \frac{1}{\varphi(x,0)} \int_{0^-}^{x} \frac{g(s)}{\varphi(s,0)} ds \le \frac{1}{4h}, \quad \forall x \in [-a,0). \end{gathered} \tag{2.33}$$

We remark that for the Liénard system (1.2) we may take $k = h = 1$, and (2.32) and (2.33) can be found in [2] or in [5].

Acknowledgements. The author expresses his gratitude to the referees for their proper remarks which improved considerably this paper.

The author was partially supported by a grant from the Ministry of National Education of România.

References

[1] W. Hahn, *Stability of Motion*, Die Grundlehren der mathematischen Wissenschaften, vol. 138, Springer-Verlag, Berlin, 1967. MR 36#6716. Zbl 189.38503.

[2] T. Hara, *On the Vinograd type theorems for Liénard system*, Nonlinear Anal. **20** (1993), no. 6, 647–658. MR 94a:34068. Zbl 773.34040.

[3] ______, *Notice on the Vinograd type theorems for Liénard system*, Nonlinear Anal. **22** (1994), no. 12, 1437–1443. MR 95e:34047. Zbl 803.34043.

[4] N. N. Krasovskiĭ, *On stability of solutions of a system of two differential equations*, Akad. Nauk SSSR. Prikl. Mat. Meh. **17** (1953), no. 6, 651–672 (Russian). MR 15,624e.

[5] M. Mureşan, *Ultimately positive solutions of Liénard type equations*, Libertas Math. **15** (1995), 165–173. MR 96m:34051. Zbl 840.34022.

[6] M. Mureşan and N. Vornicescu, *On oscillation conditions for Liénard-type equation*, Anal. Numér. Théor. Approx. **17** (1988), no. 2, 157–169. MR 90k:34037. Zbl 673.34042.

[7] N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method*, Applied Mathematical Sciences, vol. 22, Springer-Verlag, New York, 1977. MR 56 #9008. Zbl 364.34022.

[8] R. E. Vinograd, *The inadequacy of the method of characteristic exponents for the study of nonlinear differential equations*, Math. Sb. (N.S) **41** (1957), 431–438 (Russian).

Marian Mureşan: Faculty of Mathematics and Computer Science, Babeş-Bolyai University, M. Kogalniceanu 1, 3400 Cluj-Napoca, Romania

E-mail address: mmarian@math.ubbcluj.ro