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# QUASI-PSEUDOMETRIZABILITY OF THE POINT OPEN ORDERED SPACES AND THE COMPACT OPEN ORDERED SPACES

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Abstract. We determine conditions for quasi-pseudometrizability of the point open ordered spaces and the compact open ordered spaces. This generalizes the results on metrizability of the point open topology and the compact open topology for function spaces. We also study conditions for complete quasi-pseudometrizability.

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**1. Introduction.** An ordered topological space (or simply, an ordered space) is a triple $(X,\tau,\le)$, where $\tau$ is a topology on $X$ and $\le$ is a closed partial order on $X$. We will use $\mathbb{R}_0$ to denote the set of real numbers with the usual topology and the usual order. For a subset $A$ of the set of real numbers, we use $\mathbf{A}_0$ to denote $\mathbf{A}$ as an ordered subspace of $\mathbb{R}_0$, for example $\mathbf{I}_0$ denote the closed interval with the usual topology and the usual order. A mapping $f:(X,\tau,\le)\to(Y,\tau',\le')$ between two ordered topological spaces $(X,\tau,\le)$ and $(Y,\tau',\le')$ is said to be order preserving (order reversing) if $f(x)\le' f(y)$ $(f(y)\le' f(x))$ whenever $x,y\in X$ and $x\le y$, and is continuous if it is continuous with respect to the given topologies. If $(X,\le)$ is a partially ordered set and $A$ is a nonempty subset of $X$, we define $d(A)=\{y\in X : y\le x \text{ for some } x\in A\}$ to be the decreasing hull of $A$; the increasing hull $i(A)$ is defined dually. We will write $d(x)(i(x))$ in place of $d(\{x\})(i(\{x\}))$. If $A=d(A)\cap i(A)$, then $A$ is said to be convex. An ordered space $(X,\tau,\le)$ is $T_2$-ordered if the order $\le$ is closed in $X\times X$. A topological space $(Y,\tau')$ contains a path if there is a continuous function $p:(\mathbf{I}$, usual topology$)\to(Y,\tau')$ such that $p(0)\ne p(1)$. Similarly, an ordered space $(Y,\tau',\le')$ contains an ordered path if there is a continuous order-preserving function $p:\mathbf{I}_0\to(Y,\tau',\le')$ such that $p(0)\ne p(1)$. For ordered spaces we refer the reader to [3], and for quasi-uniformities we refer the reader to [1].

Let $(X,\tau,\le)$ and $(Y,\tau',\le')$ be two ordered topological spaces. Let $C(X,Y)$ denote the set of all continuous functions from $(X,\tau)$ to $(Y,\tau')$, and let $C^{\uparrow}(X,Y)$ denote the subset of $C(X,Y)$ consisting of all continuous order-preserving functions from $(X,\tau,\le)$ to $(Y,\tau',\le')$. Let $\mathcal{F}=\{F\subseteq X : F \text{ is finite}\}$. For each $F\in\mathcal{F}$ and $G\in\tau'$, consider the set $[F,G]=\{f\in C(X,Y) : f(F)\subseteq G\}$. The collection $\{[F,G] : F\in\mathcal{F} \text{ and } G\in\tau\}$ is a subbase for a topology $T_p$ on $C(X,Y)$. Define an order on $C(X,Y)$ as follows: for each $f,g\in C(X,Y)$, $f\le_s g \Leftrightarrow f(x)\le' g(x)$ for all $x\in X$ (i.e., $\le_s$ is the order on $C(X,Y)$ defined pointwise). We refer to the ordered space $(C(X,Y),T_p,\le_s)$ as the point open ordered space. The subset $C^{\uparrow}(X,Y)$ of $C(X,Y)$ with the restriction of this topology and order will be denoted by $(C^{\uparrow}(X,Y),T_p,\le_s)$.

If $Y$ is a set and $\mathcal{U}$ is a quasi-uniformity on $Y$ such that $(Y, T(\mathcal{U}))$ is $T_0$, then $(Y, T(\mathcal{U} \vee \mathcal{U}^{-1}), \cap\mathcal{U})$ is an ordered space. In fact, $(Y, T(\mathcal{U} \vee \mathcal{U}^{-1}), \cap\mathcal{U})$ is a completely regular ordered space. An ordered space $(Y, \tau', \leq')$ is said to admit a quasi-uniformity $\mathcal{U}$ on $Y$ if $\tau' = T(\mathcal{U} \vee \mathcal{U}^{-1})$ and $\leq' = \cap\mathcal{U}$, in this case we say the quasi-uniformity $\mathcal{U}$ is compatible with $(\tau', \leq')$, and $(Y, \tau', \leq')$ is also said to be quasi-uniformizable. It is well known (see [1, 3]) that an ordered space is quasi-uniformizable if and only if it is a completely regular ordered space.

Let $(Y, \tau', \leq')$ be quasi-uniformizable and $\mathcal{U}$ a quasi-uniformity compatible with $(\tau', \leq')$. For each $F \in \mathcal{F}$ and each $U \in \mathcal{U}$ we consider the set $(F, U) = \{(f, g) \in C(X, Y) \times C(X, Y) : (f(x), g(x)) \in U$ for all $x \in F\}$. Then $\{(F, U) : F \in \mathcal{F}$ and $U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_p$ on $C(X, Y)$ called the quasi-uniformity of quasi-uniform pointwise convergence induced by $\mathcal{U}$. It was shown in [4] that $(C(X, Y), \mathcal{U}_p^*, \cap\mathcal{U}_p)$, where $\mathcal{U}_p^* = \mathcal{U}_p \vee \mathcal{U}_p^{-1}$, coincides with the point open ordered space.

Let $(X, \tau, \leq)$ and $(Y, \tau', \leq')$ be two ordered topological spaces. Let $\mathcal{K} = \{K \subseteq X : K$ is compact$\}$. For each $K \in \mathcal{K}$ and $G \in \tau'$; consider the set $[K, G] = \{f \in C(X, Y) : f(K) \subseteq G\}$. The collection $\{[K, G] : F \in \mathcal{K}$ and $G \in \tau\}$ is a subbase for a topology $T_k$ on $C(X, Y)$. Define an order on $C(X, Y)$ as follows: for each $f, g \in C(X, Y)$, $f \leq_s g \Leftrightarrow f(x) \leq' g(x)$ for all $x \in X$ (i.e., $\leq_s$ is the order on $C(X, Y)$ defined pointwise). We refer to the ordered space $(C(X, Y), T_k, \leq_s)$ as the compact open ordered space. The subset $C^{\uparrow}(X, Y)$ of $C(X, Y)$ with the restriction of this topology and order will be denoted by $(C^{\uparrow}(X, Y), T_k, \leq_s)$.

Analogously, let $(Y, \tau', \leq')$ be quasi-uniformizable and $\mathcal{U}$ be a quasi-uniformity compatible with $(\tau', \leq')$. For each $K \in \mathcal{K}$ and each $U \in \mathcal{U}$ we consider the set $(K, U) = \{(f, g) \in C(X, Y) \times C(X, Y) : (f(x), g(x)) \in U$ for all $x \in K\}$. Then $\{(K, U) : K \in \mathcal{K}$ and $U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_k$ on $C^{\uparrow}(X, Y)$, which we refer to as the quasi-uniformity of quasi-uniform convergence on compacta. It was shown in [4] that the ordered space $(C(X, Y), T(\mathcal{U}_k^*), \cap\mathcal{U}_k)$, where $\mathcal{U}_k^* = \mathcal{U}_k \vee \mathcal{U}_k^{-1}$, coincides with the compact open ordered space.

The following strengthening of complete regularity was introduced in [2].

Let $(X, \tau, \leq)$ be a topological ordered space. Then $X$ is said to be strictly completely regular ordered space if (i) the order on $X$ is semi-closed, that is, $d(a)$ and $i(a)$ are closed for all $a \in X$. (ii) $X$ is strongly order convex, that is, the open upper sets and the open lower sets form a subbasis for the topology. (iii) Given a closed lower (resp., upper) set $A$ and a point $x \notin A$, there exists a continuous order-preserving function $f : (X, \tau, \leq) \to \mathbf{I}_0$ such that $f(A) = 0$ and $f(x) = 1$ (resp., $f(A) = 1$ and $f(x) = 0$).

Recall that a quasi-pseudometric on a set $X$ is a nonnegative real-valued function $d$ on $X \times X$ such that for all $x, y, z \in X$,

(i) $d(x, x) = 0$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-pseudometric $d$ will be called separating if $d(x, y) + d(y, x) > 0$ whenever $x \neq y$ and is called a quasi-metric if $d(x, y) > 0$ whenever $x \neq y$.

Each quasi-(pseudo)metric $d$ on $X$ generates a topology $T(d)$ on $X$ which has a base family of $d$-balls $\{B_d(x, r) : x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$. The conjugate quasi-(pseudo)metric $d^{-1}$ of $d$ given by $d^{-1}(x, y) = d(y, x)$ is also a quasi-(pseudo)metric. The topology generated by the conjugate quasi-(pseudo)metric is denoted by $T(d^{-1})$. A quasi-(pseudo)metric $d$ on $X$ is said to be compatible with an ordered space $(X,\tau,\leq)$ if $\tau = T(d\vee d^{-1})$ and $\leq=\leq_{T(d)}$, where $\leq_{T(d)}$ is the specialization order with respect to the topology $T(d)$, that is, $x\leq_{T(d)} y \Leftrightarrow x\in \mathrm{cl}_{T(d)}(y)\Leftrightarrow d(x,y)=0$.

An ordered space $(X,\tau,\leq)$ is said to be (completely) (separated) quasi-(pseudo)-metrizable if there is a (completely) (separating) quasi-(pseudo)metric on $X$ compatible with $(X,\tau,\leq)$. It is known that a quasi-pseudometric $d$ on $X$ is separating if and only if $(X,T(d)\vee T(d^{-1}))$ is a Hausdorff space. If $d$ is a separating quasi-pseudometric on a set $X$, then $d\vee d^{-1}$, defined by $d\vee d^{-1}(x,y) = \max\{d(x,y),d^{-1}(x,y)\}$, is a metric on $X$ compatible with $(T(d)\vee T(d^{-1}))$.

The following are the well-known results which will be generalized to the ordered case.

**Theorem 1.1.** *Let $(X,\tau)$ and $(Y,\tau')$ be two topological spaces such that $(Y,\tau')$ contains a nontrivial path. The topology of pointwise convergence is metrizable if and only if $(X,\tau)$ is countable and $(Y,\tau')$ is metrizable.*

**Theorem 1.2.** *Let $(X,\tau)$ and $(Y,\tau')$ be two topological spaces such that $(Y,\tau')$ contains a nontrivial path. The topology of pointwise convergence is completely metrizable if and only if $(X,\tau)$ is countable and discrete and $(Y,\tau')$ is completely metrizable.*

**Theorem 1.3.** *Let $(X,\tau)$ and $(Y,\tau')$ be two topological spaces such that $(Y,\tau')$ contains a nontrivial path. The topology of compact convergence is metrizable if and only if $(X,\tau)$ is hemicompact and $(Y,\tau')$ is metrizable.*

**Theorem 1.4.** *Let $(X,\tau)$ and $(Y,\tau')$ be two topological spaces such that $(Y,\tau')$ contains a nontrivial path. The topology of compact convergence is completely metrizable if and only if $(X,\tau)$ is a hemicompact $k$-space and $(Y,\tau')$ is completely metrizable.*

**2. Quasi-pseudometrizability of the point open ordered spaces.** In this section, we discuss the quasi-pseudometrizability of the point open ordered spaces. The results generalize the classical ones in topological spaces.

**Theorem 2.1.** *Let $(X,\tau,\leq)$ be a $T_2$-ordered quasi-uniformizable ordered space and $(Y,\tau^{'},\leq^{'})$ an ordered topological space such that $(Y,\tau^{'})$ contains a nontrivial path. Then the point open ordered space is separated quasi-pseudometrizable if and only if $X$ is countable and $(Y,\tau^{'},\leq^{'})$ is separated quasi-pseudometrizable.*

**Proof.** Suppose that the point open ordered space is separated quasi-pseudometrizable and let $d$ be a separating quasi-pseudometric on $C(X,Y)$ compatible with $(T_p,\leq_s)$. For each $y\in Y$ we define $f_y : X\to Y$ by $f_y(x) = y$ for all $x\in X$. Since $f_y$ is a constant, $f_y\in C(X,Y)$. For each $y,z\in Y$, define $\upsilon(y,z) = d(f_y,f_z)$. It is clear that $\upsilon$ is a separating quasi-pseudometric. We now show that $\upsilon$ is compatible with $(\tau^{'},\leq^{'})$, that is, $\tau^{'} = T(\upsilon\vee\upsilon^{-1})$ and $\leq^{\dagger}=\leq_{T(\upsilon)}$. Let $y\in U\in\tau^{'}$. Now for each $x\in X$, $f_y\in[x,U]$. Since $[x,U]$ is open in $T_p$ and $(T_p,\leq_s)$ is separated quasi-pseudometrizable, there is $r>0$ such that $B_{d^*}(f_y,r)\subseteq[x,U]$. Then

$$\begin{aligned} z\in B_{\upsilon^*}(y,r) &\Longrightarrow \upsilon^*(y,z)<r\\ &\Longrightarrow d^*(f_y,f_z)<r\\ &\Longrightarrow f_z\in B_{d^*}(f_y,r) \end{aligned}$$

$$\begin{aligned} &\Longrightarrow f_z \in [x,U] \\ &\Longrightarrow f_z(x) \in U \\ &\Longrightarrow z \in U. \end{aligned} \tag{2.1}$$

Therefore $\tau' \subseteq T(\upsilon \vee \upsilon^{-1})$. On the other hand, we have

$$\begin{aligned} z \in B_{\upsilon^*}(y,r) &\Longrightarrow \upsilon^*(y,z) < r \\ &\Longrightarrow d^*(f_y,f_z) < r \\ &\Longrightarrow f_z \in B_{d^*}(f_y,r) \\ &\Longrightarrow f_z \in \bigcap_{i=1}^{n}[x_i,U_i] \subseteq B_{d^*}(f_y,r) \\ &\Longrightarrow z \in \bigcap_{i=1}^{n} U_i \subseteq B_{\upsilon^*}(y,r) \\ &\Longrightarrow B_{\upsilon^*}(y,r) \text{ is open in } \tau'. \end{aligned} \tag{2.2}$$

Therefore $\tau' = T(\upsilon \vee \upsilon^{-1})$.
Now we show that $\le' = \le_{T(\upsilon)}$,

$$\begin{aligned} y \le^{\uparrow} z &\Longleftrightarrow f_y \le_s f_z \\ &\Longleftrightarrow f_y \le_{T(d)} f_z \quad \text{since } \le_s = \le_{T(d)} \\ &\Longleftrightarrow f_y \in \mathrm{cl}_{T(d)}\{f_z\} \\ &\Longleftrightarrow f_z \in B_d(f_y,r) \quad \forall r > 0 \\ &\Longleftrightarrow z \in B_\upsilon(y,r) \quad \forall r > 0 \\ &\Longleftrightarrow y \le_{T(\upsilon)} z. \end{aligned} \tag{2.3}$$

Since $\upsilon^*$ is a metric on $C(X,Y)$ compatible with $T_p$, it follows from Theorem 1.1 that $X$ is countable.

Conversely, suppose $(Y,\tau',\le')$ is quasi-pseudometrizable and $X$ is countable. Then $(\Pi_{x\in X} Y_x, \Pi_{x\in X}(\tau')_x, \Pi_{x\in X}(\le')_x)$, where $Y_x = Y$, $(\tau')_x = \tau'$, and $(\le')_x = \le'$ for all $x \in X$, is a product of countable separated quasi-pseudometrizable ordered spaces and therefore is a separated quasi-pseudometrizable ordered space. Then the point open ordered space is an ordered subspace of a separated quasi-pseudometrizable ordered space and hence it is a separated quasi-pseudometrizable ordered space. □

The following theorem in bitopological space is useful for the results on bicomplete separated quasi-pseudometrizability. It was proved by Romaguera and Salbany in [6].

**Theorem 2.2** [6]. *A (separated) quasi-(pseudo)metrizable bitopological space $(X,\tau_1,\tau_2)$ is bicompletely (separated) quasi-(pseudo)metrizable if and only if $(X,\tau_1 \vee \tau_2)$ is completely metrizable.*

**Corollary 2.3.** *A (separated) quasi-(pseudo)metrizable ordered space $(X,\tau,\le)$ is bicompletely (separated) quasi-(pseudo)metrizable if and only if $(X,\tau)$ is completely metrizable.*

**Proof.** If $(X,\tau,\le)$ is bicompletely (separated) quasi-(pseudo)metrizable, then $(X,\tau)$ is obviously completely metrizable. Conversely, suppose that $(X,\tau,\le)$ is (separated) quasi-(pseudo)metrizable ordered space and $(X,\tau)$ is completely metrizable. Let $d$ be a (separated) quasi-(pseudo)metric compatible with $(\tau,\le)$ and let $(X,T(d),T(d^{-1}))$ be the corresponding bitopological space. Since $(X,T(d)\vee T(d^{-1}))=(X,\tau)$ is completely metrizable, by Theorem 2.2 $(X,T(d),T(d^{-1}))$ is bicompletely (separated) quasi-(pseudo)metrizable. Therefore, $(X,T(d)\vee T(d^{-1}),\le_{T(d)})=(X,\tau,\le)$ is bicompletely (separated) quasi-(pseudo)metrizable. □

**Theorem 2.4.** *Let $(X,\tau,\le)$ be a $T_2$-ordered quasi-uniformizable ordered space and $(Y,\tau',\le')$ an ordered topological space such that $(Y,\tau')$ contains a nontrivial path. Then the point open ordered space is bicompletely separated quasi-pseudometrizable if and only if $X$ is countable and discrete and $(Y,\tau',\le')$ is bicompletely separated quasi-pseudometrizable.*

**Proof.** Suppose that the point open ordered space is bicompletely separated quasi-pseudometrizable. Since $T_p$ is completely metrizable, by Theorem 1.2 $(X,\tau)$ is countable and discrete and $(Y,\tau')$ is completely metrizable. By Theorem 2.1, $(Y,\tau',\le')$ is separated quasi-pseudometrizable. By Corollary 2.3, $(Y,\tau',\le')$ is bicompletely separated quasi-pseudometrizable.

Conversely, suppose that $X$ is countable and discrete and $(Y,\tau',\le')$ is bicompletely separated quasi-pseudometrizable. Since $(Y,\tau')$ is completely metrizable, by Theorem 1.2, $T_p$ is completely metrizable. By Theorem 2.1, $(C(X,Y),T_p,\le_s)$ is separated quasi-pseudometrizable. By Corollary 2.3, the point open ordered space is bicompletely separated quasi-pseudometrizable. □

**Proposition 2.5.** *Let $(X,\tau,\le)$ be a strictly completely regular ordered space in which every closed set is convex, and let $(Y,\tau',\le')$ be an ordered topological space containing an ordered path. Then the following statements are equivalent:*

(i) *$X$ is countable and $(Y,\tau',\le')$ is separated quasi-pseudometrizable.*
(ii) *$(C(X,Y),T_p,\le_s)$ is separated quasi-pseudometrizable.*
(iii) *$(C^{\uparrow}(X,Y),T_p,\le_s)$ is separated quasi-pseudometrizable.*

**Proof.** (i)⇒(ii) follows from Theorem 2.1.

(ii)⇒(iii) follows from the fact that an ordered subspace of a separated quasi-pseudometrizable ordered space is separated quasi-pseudometrizable.

(iii)⇒(i). Let $d$ be a separating quasi-pseudometric on $C^{\uparrow}(X,Y)$ compatible with $(T_p,\le_s)$. For each $y\in Y$ define $f_y:X\to Y$ by $f_y(x)=y$ for all $x\in X$. Since $f_y\in C^{\uparrow}(X,Y)$, as in the proof of Theorem 2.1, it can be shown that the real-valued function $\rho$, defined on $Y\times Y$ by $\rho(y,z)=d(f_y,f_z)$, is a separating quasi-pseudometric compatible with $(\tau',\le')$.

We now show that $X$ is countable. Let $p:\mathbf{I}_0\to(Y,\tau',\le')$ be an ordered path. Now define $f:X\to Y$ by $f(x)=p(0)$ for all $x\in X$. Then $f\in C^{\uparrow}(X,Y)$. Since $f$ is a constant function, for each $n\in\mathbb{N}$ there is a finite set $F_n\subseteq X$ and a $\tau'$ open set $H_n$ such that $f\in[F_n,H_n]\subseteq B_{\rho*}(f,2^{-n})$. We now show that $X=\bigcup\{F_n:n\in\mathbb{N}\}$. Suppose on the contrary that we have $x\in X\setminus\bigcup\{F_n:n\in\mathbb{N}\}$. Since for each $F_n$, we have $F_n=i(F_n)\cap d(F_n)$, without loss of generality, we can assume that $x\notin d(F_n)$. By strict complete regularity of $(X,\tau,\leq)$ there is a continuous order-preserving function $\varphi:(X,\tau,\leq)\to \mathbf{I}_0$ such that $\varphi(x)=1$ and $\varphi(d(F_n))=0$. Now $p\circ\varphi\in C^{\uparrow}(X,Y)$. Since $p(0)\neq p(1)$ and $\rho^*$ is a metric, we have $\rho*(p(0),p(1))>2^{-m}$ for some $m$. Therefore $p\circ\varphi\in[F_n,H_n]\subseteq B_{\rho*}(f,2^{-n})$ for all $n$. But for $n=m$ we have $p\circ\varphi\notin B_{\rho*}(f,2^{-m})$ which is a contradiction. Therefore $X=\bigcup\{F_n:n\in\mathbb{N}\}$ and this completes the proof. □

**3. Quasi-pseudometrizability of the compact open ordered spaces.** In this section, we discuss the quasi-pseudometrizability of the compact open ordered spaces. We generalize Theorems 1.1 and 1.2.

A topological space $X$ is said to be hemicompact if there is a sequence $K_1,K_2,\dots$ of compact subsets of $X$ such that if $K$ is any compact subset of $X$, then $K\subseteq K_n$ for some $n$.

**Theorem 3.1.** *Let $(X,\tau,\leq)$ be a $T_2$-ordered quasi-uniformizable ordered space, and let $(Y,\tau',\leq')$ be an ordered topological space containing a nontrivial path. Then the compact open ordered space is separated quasi-pseudometrizable if and only if $(X,\tau)$ is hemicompact and $(Y,\tau',\leq')$ is separated quasi-pseudometrizable.*

**Proof.** Suppose that there is a separating quasi-pseudometric $d$ on $C(X,Y)$ compatible with $(T_k,\leq_s)$. For each $y\in Y$, we define $f_y:X\to Y$ by $f_y(x)=y$ for all $x\in X$. Since $f_y$ is a constant, $f_y\in C(X,Y)$. For each $y,z\in Y$, define $\rho(y,z)=d(f_y,f_z)$. As in the proof of Theorem 2.1, $\rho$ is a separating quasi-pseudometric compatible with $(\tau',\leq')$. Since $d^*$ is a metric on $C(X,Y)$ compatible with $T_k$, it follows from Theorem 1.3 that $(X,\tau)$ is hemicompact.

Conversely, suppose that $(X,\tau)$ is hemicompact and $(Y,\tau',\leq')$ is a separated quasi-pseudometrizable. Let $(K_n)$ be a sequence of $\tau$-compact subsets of $X$ satisfying the condition for hemicompactness of $(X,\tau)$ and let $d$ be a separating quasi-pseudometric on $Y\times Y$ compatible with $(\tau',\leq')$. For each $m\in\mathbb{N}$, put $V_m=\{(y,z)\in Y\times Y : d(y,z)<1/m\}$. Then $\mathcal{B}=\{(K_n,V_m):n,m\in\mathbb{N}\}$ is a base for the quasi-uniformity of quasi-uniform convergence of $(\mathcal{U}(d))$ on compacta. Then there is a quasi-pseudometric $\rho$ on $C(X,Y)$ compatible with the bitopology of quasi-uniform convergence (of $\mathcal{U}(d)$) on 2 compacta. Then the corresponding ordered space is the compact open ordered space. By Theorem 11 of [5], $\rho$ is compatible with the compact open ordered space. It is obvious that $\rho$ is a separating quasi-pseudometric. □

**Theorem 3.2.** *Let $(X,\tau,\leq)$ be a $T_2$-ordered quasi-uniformizable ordered space and $(Y,\tau',\leq')$ an ordered topological space containing a nontrivial path. Then the compact open ordered space is bicompletely separated quasi-pseudometrizable if and only if $X$ is a hemicompact $k$-space and $(Y,\tau',\leq')$ is bicompletely separated quasi-pseudometrizable.*

**Proof.** Suppose that $d$ is a bicompletely separating quasi-pseudometric on $C(X,Y)$ compatible with $(T_k,\leq_s)$. By Theorem 3.1, $X$ is hemicompact and $(Y,\tau',\leq')$ is a separated quasi-pseudometrizable. Since $d^*$ is a complete metric on $C(X,Y)$ compatible with the compact open topology $T_k$, $(X,\tau)$ is a $k$-space and $(Y,\tau')$ is completely metrizable. By Corollary 2.3, $(Y,\tau',\leq')$ is bicompletely separated quasi-pseudometrizable.

Conversely, suppose that $X$ is hemicompact $k$-space and $(Y,\tau',\leq')$ is bicompletely separated quasi-pseudometrizable. Since $(Y,\tau')$ is completely metrizable, the compact open ordered space $T_k$ is completely metrizable. By Theorem 3.1, $(C(X,Y), T_k, \leq_s)$ is a separated quasi-pseudometrizable. By Corollary 2.3, $(C(X,Y), T_k, \leq_s)$ is bicompletely separated quasi-pseudometrizable. □

**Proposition 3.3.** *Let $(X,\tau,\leq)$ be a strictly completely regular ordered space in which every closed set is convex, and let $(Y,\tau',\leq')$ be an ordered topological space containing a nontrivial ordered path. Then the following statements are equivalent:*

(i) *$(X,\tau)$ is hemicompact and $(Y,\tau',\leq')$ is separated quasi-pseudometrizable.*

(ii) *$(C(X,Y), T_k, \leq_s)$ is separated quasi-pseudometrizable.*

(iii) *$(C^{\uparrow}(X,Y), T_k, \leq_s)$ is separated quasi-pseudometrizable.*

**Proof.** (i)⇒(ii) follows from Theorem 3.1.

(ii)⇒(iii) follows from the fact that an ordered subspace of a separated quasi-pseudometrizable ordered space is separated quasi-pseudometrizable.

(iii)⇒(i). Let $d$ be a separating quasi-pseudometric on $C^{\uparrow}(X,Y)$ compatible with $(T_k, \leq_s)$. For each $y \in Y$ define $f_y : X \to Y$ by $f_y(x) = y$ for all $x \in X$. Since $f_y \in C^{\uparrow}(X,Y)$, as in the proof of Theorem 2.1, it can be shown that the real-valued function $\rho$, defined on $Y \times Y$ by $\rho(y,z) = d(f_y, f_z)$, is a separating quasi-pseudometric compatible with $(\tau', \leq')$.

We now show that $(X,\tau)$ is a hemicompact. Let $p : \mathbf{I}_0 \to (Y,\tau',\leq')$ be an ordered path. Define $f : X \to Y$ by $f(x) = p(0)$ for all $x \in X$. Let $V_n = \{h \in C^{\uparrow}(X,Y) : d^*(f,h) < 1/n\}$ for all $n \in \mathbb{N}$. Then $V_n$ is open in $T(d^*)$. Since $T(d^*) = T_k$, each $V_n$ is a $T_k$-neighborhood of $f$. Then there is a sequence $(K_n)$ of $\tau$-compact subsets of $X$ and a decreasing sequence $(\sigma_n)$ of positive numbers such that $(\sigma_n) \to 0$ and $f \in [K_n, B_{\rho^*}(p(0), \sigma_m)] \subseteq V_n$. Since $p(1) \neq p(0)$ and $\rho^*$ is a metric, we have $\rho^*(p(0), p(1)) > \sigma_m$ for some $m$. Let $K \in \mathcal{K}$ and put $A = [K, B_{\rho^*}(p(0), \sigma_m)]$. Then there is $n$, such that $V_n \subseteq A$. We now show that $K \subseteq K_n$. Suppose on the contrary that there is $x_0 \in K \setminus K_n$. Since $K_n$ is compact by [3, Proposition 4, page 44], $d(K_n)$ is closed. Since $K_n = i(K_n) \cap d(K_n)$, without loss of generality, we can assume that $x_0 \notin d(K_n)$. By strict complete regularity of $(X,\tau,\leq)$ there is a continuous order-preserving function $\Psi : (X,\tau,\leq) \to \mathbf{I}_0$ such that $\Psi(x_0) = 1$ and $\Psi(d(K_n)) = 0$. Then $p \circ \Psi \in C^{\uparrow}(X,Y)$ since both $p$ and $\Psi$ are continuous order-preserving functions. We also have that $p \circ \Psi \in [K_n, B_{\rho^*}(p(0), \sigma_m)] \subseteq V_n \subseteq A$ since $p \circ \Psi(K_n) = p(0)$. But on the other hand, $p \circ \Psi(x_0) = 1$ implies $p \circ \Psi \notin A$. This contradiction shows that $K \subseteq K_n$ and this concludes the proof. □

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