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ON NONAUTONOMOUS SECOND-ORDER DIFFERENTIAL EQUATIONS ON BANACH SPACE

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ABSTRACT. We show the existence and uniqueness of classical solutions of the nonautonomous second-order equation: $u''(t) = A(t)u'(t)+B(t)u(t)+f(t)$, $0 \le t \le T$; $u(0) = x_0$, $u'(0) = x_1$ on a Banach space by means of operator matrix method and apply to Volterra integrodifferential equations.

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1. Introduction. In this paper, we study nonautonomous second-order Cauchy problems

$$u''(t) = A(t)u'(t)+B(t)u(t)+f(t), \quad 0 \le t \le T, \qquad u^{(i)}(0) = x_i, \quad i = 0,1 \tag{1.1}$$

on a Banach space E, where $A(t)$ and $B(t)$, $0 \le t \le T$, are linear operators on E and f is a continuous function from $[0,T]$ to E.

Definition 1.1. A function $u(\cdot) : [0,T] \to E$ is said to be a solution of (1.1) if it is twice continuously differentiable on $[0,T]$, $A(t)u'(t)$, and $B(t)u(t)$ are defined and continuous in t, and (1.1) is satisfied.

Our idea is to reduce (1.1) to a differential equation of first-order. It is motivated by the work of N. Tanaka [12], who studied the first-order abstract Cauchy problem

$$u'(t) = A(t)u(t), \quad 0 \le t \le T, \qquad u(0) = x_0. \tag{1.2}$$

In his paper, Tanaka showed the existence and uniqueness of classical solutions of (1.2), when family $\{A(t)\}_{0 \le t \le T}$ of linear operators in E satisfies the conditions which are usually referred as the "hyperbolic" condition, except for the density of the common domain D of $A(t)$.

The purpose of this paper is to show the existence and uniqueness of classical solutions of (1.1) on the basis of Tanaka's result in [12] and the operator matrix method. We will consider two cases: the damped case, when $A(t)$ is more unbounded than $B(t)$ and the undamped one, when $B(t)$ is more unbounded than $A(t)$. For both cases, we use an operator matrix method to reduce (1.1) into a first-order differential equation of the form of (1.2) and then apply Tanaka's result. The two cases reduce in different ways, but the technique in each reduction is quite straightforward. In the undamped case, our result obtained improves Kozak's one [4] by requiring much weaker assumptions. In the damped case, we generalize Neubrander's result [6] to the nonautonomous version. Our proof is simpler and more natural than Oka's one in [8]. This proof creates a new framework to deal with the abstract higher-order differential equations on Banach spaces, which we will discuss in a subsequent paper.

In the following, for a linear operator A on a Banach space E, we denote the resolvent set of an operator A by $\rho(A)$ and the resolvent $(\lambda - A)^{-1}$ by $R(\lambda, A)$. By $L(E,F)$ we denote the set of all linear, bounded operators from E to F. Finally, for short, we write the family of $A(t)$, $0 \le t \le T$ by $\{A(t)\}$. First we recall the fundamental results obtained by Tanaka [12].

Theorem 1.2 (see [12, Theorem 1.8]). *A family of operators $A(t)$, $(0 \le t \le T)$ satisfies the hyperbolic condition if*

(H1) *The common domain $D := D(A(t))$ is a Banach space for the norm $\|\cdot\|_D$. Moreover, there exists $c_0 > 0$ such that*

$$c_0^{-1}\|x\|_D \le \|x\| + \|A(t)x\| \le c_0\|x\|_D \tag{1.3}$$

for all $t \in [0,T]$ and $x \in D$.

(H2) *The family $(A(t))_{t\in[0,T]}$ is stable, that is, there exist constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that*

$$(\omega, \infty) \subset \rho(A(t)) \quad \forall t \in [0,T], \tag{1.4}$$

$$\left\| \prod_{j=1}^{k} R(\lambda A(t_j)) \right\| \le M(\lambda - \omega)^{-k} \quad \forall \lambda > \omega, \tag{1.5}$$

and any finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_k \le T$.

(H3) *The mapping $t \mapsto A(t)y$ is continuously differentiable in E for every $y \in D$.*

If the family $\{A(t)\}$ satisfies the hyperbolic condition, then there is an evolution family $\{U(t,s)\}_{0\le s\le t\le T}$ on $\bar{D}$ with the following properties.

(1) *$U(t,s)D(s) \subset D(t)$ for all $0 \le s \le t \le T$, where the set $D(r)$ is defined by*

$$D(r) := \{x \in D : A(r)x \in \bar{D}\}. \tag{1.6}$$

(2) *The mapping $t \to U(t,s)x$ is continuously differentiable in E on $[s,T]$ and $(\partial/\partial t)U(t,s)x = A(t)U(t,s)x$ for $x \in D(s)$ and $t \in [s,T]$.*

If there is such an evolution family $\{U(t,s)\}_{0\le s\le t\le T}$, then, for every initial value $u_0 \in D(0)$, $u(t) := U(t,0)u_0$ is the unique solution of (1.2).

Generally, it is not trivial to show the stability of a family of operators. Thus, the following two lemmas, which will be used frequently, are very useful tools to verify this condition.

Lemma 1.3 (see [11, Theorem 5.2.3]). *Let $\{A(t)\}_{0\le t\le T}$ be stable and $\{B(t)\}_{0\le t\le T}$ be a family of uniformly bounded operators. Then the family $\{A(t)+B(t)\}_{0\le t\le T}$ with $D(A(t)+B(t)) := D(A(t))$ is stable.*

Lemma 1.4 (see [2, Proposition 4.4]). *Let $\{A(t)\}_{0\le t\le T}$ be a stable family and $\{S(t)\}_{0\le t\le T}$ be a family of isomorphisms $S(t) \in L(E)$, which is strongly continuously differentiable. Then the family $\{\tilde{A}(t)\}_{0\le t\le T} := \{S(t)A(t)S^{-1}(t)\}_{0\le t\le T}$ with $D(\tilde{A}(t)) = \{x : S^{-1}(t)x \in D(A(t))\}$ is stable.*

For further information on evolution equations, evolution family and the theory of operator matrices, we refer to, for example, [5, 11, 12].

2. The damped second-order equations. We now consider the damped second-order differential equations. First, we start with the homogeneous version

$$u''(t)=A(t)u'(t)+B(t)u(t), \quad 0\le t\le T, \qquad u(0)=x_0\in E, \qquad u'(0)=x_1\in E, \tag{2.1}$$

where operators $A(t)$ have common domain D and $D(B(t))\supset D$. For our purpose, we assume that there exists an invertible operator A from D onto E with $A^{-1}\in L(E)$. We introduce new variables by defining

$$v_0:=Au, \qquad v_1:=u'. \tag{2.2}$$

Then we have $v_0'=Au'=v_1$ and $v_1'=u''=A(t)u'+B(t)u=A(t)v_1+B(t)A^{-1}v_0$. Moreover, $v_0(0)=Ax_0$ and $v_1(0)=x_1$. Thus, we can write a differential equation for $\mathcal{V}:=(v_0,v_1)^T$ in the Banach space E^2 as follows:

$$\mathcal{V}'(t)=\mathcal{A}(t)\mathcal{V}(t), \quad 0\le t\le T, \qquad \mathcal{V}(0)=\mathcal{V}_0\in E^2, \tag{2.3}$$

where $\mathcal{A}(t):=\left(\begin{smallmatrix}0 & A\\ B(t)A^{-1} & A(t)\end{smallmatrix}\right)$ with $D(\mathcal{A}(t)):=E\times D$ and $\mathcal{V}_0:=(Ax_0,x_1)^T$. We easily see that if $\{A(t)\}$ satisfies the hyperbolic condition, we can choose $A:=(A(0)-\lambda)$ for a $\lambda>\omega$. We have the following lemma.

Lemma 2.1. *For $x_i\in D$ $(i=0,1)$, the following statements hold true.*

(1) *If $u(t)$ is a solution of (2.1), then $u^{(i)}(t)\in D$, $(i=0,1)$ and $(Au(t),u'(t))^T$ is a solution of (2.3).*

(2) *Conversely, if $(v_0(t),v_1(t))^T$ is a solution of (2.3), then $u(t):=\int_0^t v_1(s)\,ds+x_0$ is a solution of (2.1).*

Proof. (1)⇒(2). Let $u(t)$ be a solution of (2.1) with $u_0\in D$ and $u_1\in D$. In view of the closedness of A we have

$$\int_0^t Au'(\tau)\,d\tau=A\int_0^t u'(\tau)\,d\tau=A[u(t)-x_0]. \tag{2.4}$$

Since $x_0\in D$, it follows $u(t)\in D$, hence the function $t\mapsto Au(t)$ is continuously differentiable and $(d/dt)Au(t)=Au'(t)$. Therefore, $(v_0(t),v_1(t))^T:=(Au(t),u'(t))^T$ is continuously differentiable and satisfies (2.3), and thus, is a solution of (2.3).

(2)⇒(1). Conversely, suppose that $(v_0(t),v_1(t))^T$ is a solution of (2.3). We define the function u by

$$u(t):=\int_0^t v_1(r)\,dr+x_0. \tag{2.5}$$

Then $u(t)$ is twice continuously differentiable. Furthermore, from (2.3) we have

$$v_0(t)=\int_0^t Av_1(r)\,dr+x_0=A\left(\int_0^t v_1(r)\,dr+A^{-1}v_0(0)\right)=Au(t). \tag{2.6}$$

Thus,

$$u''(t)=v_1'(t)=B(t)A^{-1}v_0(t)+A(t)v_1(t)=A(t)u'(t)+B(t)u(t). \tag{2.7}$$

Finally, it is easy to see that $u'(0) = x_0$ and $u(0) = x_1$. Therefore, $u(t)$ is a solution of (2.1), and the lemma is proved. □

Now we are in a position to express the main result of this section.

Theorem 2.2. *For the second-order differential equation (2.1) we assume that $\{A(t)\}$ satisfies the hyperbolic condition and $\{B(t)\}$ is a family of linear operators with $D(B(t)) \supseteq D$ such that $B(t) \in L(D,E)$ and $t \mapsto B(t)x$ is continuously differentiable for each $x \in D$. Then it has a unique solution for every initial $x_0 \in D$, $x_1 \in D$, such that $(B(0)x_0 + A(0)x_1) \in \bar{D}$.*

Proof. By Lemma 2.1 and Tanaka's theorem (Theorem 1.2), to show the existence of solutions of (2.1), we only have to prove that the family $\{\mathcal{A}(t)\}$ satisfies the hyperbolic condition. It is easy to see that $\{\mathcal{A}(t)\}$ satisfies items (H1) and (H3) of this condition. It remains to show its stability. To do that, we assume, without loss of generality, that $\omega < 0$. Then $A(t)$ is invertible for every $t \in [0,T]$. Since $B(t) \in L(D,E)$, $B(t)A^{-1}$ is bounded in E. Because of the strongly continuous differentiability of $\{B(t)A^{-1}\}$, by the principle of uniform boundedness, this family is uniformly bounded. In addition, if $A(t)$ is strongly continuously differentiable, then so is $A^{-1}(t)$. Thus, the strongly continuous differentiability of $AA^{-1}(t)$ follows from that of its inverse $A(t)A^{-1}$. Using elementary matrix rules we have

$$\begin{pmatrix} 0 & A \\ B(t)A^{-1} & A(t) \end{pmatrix} = \begin{pmatrix} I & AA^{-1}(t) \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix} \begin{pmatrix} I & -AA^{-1}(t) \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B(t)A^{-1} & 0 \end{pmatrix}. \tag{2.8}$$

By the stability of family $\{A(t)\}$, the family $\tilde{\mathcal{A}}(t) := \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & A(t) \end{smallmatrix}\right)\}$ with $D(\tilde{\mathcal{A}}(t)) := E \times D$ is stable in E^2. From the above observation, the family of isomorphisms $\left(\begin{smallmatrix} I & AA^{-1}(t) \\ 0 & I \end{smallmatrix}\right)$ and of their inverses, $\left(\begin{smallmatrix} I & -AA^{-1}(t) \\ 0 & I \end{smallmatrix}\right)$, are strongly continuously differentiable. Using Lemmas 1.3 and 1.4 we conclude that the family $\{\mathcal{A}(t)\}$ is stable.

The uniqueness of the solutions of (2.1) follows, by Lemma 2.1, from that of the solutions of (2.3), completing the proof of the theorem. □

Remark 2.3. In the above proof, for convenience, we assumed $\omega < 0$, that is, $A(t)$ is invertible. Actually, this assumption can be removed. Indeed, if family $\{A(t)\}$ satisfies the hyperbolic condition and if $A \in L(D,E)$, then by the identity

$$\begin{pmatrix} 0 & A \\ 0 & A(t) \end{pmatrix} = \begin{pmatrix} I & -AR(\lambda, A(t)) \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & A(t) - \lambda \end{pmatrix} \begin{pmatrix} I & AR(\lambda, A(t)) \\ 0 & I \end{pmatrix} \tag{2.9}$$

for a $\lambda > \omega$, we see that $\left(\begin{smallmatrix} 0 & A \\ 0 & A(t) \end{smallmatrix}\right)$ is stable by the same argument.

We now consider the inhomogeneous equation (1.1). To this end, we recall that in [7] we considered the first-order inhomogeneous equation

$$u'(t) = A(t)u(t) + f(t), \quad 0 \le t \le T, \quad u(0) = x_0, \tag{2.10}$$

for a hyperbolic family $\{A(t)\}$ and $f \in W^{1,1}(\mathbb{R}_+,E)$. We showed that the family $\{\mathcal{A}(t)\}_{0\le t\le T} := \left\{\begin{pmatrix} A(t) & \delta_0 \\ 0 & d/dx\end{pmatrix}\right\}_{0\le t\le T}$ with $D(\mathcal{A}(t)) := D\times W^{1,1}(\mathbb{R}_+,E)$ satisfies the hyperbolic condition. Moreover, the first component of the solution of the problem

$$\mathcal{U}'(t) = \mathcal{A}(t)\mathcal{U}(t), \quad 0 \le t \le T, \quad \mathcal{U}(0) = \begin{pmatrix} x_0 \\ f \end{pmatrix}, \tag{2.11}$$

is the solution of (2.10). By combining this result and Lemma 2.1 we have the following.

Theorem 2.4. *Suppose that the assumptions of Theorem 2.2 are satisfied. Then (1.1) has a unique solution for every $f \in W^{1,1}(\mathbb{R}_+,E)$ and $x_i \in D$, $(i = 0,1)$, satisfying $(B(0)x_0 + A(0)x_1 + f(0)) \in \bar{D}$.*

Proof. On $E\times E\times L^1(\mathbb{R}_+,E)$ we consider the equation

$$\begin{pmatrix} v \\ w \\ \phi \end{pmatrix}'(t) = \mathcal{C}(t)\begin{pmatrix} v \\ w \\ \phi \end{pmatrix}(t), \quad 0 \le t \le T, \quad \begin{pmatrix} v \\ w \\ \phi \end{pmatrix}(0) = \begin{pmatrix} Au_0 \\ u_1 \\ f \end{pmatrix}, \tag{2.12}$$

where

$$\mathcal{C}(t) := \begin{pmatrix} 0 & A & 0 \\ B(t)A^{-1} & A(t) & \delta_0 \\ 0 & 0 & \dfrac{d}{dx} \end{pmatrix}, \tag{2.13}$$

and $D(\mathcal{C}(t)) := E\times D\times W^{1,1}(\mathbb{R}_+,E)$.

We write $\mathcal{C}(t) = \begin{pmatrix} 0 & \mathcal{A}_0 \\ \mathcal{B}(t) & \mathcal{A}(t)\end{pmatrix}$ with $\mathcal{A}_0 := (A,0)$, $\mathcal{B}(t) := \begin{pmatrix} B(t)A^{-1} \\ 0 \end{pmatrix}$, and $\mathcal{A}(t) := \begin{pmatrix} A(t) & \delta_0 \\ 0 & d/dx \end{pmatrix}$.

From the above consideration, the family $\{\mathcal{A}(t)\}$ satisfies the hyperbolic condition. As in Theorem 2.2, and in view of Remark 2.3, we conclude that $\{\mathcal{C}(t)\}$ is stable and thus satisfies the hyperbolic condition. Therefore, by Tanaka's theorem, problem (2.12) has a unique solution for every initial value $\mathcal{V}(0) := (Ax_0,x_1,f)^T(0) \in \mathfrak{D} := E\times D\times W^{1,1}(\mathbb{R}_+,E)$, such that $\mathcal{C}(0)\mathcal{V}(0) \in \bar{\mathfrak{D}}$, or in other words, $(B(0)x_0 + A(0)x_1 + f(0)) \in \bar{D}$.

Let $\mathcal{V}(t) := (v(t),w(t),\phi(t))^T$ be a solution of (2.12). Obviously $\phi(t) = T_r(t)f$, where $T_r(t)f(\theta) := f(t+\theta)$. We now define a function u by

$$u(t) := \int_0^t w(r)\,dr + x_0. \tag{2.14}$$

Then, with the same procedure as in Theorem 2.2, we have that $u(\cdot)$ is twice continuously differentiable, $v(t) = Au(t)$, $u'(t) = w(t)$, and

$$\begin{aligned} u''(t) = w'(t) &= B(t)A^{-1}(0)v(t) + A(t)w(t) + \phi(t)(0) \\ &= A(t)u'(t) + B(t)u(t) + f(t). \end{aligned} \tag{2.15}$$

Moreover, $u(0) = x_0$ and $u'(0) = w(0) = x_1$. Therefore, $u(t)$ is a solution of (1.1). The uniqueness of this solution follows from the uniqueness of the solution of the homogeneous equation and the theorem is proved. □

3. The undamped second-order equations. This section is devoted to second-order differential equations in which $B(t)$ is more unbounded than $A(t)$. We start with the following problem:

$$u''(t) = B(t)u(t) + f(t) \quad 0 \le t \le T, \quad u(0) = x_0, \quad u'(0) = x_1, \tag{3.1}$$

and carry out the substitution:

$$v_0 := u, \qquad v_1 := u'. \tag{3.2}$$

Then we can rewrite (3.1) in matrix form as

$$\begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix}' = \begin{pmatrix} 0 & I \\ B(t) & 0 \end{pmatrix}\begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad 0 \le t \le T, \quad \begin{pmatrix} v_0(0) \\ v_1(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \tag{3.3}$$

on E^2. To investigate the Cauchy problem (3.3), we make the following assumptions to $B(t)$.

Assumption 3.1. (A1) *For each $t \in [0,T]$, there exists a linear operator $C(t) : E \to E$ such that $B(t) = C^2(t)$ with $D(C^i(t)) \equiv D_i$, $i = 1,2$, independent of t.*
(A2) *$\{C(t)\}_{0\le t\le T}$ and $\{-C(t)\}_{0\le t\le T}$ with $D(C(t)) = D_1$ are stable families.*
(A3) *The map $t \mapsto C^i(t)x$ is continuously differentiable for every $x \in D_i$ and $i = 1,2$.*

Without loss of generality, we assume $\omega < 0$, where (M,ω) are the stability constants of the family $\{C(t)\}_{0\le t\le T}$. On the subsets D_1 and D_2 of E, we establish the following norms

$$\|x\|_{[D_1]} := \|C(0)x\| \quad \text{for } x \in D_1, \qquad \|x\|_{[D_2]} := \|C^2(0)x\| \quad \text{for } x \in D_2. \tag{3.4}$$

Then it is easy to see that $([D_1], \|\cdot\|_{[D_1]})$ and $([D_2], \|\cdot\|_{[D_2]})$ are Banach spaces. Moreover, $C(t)$ and $C^2(t)$ are bounded operators from $[D_1]$ and $[D_2]$ to E, respectively, for every $t \in [0,T]$. From the above assumptions, we obtain some information.

Lemma 3.2. (i) *Each $\mathcal{B}(t)$, where*

$$\mathcal{B}(t) := \begin{pmatrix} 0 & I \\ B(t) & 0 \end{pmatrix} \tag{3.5}$$

with $D(\mathcal{C}(t)) := D_2 \times D_1$ on the Banach space $[D_1] \times E$ is similar to the operator

$$\mathcal{C}(t) := \begin{pmatrix} 0 & C(t) \\ C(t) & 0 \end{pmatrix} \tag{3.6}$$

with $D(\mathcal{C}(t)) := D_1 \times D_1$ on E^2.
(ii) *Let $Q := (1/\sqrt{2})\left(\begin{smallmatrix} I & -I \\ I & I \end{smallmatrix}\right)$ and $Q^{-1} := (1/\sqrt{2})\left(\begin{smallmatrix} I & I \\ -I & I \end{smallmatrix}\right)$ on E^2. Then*

$$\begin{pmatrix} 0 & C(t) \\ C(t) & 0 \end{pmatrix} = Q\begin{pmatrix} C(t) & 0 \\ 0 & -C(t) \end{pmatrix}Q^{-1} \tag{3.7}$$

with the same domain.

Proof. (ii) is trivial and (i) follows from the identity

$$\begin{pmatrix} 0 & C(t) \\ C(t) & 0 \end{pmatrix} = \begin{pmatrix} C(t) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ C^2(t) & 0 \end{pmatrix} \begin{pmatrix} C(t)^{-1} & 0 \\ 0 & I \end{pmatrix}, \tag{3.8}$$

where $\left(\begin{smallmatrix} C(t) & 0 \\ 0 & I \end{smallmatrix}\right)$ are isomorphisms from $[D_1]\times E$ to E^2. □

Now we prove the main results of this section.

Theorem 3.3. *Let the operators $B(t)$ satisfy Assumption 3.1. Then the second-order Cauchy problem (3.1) has a unique solution with $u(t) \in D_2$, $u'(t) \in \bar{D}_2$ in $[D_1]$-norm and $u''(t) \in \bar{D}_1$ for every initial value $(u(0),u'(0))^T = (x_0,x_1)^T \in D_2 \times D_1$ and $f \in W^{1,1}(\mathbb{R}_+,E)$ such that $B(0)x_0 + f(0) \in \bar{D}_1$ and $x_1 \in \bar{D}_2$ in $[D_1]$-norm.*

Proof. Consider the family $\{\mathcal{B}(t)\}_{0\le t\le T}$ in problem (3.3). As $\{B(t)\}$ is strongly continuously differentiable, so is $\{\mathcal{B}(t)\}$. We now show that $\{\mathcal{B}(t)\}_{0\le t\le T}$ is stable.

By assumption, $\{C(t)\}$, and $\{-C(t)\}$ are stable families, and so is the family $\left\{\left(\begin{smallmatrix} C(t) & 0 \\ 0 & -C(t) \end{smallmatrix}\right)\right\}$. By Lemma 3.2(ii), the family $\left\{\left(\begin{smallmatrix} 0 & C(t) \\ C(t) & 0 \end{smallmatrix}\right)\right\}$ is stable since it is similar to a stable family. Now, using Lemmas 1.4 and 3.2(i), for which we notice that the families $\left\{\left(\begin{smallmatrix} C(t) & 0 \\ 0 & I \end{smallmatrix}\right)\right\}$ and $\left\{\left(\begin{smallmatrix} C^{-1}(t) & 0 \\ 0 & I \end{smallmatrix}\right)\right\}$ are strongly continuously differentiable by assumption, we see that the family $\{\mathcal{B}(t)\}$ is stable and therefore satisfies the hyperbolic condition. By Tanaka's theorem, equation (3.3) has a unique solution for each $(x_0,x_1)^T \in D_2 \times D_1$ and $f \in W^{1,1}(\mathbb{R}_+,E)$ such that $B(0)x_0 + f(0) \in \bar{D}_1$ and $x_1 \in \bar{D}_2$ in $[D_1]$-norm.

Let $\mathcal{V}(\cdot) = (v_0,v_1)^T$ be a solution of (3.3). Then we have $v_0'(t) = v_1(t)$ in $[D_1]$ and $v_1'(t) = B(t)v_0 + f(t)$ in E. Since the norm in $[D_1]$ is finer than the norm of E, the above equations also hold in E. This implies $v_0(t) \in D_2$, $v_1(t) \in D_1$, $t \mapsto v_0(t)$ is twice continuously differentiable and $v_0''(t) = B(t)v_0(t) + f(t)$ for $t \in [0,T]$. That means that v_0 is a solution of the second-order Cauchy problem (3.1).

To prove the uniqueness of the solution of (3.1), we again apply Tanaka's theorem. We first assume that $f \equiv 0$. On $\overline{D_2 \times D_1}$, the closure of $D_2 \times D_1$ on Banach space $[D_1]\times E$, we consider the evolution family $\{\mathcal{V}(t,s)\}_{0\le s\le t\le T}$ generated by $\{\mathcal{C}(t)\}$, where

$$\mathcal{V}(t,s) = \begin{pmatrix} V_{1,1}(t,s) & V_{1,2}(t,s) \\ V_{2,1}(t,s) & V_{2,2}(t,s) \end{pmatrix}. \tag{3.9}$$

Then we have $V_{1,1}(t,t) \equiv I$ on $[D_1]$ and $V_{1,2}(t,t) \equiv 0$. Moreover, by [3, Lemma 4], $V_{1,1}(t,s)$ is bounded in E. From the identity

$$\frac{\partial}{\partial \tau}\mathcal{V}(t,\tau)\mathcal{U} = \mathcal{V}(t,\tau)\mathcal{B}(\tau)\mathcal{U} \tag{3.10}$$

for $\mathcal{U} \in D(\tau) := \{(u_1,u_2)^T \in D_2 \times D_1,\ \mathcal{B}(\tau)(u_1,u_2)^T \in \overline{D_2 \times D_1}\}$, we obtain

$$\frac{\partial}{\partial \tau}V_{1,1}(t,\tau)u_1 = -V_{1,2}(t,\tau)B(\tau)u_1 \tag{3.11}$$

for $u_1 \in D_2$ and $B(\tau)x_1 \in \bar{D}_1$, and

$$\frac{\partial}{\partial \tau}V_{1,2}(t,\tau)u_2 = -V_{1,1}(t,\tau)u_1 \tag{3.12}$$

for $u_2 \in \bar{D}_2$ in $[D_1]$-norm.

Now let u be a solution of (3.1), then $u(\tau) \in D_2$, $B(\tau)u(\tau) = u''(\tau) \in \bar{D}_1$, and $u'(\tau) \in \bar{D}_2$ in $[D_1]$-norm for $\tau \in [0,T]$. From the above equations it follows that

$$\begin{aligned} \frac{\partial}{\partial \tau}\left[V_{1,1}(t,\tau)u(\tau)\right] &= -V_{1,2}(t,\tau)B(\tau)u(\tau) + V_{1,1}(t,\tau)u'(\tau), \\ \frac{\partial}{\partial \tau}\left[V_{1,2}(t,\tau)u'(\tau)\right] &= -V_{1,1}(t,\tau)u'(\tau) + V_{1,2}(t,\tau)u''(\tau) \\ &= -V_{1,1}(t,\tau)u'(\tau) + V_{1,2}(t,\tau)B(\tau)u(\tau). \end{aligned} \tag{3.13}$$

Adding these equations, we obtain

$$\frac{\partial}{\partial \tau}\left[V_{1,1}(t,\tau)u(\tau)\right] + \frac{\partial}{\partial \tau}\left[V_{1,2}(t,\tau)u'(\tau)\right] = 0. \tag{3.14}$$

Integrating both sides from 0 to t, we have

$$u(t) = V_{1,1}(t,0)x_0 + V_{1,2}(t,0)x_1. \tag{3.15}$$

Thus, u is uniquely determined by x_0 and x_1. The uniqueness of the solutions for inhomogenous Cauchy problem follows from that of the solutions for the homogenous one, and the theorem is proved. □

Corollary 3.4. *For the complete second-order Cauchy problem*

$$u''(t) = A(t)u'(t) + B(t)u(t) + f(t), \quad 0 \le t \le T, \qquad u^{(0)}(s) = x_i, \quad i = 0,1, \tag{3.16}$$

we assume that the operators $B(t)$ satisfy Assumption 3.1 and that $\{A(t)\}_{0\le t\le T}$ is a family of bounded operators such that $t \mapsto A(t)x$ is continuously differentiable for all $x \in D_1$. Then the second-order Cauchy problem (3.16) has a unique solution with $u(t) \in D_2$, $u'(t) \in \bar{D}_2$ in $[D_1]$, and $u''(t) \in \bar{D}_1$ for every initial value $(u(0),u'(0))^T = (x_0,x_1)^T \in D_2 \times D_1$ and $f \in W^{1,1}(\mathbb{R}_+,E)$ satisfying $B(0)x_0 + A(0)x_1 + f(0) \in \bar{D}_1$ and $x_1 \in \bar{D}_2$ in $[D_1]$-norm.

Proof. On the Banach space $[D_1] \times E$, we consider the initial value problem

$$\mathcal{V}'(t) = \mathcal{B}_1(t)\mathcal{V}(t) + F(t), \quad 0 \le t \le T, \quad \mathcal{V}(s) = (x_0,x_1)^T, \tag{3.17}$$

where $\mathcal{V}(t) := (v_0(t),v_1(t))^T$, $F(t) := (0,f(t))^T$, and

$$\mathcal{B}_1(t) := \begin{pmatrix} 0 & I \\ B(t) & A(t) \end{pmatrix} \tag{3.18}$$

with $D(\mathcal{B}(t)) := D_2 \times D_1$. We can write $\mathcal{B}_1(t)$ as

$$\mathcal{B}_1(t) = \begin{pmatrix} 0 & I \\ B(t) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix} = \mathcal{B}(t) + \mathcal{A}(t), \tag{3.19}$$

that is, the sum of $\mathcal{B}(t)$ and a bounded operator $\mathcal{A}(t)$. Applying Lemma 1.3 and Theorem 3.3, we see that $\{\mathcal{B}_1(t)\}$ is stable and thus satisfies the hyperbolic condition. As in the proof of Theorem 3.3, the first component of the solution of (3.17) is a solution of the second-order Cauchy problem (3.16). □

Remark 3.5. As in the proof of Theorem 3.3 we show that in the inhomogenous case, the solution of (3.1) and (3.16) has the form

$$u(t) = V_{1,1}(t,s)x_0 + V_{1,2}(t,s)x_1 + \int_s^t V_{1,2}(t,\tau)f(\tau)\,d\tau. \tag{3.20}$$

Remark 3.6. If D_1 is dense in E, then D_2 is dense in $[D_1]$. Therefore, in Theorem 3.3 and Corollary 3.4, we can drop all compatibility conditions.

Applications. (1) We first consider the autonomous second-order Cauchy problem

$$u''(t) = Bu(t) + f(t), \quad t \geq 0, \quad u(0) = x_0, \quad u'(0) = x_1. \tag{3.21}$$

By Theorem 3.3, if $B = C^2$, such that C is the generator of a C_0-group, or, equivalently, if B is the generator of a cosinus family, then (3.21) has a unique solution for every initial value $(x_0, x_1)^T \in D(C^2) \times D(C)$. This is a classical result on the "wellposedness" of second-order Cauchy problems (see [1]).

(2) We are now concerned with the second-order Volterra integrodifferential equation

$$u''(t) = B(t)u(t) + \int_0^t C(t,s)u(s)\,ds + f(t), \quad 0 \leq t \leq T, \qquad u^{(i)}(0) = x_i, \quad i = 0,1. \tag{3.22}$$

The autonomous version of (3.22) was studied by Oka [9] for $B(t) \equiv B$ and $C(t,s) = C(t-s)$. For the first-order Volterra integrodifferential equations, Oka and Tanaka [10] showed that under the conditions

(A) the family $\{B(t)\}_{0 \leq t \leq T}$ satisfies the hyperbolic condition with constant domain D, which is not necessarily dense in E,

(B) $\{C(t,s)\}_{0 \leq s \leq t \leq T}$ is a family of bounded linear operators from D to E such that for every $y \in D$, $C(t,s)y$ is continuous on the set $\Delta := \{(t,s) : 0 \leq s \leq t \leq T\}$ and continuously differentiable with respect to t, then the Volterra integrodifferential equation

$$u'(t) = B(t)u(t) + \int_0^t C(t,s)u(s)\,ds + f(t), \quad 0 \leq t \leq T, \quad u(0) = x_0 \tag{3.23}$$

has a unique solution for every initial value $x_0 \in D$ and $f \in W^{1,1}(\mathbb{R}_+, E)$, such that $B(0)x_0 + f(0) \in \bar{D}$.

Combining this and our result, we obtain the existence and uniqueness of the solutions of (3.22). More precisely, we have the following theorem.

Theorem 3.7. *Consider the nonautonomous second-order Volterra integrodifferential equation (3.22), where the families $\{B(t)\}_{0 \leq t \leq T}$ and $\{C(t,s)\}_{0 \leq s \leq t \leq T}$ have the properties*

(i) *the family $\{B(t)\}$ satisfies Assumption 3.1,*

(ii) *$\{C(t,s)\}$ is a family of bounded linear operators from $[D_2]$ to E such that for every $y \in D_2$, $C(t,s)y$ is continuous on the set $\Delta := \{(t,s) : 0 \leq s \leq t \leq T\}$ and continuously differentiable with respect to t.*

Then (3.22) has a unique solution for every initial value $(x_0, x_1)^T \in D_2 \times D_1$ and every inhomogenous term $f \in W^{1,1}(\mathbb{R}_+, E)$ satisfying $B(0)x_0 + f(0) \in \bar{D}_1$ and $x_1 \in \bar{D}_2$ in $[D_1]$.

PROOF. On the basis of our substitution, we convert our second-order problem into a first-order system on $[D_1]\times E$ as follows:

$$\mathcal{U}'(t) = \mathcal{B}(t)\mathcal{U}(t) + \int_0^t \mathcal{C}(t,s)\mathcal{U}(s)\,ds + F(t), \quad 0 \le t \le T, \quad \mathcal{U}(0) = (x_0, x_1)^T \tag{3.24}$$

on $[D_1]\times E$, where $\mathcal{U} := (u,u')^T$, $F := (0,f)^T$, $\mathcal{B}(t)$ as defined in (3.5) and

$$\mathcal{C}(t,s) := \begin{pmatrix} 0 & 0 \\ C(t,s) & 0 \end{pmatrix}. \tag{3.25}$$

We can now check that the families $\{\mathcal{B}(t)\}$ and $\{\mathcal{C}(t,s)\}_{0\le s\le t\le T}$ satisfy the conditions (A) and (B). Using the result of Oka and Tanaka, we obtain the existence and uniqueness of the solutions of (3.24) and then those of (3.22). □

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