IJMMS 25:8 (2001) 497–507
PII. S0161171201005312
http://ijmms.hindawi.com

FIXED POINTS VIA A GENERALIZED LOCAL COMMUTATIVITY

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(Received 24 May 2000)

ABSTRACT. Let $g : X \to X$. The concept of a semigroup of maps which is "nearly commutative at g" is introduced. We thereby obtain new fixed point theorems for functions with bounded orbit(s) which generalize a recent theorem by Huang and Hong, and results by Jachymski, Jungck, Ohta, and Nikaido, Rhoades and Watson, and others.

2000 Mathematics Subject Classification. Primary 47H10, 54H25.

1. Introduction. By a *semi-group of maps* we mean a family H of self maps of a set X which is closed with respect to composition of maps ($f \circ g = fg$) and includes the identity map $i_d(x) = x$, for $x \in X$. We often associate with a function $g : X \to X$ following semi-groups:

$$O_g = \{g^n \mid n \in \mathbb{N} \cup \{0\}\}, \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers and $g^0 = i_d$, and

$$C_g = \{f : X \longrightarrow X \mid fg = gf\}. \tag{1.2}$$

A quick check confirms that C_g is a semi-group.

If H is a semi-group of self maps of a set X and $a \in X$, $H(a) = \{h(a) \mid a \in H\}$. In particular, if $H = O_g$, $O_g(a) = \{g^n(a) \mid n \in \mathbb{N} \cup \{0\}\}$ and is called the orbit of g at a.

In general, Lemma 3.2 and some theorems in Section 3 will be stated in the context of semi-metric spaces. A *semi-metric* on a set X is a function $d : X \times X \to [0,\infty)$ such that $d(x,y) = d(y,x)$ for $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$. A *semi-metric space* is a pair $(X;d)$, where X is a topological space and d is a semi-metric on X. The topology $t(d)$ on X is generated by the sets $S(p,\epsilon) = \{x \mid d(x,p) < \epsilon\}$ with the requirement that p is an interior point of $S(p,\epsilon)$. A sequence $\{x_n\}$ in X converges in $t(d)$ to $p \in X$ (denoted as $x_n \to p$) if and only if $d(x_n,p) \to 0$. We let $t(d)$ be T_2 (Hausdorff) to ensure unique limits. Thus, a metric space (X,d) is a semi-metric space having the triangle inequality. For further details on semi-metric spaces, see, for example, [1, 4, 6].

If $g : X \to X$, a semi-metric space $(X;d)$ is *complete* (*g-orbitally complete*) if and only if every Cauchy sequence (in the usual sense) in X ($O_g(x)$) converges to a point of X. g is *continuous* at $p \in X$ if and only if whenever $\{x_n\}$ is a sequence in X and $x_n \to p$, then $f(x_n) \to f(p)$. And if S is a bounded subset of X, $\delta(S) = \sup\{d(x,y) \mid x, y \in S\}$.

We are now ready to focus on the intent of this paper, namely, to introduce a generalized "local commutativity" and to demonstrate the concept's usefulness.

2. Nearly commutative semi-groups. In [2], a semi-group H of maps is said to be *near-commutative* if and only if for each pair $f,g \in H$, there exists $h \in H$ such that $fg = gh$. We generalize as follows.

Definition 2.1. A semi-group H of self maps of a set X is *nearly commutative (n.c.) at* $g : X \to X$ if and only if $(f \in H)$ implies that there exists $h \in H$ such that $fg = gh$.

Of course, O_g and C_g are n.c. at g. Observe also that a *near-commutative semi-group* H of self maps of a set X is n.c. at *each* $g \in H$. The following provides for each $a \in (0,\infty)$ an example of a semi-group $H = S_a$ of self maps which is not near-commutative but is n.c. at a particular $g : X \to X$.

Example 2.2. Let $X = [0,\infty)$ and $a \in (0,\infty)$. Let $g(x) = ax$ and define

$$S_a = \{a^m x^n \mid x \in [0,\infty),\ n \in \mathbb{N},\ m \in \mathbb{N} \cup \{0\}\}, \tag{2.1}$$

where S_a is *nearly commutative (n.c.) at* g. For if $f(x) = a^m x^n$ is a representative element of S_a, then $fg(x) = f(g(x)) = a^m(ax)^n = a^{m+n}x^n$. We want $h(x) = a^r x^s \in S_a$ such that $fg = gh$. Now, $g(h(x)) = a(a^r x^s) = a^{r+1}x^s$, so we can let $s = n$ and $r+1 = m+n$; that is, $r = m+(n-1)$. Since $n \in \mathbb{N}$ and $(n-1)$, $m \in \mathbb{N}\cup\{0\}$, s and r so designated imply $h \in S_a$. Thus, $(f \in H = S_a)$ implies that there exists $h \in H$ such that $fg = gh$. Since $i_d \in S_a$, S_a is clearly a semi-group, and we are finished. On the other hand, S_a is not a *near-commutative* semi-group. For example, let $f(x) = a^2x^2$ and $h(x) = a^2x^3$. We want $t(x) = a^r x^s$ such that $fh = ht$. So we must have $3s = 6$ and $(2+3r) = 6$. But then $r = 4/3$, and $r \notin \mathbb{N}\cup\{0\}$.

Now, let $\mathcal{M}_n$ and $\mathcal{N}_n$ denote the set of all $n \times n$ real matrices and the set of all nonsingular $n \times n$ real matrices, respectively. Then, both sets $\mathcal{M}_n$ and $\mathcal{N}_n$ are semi-groups of linear transformations $A : \mathbb{R}^n \to \mathbb{R}^n$ relative to composition of maps (matrix multiplication).

Example 2.3. $\mathcal{N}_n$ is n.c. For if $A,B \in \mathcal{N}_n$, there exists $C = B^{-1}(AB) \in \mathcal{N}_n$ such that $AB = BC$.

Example 2.4. $\mathcal{M}_n$ is n.c. at any $B \in \mathcal{N}_n$, by Example 2.3. But $\mathcal{M}_n$ is not near commutative. For instance, if $n = 2$, $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$, there exists no 2×2 matrix C such that $AB = BC$.

Now, let $g : X \to X$. Since any semi-group of self maps which commute with g is a subset of C_g, we might hope that $H_g = \{f : X \to X \mid fg = gh \text{ for some } h : X \to X\}$ would be a maximal semi-group which is n.c. at g. However, H_g so defined need not be n.c. at g! For example, let $X = [0,\infty)$, $g(x) = 1/(x+1)$, and $f(x) = x/2$. Then $h(x) = 2x+1$ satisfies $f(g(x)) = g(h(x))$ for $x \in [0,\infty)$. However, there exists no $k \in H_g$ such that $h(g(x)) = g(k(x))$; that is, $2(x+1)^{-1}+1 = (k(x)+1)^{-1}$ (note that x, $k(x) \geq 0$).

Note that the map $g(x) = 1/(x+1)$ was not surjective. So consider the following example.

Example 2.5. Let X be any set and let $g : X \to X$ be surjective. Then the family of all self mappings of X, $\mathcal{F} = \{f \mid f : X \to X\}$, is n.c. at g. For suppose $f \in \mathcal{F}$; we need $h \in \mathcal{F}$ such that $fg(x) = gh(x)$ for all $x \in X$. So let $a \in X$. Since g is onto, we can choose $x_a \in X$ such that $g(x_a) = f(g(a))$. Choose such an x_a for each $a \in X$ and define $h(a) = x_a$. Then $h : X \to X$ and $g(h(a)) = g(x_a) = f(g(a))$ for $a \in X$; that is, $fg = gh$.

Proposition 2.6. *Suppose that H is a semigroup of maps which is n.c. at $g : X \to X$. If $f \in H$ and $n \in \mathbb{N}$, there exists $h_n \in H$ such that $fg^n = g^n h_n$ (i.e., H is n.c. at g^n).*

Proof. Let $f \in H$. Since, H is n.c. at g, there exists $h_1 \in H$ such that $fg = gh_1$. So suppose that $k \in \mathbb{N}$ such that $fg^k = g^k h_k$ for some $h_k \in H$. Then

$$fg^{k+1} = (fg^k)g = (g^k h_k)g = g^k(h_k g). \tag{2.2}$$

Since $h_k \in H$, there exists $h_{k+1} \in H$ such that $h_k g = gh_{k+1}$, and therefore (2.2) implies $fg^{k+1} = g^k(gh_{k+1}) = g^{k+1}h_{k+1}$, as desired. □

Throughout this paper, P denotes a function $P : [0,\infty) \to [0,\infty)$ which is nondecreasing, and satisfies $\lim_{n\to\infty} P^n(t) = 0$ for $t \in [0,\infty)$. (For example, we could let $P(t) = \alpha t$ for some $\alpha \in (0,1)$, or $t/(t+1)$.) And throughout this paper, we appeal to the following lemma.

Lemma 2.7. *Let H be a semi-group of self maps of a set X and suppose that H is nearly commutative at $g : X \to X$. Let $d : X \times X \to [0,\infty)$. Suppose that for each pair $x, y \in X$ there exists a choice $r = r(\{x,y\})$, $s = s(\{x,y\}) \in H$, and $u, v \in \{x,y\}$ for which*

$$d(gx, gy) \le P(d(ru, sv)). \tag{2.3}$$

Then, if $n \in \mathbb{N}$, for each pair $x, y \in X$ there exist $r_n, s_n \in H$ and $u_n, v_n \in \{x,y\}$ such that

$$d(g^n x, g^n y) \le P^n(d(r_n u_n, s_n v_n)). \tag{2.4}$$

Proof. By (2.3), inequality (2.4) holds for $n = 1$, so suppose that $n \in \mathbb{N}$ for which (2.4) is true. Then, if $x, y \in X$,

$$d(g^{n+1}x, g^{n+1}y) = d(g(g^n x), g(g^n y)) \le P(d(ru, sv)), \tag{2.5}$$

where $r, s \in H$ and $u, v \in \{g^n x, g^n y\}$, by (2.3). Specifically, $u = g^n c$, $v = g^n d$, where $c, d \in \{x,y\}$. And since $r, s \in H$, there exist $r', s' \in H$ such that $rg^n = g^n r'$ and $sg^n = g^n s'$, by Proposition 2.6. So (2.4) implies that

$$d(ru, sv) = d(rg^n(c), sg^n(d)) = d(g^n(r'c), g^n(s'd)) \le P^n(d(r_n u_n, s_n v_v)), \tag{2.6}$$

where $r_n, s_n \in H$ and $u_n, v_n \in \{r'c, s'd\}$. Thus, $r_n u_n \in \{(r_n r')c, (r_n s')d\}$, where $r_n r'$ and $r_n s'$ are elements of H, since H is a semi-group. So $r_n u_n = r_{n+1}u_{n+1}$, where $r_{n+1} \in \{r_n r', r_n s'\}$ (i.e., $r_{n+1} \in H$) and $u_{n+1} \in \{c,d\} \subset \{x,y\}$. Similarly, $s_n v_n = s_{n+1}v_{n+1}$, where $s_{n+1} \in H$ and $v_{n+1} \in \{x,y\}$. Thus, (2.6) implies that

$$d(ru, sv) \le P^n(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1})), \quad r_{n+1}, s_{n+1} \in H,\ u_{n+1}, v_{n+1} \in \{x,y\}. \tag{2.7}$$

But P is nondecreasing, and therefore (2.7) and (2.5) yield

$$\begin{aligned} d(g^{n+1}x, g^{n+1}y) &\le P(P^n(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1}))) \\ &= P^{n+1}(d(r_{n+1}u_{n+1}, s_{n+1}v_{n+1})), \end{aligned} \tag{2.8}$$

with $r_{n+1}, s_{n+1} \in H$ and $u_{n+1}, v_{n+1} \in \{x, y\}$. So, (2.4) is true for all n by induction. □

3. Fixed point theorems

Definition 3.1. Let $(X;d)$ be a semi-metric space and let H be a semi-group of self maps of X. A map $g : X \to X$ is *P-contractive relative to H* if and only if (2.3) holds. (We will also say, "*g is a P-contraction relative to H*.")

Lemma 3.2. *Let $(X;d)$ be a T_2 semi-metric space and let H be a semi-group of self maps of X n.c. at $g : X \to X$. Suppose that g is P-contractive relative to H and that $M \subset X$ such that $B = \cup\{H(c) \mid c \in M\}$ is bounded. Then $d(g^n(x), g^n(y)) \to 0$ uniformly on B as $n \to \infty$. Specifically, if $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that*

$$(n \ge k) \Longrightarrow (d(g^n(x), g^n(y)) < \epsilon \ \forall x, y \in B). \tag{3.1}$$

Proof. By hypothesis $\delta(B) < \infty$, $P^n(\delta(B)) \to 0$ as $n \to \infty$. Let $\epsilon > 0$. We can choose $k \in \mathbb{N}$ such that

$$P^n(\delta(B)) < \epsilon \quad \text{for } n \ge k. \tag{3.2}$$

Let $x, y \in B$. If $n \in \mathbb{N}$, since g is P-contractive relative to H, Lemma 2.7 yields $r_n, s_n \in H$ and $u_n, v_n \in \{x, y\}(\subset B)$ such that

$$d(g^n(x), g^n(y)) \le P^n(d(r_nu_n, s_nv_n)). \tag{3.3}$$

Since $u_n \in B$, there exist $h \in H$ and $c \in M$ such that $u_n = h(c)$. But $r_n, h \in H$, so $r_nh \in H$. Therefore, $r_nu_n = (r_nh)(c) \in H(c) \subset B$. Likewise, $s_nv_n \in B$. But then $d(r_nu_n, s_nv_n) \le \delta(B)$ and therefore,

$$P^n(d(r_nu_n, s_nv_n)) \le P^n(\delta(B)) \quad \text{for } n \in \mathbb{N}, \tag{3.4}$$

since P is nondecreasing and n is arbitrary. Formulae (3.2), (3.3), and (3.4) imply

$$d(g^n(x), g^n(y)) < \epsilon \quad \text{for } n \ge k. \tag{3.5}$$

Since the choice of k in (3.2) was independent of x and y, (3.5) holds for all $x, y \in B$. □

Theorem 3.3. *Let $(X;d)$ be a T_2 semi-metric space, and let H be a semi-group of self maps of X which is n.c. at $g \in H$. Suppose that $H(a)$ is bounded for some $a \in X$ and X is g-orbitally complete. If g is a P-contraction relative to H, then $g^n(a) \to c$ for some $c \in X$. If g is continuous at c, $g(c) = c$.*

Proof. Since X is g-orbitally complete, to show that $g^n(a) \to c$ for some $c \in X$ it suffices to show that $\{g^n(a)\}$ is a Cauchy sequence.

To this end, let $\epsilon > 0$. Since, $H(a)$ is bounded, Lemma 3.2 with $B = H(a)$ implies that there exists $k \in \mathbb{N}$ such that

$$n \geq k \Longrightarrow d(g^n(x), g^n(y)) < \epsilon \quad \forall x, y \in H(a). \tag{3.6}$$

Therefore, if $m > n \geq k$, $m = n + r$ for some $r \in \mathbb{N}$, and

$$d(g^n(a), g^m(a)) = d(g^n(a), g^n(g^r(a))) < \epsilon, \tag{3.7}$$

since a, $g^r(a) \in H(a)$. We conclude that $\{g^n(a)\}$ is Cauchy, and there exists $c \in X$ such that $g^n(a) \to c$.

Now, if g is continuous at c, $\lim_{n\to\infty} g(g^n(a)) = g(c)$, since $g^n(a) \to c$. But then $g^{n+1}(a) \to c$ also, so $g(c) = c$ since $(X;d)$ is a T_2 semi-metric space. □

Definition 3.4. Let X and Y be topological spaces. A map $g : X \to Y$ is *closed* if and only if $g(M)$ is closed in Y whenever M is a closed subset of X.

Note that the conclusion of Lemma 3.2 asserts that $d(g^k(x_k), g^k(y_k)) \to 0$ for any sequences $\{x_k\}$ and $\{y_k\}$ in B.

Theorem 3.5. *Let $(X;d)$ be a bounded and complete T_2 semi-metric space, and let H be a semi-group of maps n.c. at $g \in H$. If g is closed and P-contractive relative to H,*
 (i) *there exists $p \in X$ such that $\{p\} = \cap\{g^n(X) \mid n \in \mathbb{N}\}$,*
 (ii) *p is the unique fixed point of g,*
 (iii) *$g^n(x) \to p$ for all $x \in X$.*

Proof. Let $x \in X$. By Theorem 3.3, $\{g^n(x)\}$ converges to p for some $p \in X$. Moreover, $p \in \cap\{g^n(X) \mid n \in \mathbb{N}\}$. Otherwise, there exists $k \in \mathbb{N}$ such that $p \notin g^k(X)$. Since $g^k(X)$ is closed, there exists $\epsilon > 0$ such that $S(p,\epsilon) \cap g^k(X) = \varnothing$. Thus, $d(g^n(x), p) \geq \epsilon$ for $n \geq k$ since $g^n(X)$ is a subset of $g^k(X)$ for $n \geq k$. This contradicts the fact that $g^n(x) \to p$.

In fact, $\{p\} = \cap\{g^n(X) \mid n \in \mathbb{N}\}$. For if $q \in \cap\{g^n(X) \mid n \in \mathbb{N}\}$, for each $k \in \mathbb{N}$ we can choose $x_k, y_k \in X$ such that $g^k(x_k) = p$ and $g^k(y_k) = q$. So

$$d(p,q) = d(g^k(x_k), g^k(y_k)) \longrightarrow 0, \tag{3.8}$$

by Lemma 3.2 with $M = X$.

Clearly, (i) implies that p is a fixed point of g, since $g(\{p\}) \subset \{p\}$. Thus, if $x \in X$, $d(g^n(x), p) = d(g^n(x), g^n(p)) \to 0$ as $n \to \infty$, so (iii) holds. Similarly, if q is a fixed point of g, then $d(p,q) = (g^n(p), g^n(q)) \to 0$, so that $q = p$. Thus, p is the only fixed point of g. □

In the following we need the triangle inequality, so we require the underlying space to be a metric space.

Theorem 3.6. *Let (X,d) be a metric space and let H be a semi-group of self maps of X n.c. at some $g \in H$. Suppose that X is g-orbitally complete and there exists $k \in \mathbb{N}$ such that for each pair $x, y \in X$, there exist $r, s \in H$ and $u, v \in \{x, y\}$ for which*

$$d(g^k x, g^k y) \leq P(d(ru, sv)). \tag{3.9}$$

(i) *If there exists* $a \in X$ *such that* $H(a)$ *is bounded, then there exists* $c \in X$ *such that* $\lim_{n\to\infty} g^n(a) = c$. *If* h *is continuous for some* $h \in H$, *then* $h(c) = c$. *(Specifically,* $g(c) = c$ *if* g *is continuous at* c.*)*

(ii) *If* $H(x)$ *is bounded for each* $x \in X$, *there exists a unique* $c \in X$ *such that* $g^n(x) \to c$ *for all* $x \in X$. *If* g *is continuous at* c, c *is a unique common fixed point for all* $h \in H$.

Proof. Suppose that $H(a)$ is bounded. Since H is n.c. at g, Proposition 2.6 says that H is n.c. at g^k. And X is g^k-orbitally complete since X is g-orbitally complete. Therefore, (3.9) and Theorem 3.3 imply that

$$\lim_{m\to\infty} (g^k)^m(a) = c \quad \text{for some } c \in X. \tag{3.10}$$

To see that $\lim_{n\to\infty} g^n(a) = c$, let $\epsilon > 0$. Then (3.10) and Lemma 3.2 (with $B = H(a)$) imply that there exists $p \in \mathbb{N}$ such that $d((g^k)^p(a),c) < \epsilon/2$ and $d(g^{kp}(x), g^{kp}(y)) < \epsilon/2$ for $x, y \in B$; that is,

$$d(g^{kp}(a),c) < \frac{\epsilon}{2}, \qquad d(g^{kp}(g^i(a)), g^{kp}(a)) < \frac{\epsilon}{2} \quad \forall i \in \mathbb{N}, \tag{3.11}$$

since $g \in H \Rightarrow g^i(a) \in H(a)$. So, if $n > kp$, $n = kp + i$ for some $i \in \mathbb{N}$, and

$$d(g^n(a),c) \le d(g^n(a), g^{kp}(a)) + d(g^{kp}(a),c), \tag{3.12}$$

or

$$d(g^n(a),c) \le d(g^{kp}(g^i(a)), g^{kp}(a)) + d(g^{kp}(a),c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \tag{3.13}$$

by (3.11). Consequently, $g^n(a) \to c$.

Now, let $h \in H$ and suppose that h is continuous at c. Then, $\lim_{n\to\infty} h(g^n(a)) = h(c)$ and

$$d(h(c),c) = \lim_{n\to\infty} d(hg^n(a), g^n(a)) = \lim_{n\to\infty} d(h(g^k)^n(a), (g^k)^n(a)). \tag{3.14}$$

But H is n.c. at g^k, so for $n \in \mathbb{N}$ there exists $h_n \in H$ such that $hg^{kn} = g^{kn}h_n$. Then, by (3.14),

$$d(h(c),c) = \lim_{n\to\infty} d((g^k)^n(h_n(a)), (g^k)^n(a)) = 0, \tag{3.15}$$

since $a, h_n(a) \in H(a)$ and Lemma 3.2 holds for g^k. Thus, (i) holds.

To prove (ii), suppose that $H(x)$ is bounded for each $x \in X$. If $a, b \in X$, $g^n(a) \to c_a$ and $g^n(b) \to c_b$ for some $c_a, c_b \in X$ by (i). But $c_a = c_b$, since $H(a) \cup H(b)$ is bounded, and therefore, Lemma 3.2 applied to g^k implies that $d(c_a, c_b) = \lim_{n\to\infty} d((g^k)^n(a), (g^k)^n(b)) = 0$.

Thus, there exists a unique $c \in X$ such that $g^n(x) \to c$ for all $x \in X$. We know that $g(c) = c$ by part (i), if g is continuous at c. Since $g^n(d) = d$ for all n if d is a fixed point of g, and therefore $g^n(d) \to d$, c must be the only fixed point of g. Moreover, $h(c) = c$ for all $h \in H$ (even though h may not be continuous). This follows, since Proposition 2.6 applied to g^k implies that for each $n \in \mathbb{N}$,

$$d(c, h(c)) = d((g^k)^n(c), h(g^k)^n(c)) = d((g^k)^n(c), (g^k)^n(h_n(c))) \tag{3.16}$$

for some $h_n \in H$. But $H(c)$ is bounded, so Lemma 3.2 applied to g^k implies that the right member of (3.16) converges to zero as $n \to \infty$, and thus, $c = h(c)$. □

Remark 3.7. Theorem 3.3 appreciably generalizes Theorem 2.1 in [5] and Theorem 3.6 generalizes Corollary 2.3 in [5]—and hence Theorem 2 in [3] and the theorems of Rhoades and Watson [9]. Note that in Theorem 3.6(ii), the mappings $h \in H$ $(h \neq g)$ need not be continuous. Remember also that C_g and O_g are special instances of H.

The following example suggests that the requirement in Theorem 3.6(ii), that $H(x)$ be bounded for each $x \in X$, is not as restrictive as may first appear.

Example 3.8. Let $S = \{$continuous functions $f : [0,\infty) \to [0,\infty)$ | there exists $a_f \in (0,\infty)$ such that $f(x) < x$ for $x > a_f\}$. (So, e.g., $\{f \mid f(x) = mx + b,\ m \in [0,1)$ and $b \geq 0\} \subset S$, and $\ln(x+b) \in S$ for $b \geq 1$.) Then (1) $S \cup \{i_d\}$ is a semi-group under composition of functions, and (2) $O_f(x)$ is bounded for $f \in S$ and $x \in [0,\infty)$.

First note that, we can let M_f denote the maximum value of f on $[0,a_f]$ for each $f \in S$ since each f is continuous. To see that (1) is true, let $f,g \in S$. We need only to show that $g \circ f = gf \in S$. Clearly, gf is a continuous self map of $[0,\infty)$. So let $a_{gf} = \max\{a_f, M_g\}$ and suppose that $x > a_{gf}$. We want $gf(x) < x$. Now, $x > a_{gf}$ implies that $x > a_f$ so that (i) $f(x) < x$. If $f(x) > a_g$, then $g(f(x)) < f(x) < x$ by (i) and the definition of a_g. If $f(x) \leq a_g$, $g(f(x)) \leq M_g \leq a_{gf} < x$. So, in any event, $(g \circ f)(x) < x$ if $x > a_{gf}$, and thus, $g \circ f \in S$. (2) follows easily by using induction to show that $(f \in S)$ implies that (if $x \in [0,\infty)$, $f^n(x) \leq \max\{x, M_f\}$ for $n \in \mathbb{N}$). We omit the details.

If we let $P(t) = \alpha t$ for fixed $\alpha \in (0,1)$ and $t \in [0,\infty)$, we have the following corollary.

Corollary 3.9. *Let (X,d) be a bounded complete metric space and let $g : X \to X$ be continuous. Suppose that H is a semi-group of self maps of X n.c. at g and $g \in H$. If there exists $\alpha \in (0,1)$ such that for any pair $x,y \in X$ there exist $r,s \in H$ and $u,v \in \{x,y\}$ for which*

$$d(gx,gy) \leq \alpha d(ru,sv), \tag{3.17}$$

then there exists a unique $c \in X$ such that $g^n(x) \to c$ for $x \in X$, and $c = gc = hc$ for all $h \in H$.

4. Some consequences

Definition 4.1. A *gauge function* is an upper semicontinuous (u.s.c.) function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

Lemma 4.2. *Let (X,d) be a metric space and let H be a semi-group of self maps of X which is n.c. at $g \in H$. Suppose that $H(x,y) = H(x) \cup H(y)$ is bounded for $x,y \in X$ and there exists a gauge function ϕ such that*

$$d(gx,gy) \leq \phi(\delta(H(x,y))) \quad \text{for } x,y \in X. \tag{4.1}$$

Then, there exists a nondecreasing continuous function $P : [0,\infty) \to [0,\infty)$ such that $P^n(t) \to 0$ for all $t > 0$ and which satisfies the following condition: for any pair $x,y \in X$ there exist $r = r(x,y)$, $s = s(x,y) \in H$, and $u,v \in \{x,y\}$ such that

$$d(gx,gy) \leq P(d(ru,sv)). \tag{4.2}$$

Proof. Let $x, y \in X$ and suppose that (4.1) holds. Since, ϕ is a gauge function, as is well known [2], there exists a nondecreasing continuous function $P : [0, \infty) \to [0, \infty)$ such that $P^n(t) \to 0$ for $t \geq 0$, and

$$\phi(t) < P(t), \quad P(t) < t \quad \forall t \in (0, \infty). \tag{4.3}$$

Since P is continuous, (4.3) implies that for any $t > 0$, there exists $\epsilon_t \in (0, t)$ such that

$$t' \in (t - \epsilon_t, t + \epsilon_t) \Longrightarrow \phi(t) < P(t'). \tag{4.4}$$

And since $H(x, y)$ is bounded, the definition of δ implies that there exist $r, s \in H$ and $u, v \in \{x, y\}$ such that, with $t = \delta(H(x, y))$,

$$t = \delta(H(x, y)) \geq d(ru, sv) > \delta(H(x, y)) - \epsilon_t. \tag{4.5}$$

So, with $t' = d(ru, sv)$, (4.4) and (4.5) imply that

$$\phi(\delta(H(x, y))) < P(d(ru, sv)). \tag{4.6}$$

Therefore, (4.1) implies that $d(gx, gy) \leq P(d(ru, sv))$. □

The following theorem provides a generalization of Theorem 2.1 in [2].

Theorem 4.3. *Let (X, d) be a complete metric space and let H be a semi-group of self maps of X which is n.c. at some $g \in H$. Suppose that the following conditions are satisfied:*

(i) *$H(x)$ is bounded for all $x \in X$, g is continuous,*
(ii) *there exists a gauge function ϕ and $k \in \mathbb{N}$ such that*

$$d(g^k x, g^k y) \leq \phi(\delta(H(x, y))) \quad \textit{for } x, y \in X. \tag{4.7}$$

Then

(a) *H has a unique common fixed point c and $g^n(x) \to c$ for $x \in X$.*
(b) *If for each $h \in H - \{i_d\}$ there exists $k = k_h \in \mathbb{N}$ such that (4.7) holds with $g = h$, then*

$$h^n(x) \longrightarrow c \quad \forall x \in X,\ h \in H - \{i_d\}. \tag{4.8}$$

Proof. Now, (i) implies that $H(x, y) = H(x) \cup H(y)$ is bounded for $x, y \in X$. To see that (a) is true, note that H is n.c. at g^k by Proposition 2.6 and substitute g^k for g in Lemma 4.2 to conclude that (3.9) holds. Consequently, we can appeal to Theorem 3.6(ii) to obtain a $c \in X$ such that $g^n(x) \to c$ for $x \in X$. And since g is continuous, c is the unique fixed point of g and a fixed point for each $h \in H$. Thus, c is the unique common fixed point of H (remember, $g \in H$) and therefore (a) holds.

To prove (b) note that, by part (a), if $h \in H - \{i_d\}$, $h \neq g$, $h^n(c) = g(c) = c$ for $n \in \mathbb{N}$. But Theorem 3.6 applied to h yields a unique $c_1 \in X$ such that $h^n(x) \to c_1$ for all $x \in X$. Since $h^n(c) = c$ for all $n, c_1 = c$. □

Remark 4.4. Theorem 4.3 generalizes Theorem 2.1 in [2] in the following ways:

(i) The semi-group H is not required to be near-commutative (i.e., n.c. at each $h \in H$), but n.c. only at g,

(ii) g is the only member of H required to be continuous,

(iii) in (b), (4.7) is required to hold only for $k = k_h$, not for all $k \geq k_h$.

Theorem 4.3 yields the following corollary, which generalizes the theorem of Ohta and Nikaido [8] by requiring only that the orbits of f—but not all of X—be bounded.

Corollary 4.5. *Let f be a continuous self mapping of a metric space (X,d) having bounded orbits $O_f(x)$ for all $x \in X$. If there exist $c \in (0,1)$ and $k \in \mathbb{N}$ such that*

$$d(f^k x, f^k y) \leq c\delta(\{f^i t \mid t \in \{x,y\},\ i \in \mathbb{N} \cup \{0\}\}) \tag{4.9}$$

for all $x,y \in X$, then f has a unique fixed point.

Observe that Lemma 3.2 does not require that $g \in H$, whereas the theorems in Section 3 do. The requirement that $g \in H$ was convenient in the proof, but the following proposition says that it is not necessary when $O_g(a)$ is bounded. Moreover, this result is needed for the proof of Theorem 4.7.

Proposition 4.6. *If H is a semi-group of self maps n.c. at g and $g \notin H$, then $H_g = \{g^n h \mid n \in \mathbb{N} \cup \{0\}$ and $h \in H\}$ is a semi-group which is n.c. at g. Moreover, $g \in H_g$ and $H \subset H_g$.*

Proof. H_g is a semi-group. For if $g^n h_1, g^m h_2 \in H_g$, since H is n.c. at g, we have $g^n h_1 g^m h_2 = g^n(h_1 g^m)h_2 = g^n(g^m h_3)h_2 = g^{n+m}(h_4)$, where $h_4 = h_3 h_2 \in H$.

H_g is n.c. at g, since (H n.c. at g) implies that there exists $h_2 \in H$ such that $(g^n h)g = g^n(hg) = g^n(gh_2) = g(g^n h_2)$. □

It is clear that if $g : X \to X$ is a P-contraction relative to H, then it is certainly a P-contraction relative to H_g since $H \subset H_g$. We use this fact in the proof of Theorem 4.7.

Theorem 4.7. *Let C be a compact subset of a normed linear space X which is star-shaped with respect to $q \in C$. Let $T : C \to C$ be continuous and let H be a semi-group of affine maps $I : C \to C$ n.c. at T such that $I(q) = q$. If for each pair $x,y \in C$ there exist $I,J \in H$ and $u,v \in \{x,y\}$ for which*

$$\|Tx - Ty\| \leq \|Iu - Jv\|, \tag{4.10}$$

then there exists $a \in C$ such that $a = Ta$ and $a = Ia$ for all continuous $I \in H$.

Proof. Choose a sequence $\{k_n\}$ in $(0,1)$ such that $k_n \to 1$, and for each $n \in \mathbb{N}$, let

$$T_n(x) = k_n Tx + (1-k_n)q. \tag{4.11}$$

Since C is star-shaped with respect to q, $T_n : C \to C$ for $n \in \mathbb{N}$. Moreover, if $I \in H$, there exists $J \in H$ such that

$$\begin{aligned} IT_n x &= I(k_n Tx + (1-k_n)q) = k_n I(Tx) + (1-k_n)Iq \\ &= k_n T(Jx) + (1-k_n)q = T_n Jx, \end{aligned} \tag{4.12}$$

since I is affine, H is n.c. at T, and $Iq = q$. Thus, for each $n \in \mathbb{N}$, H is a semi-group of affine maps which is n.c. at T_n. Then, by Proposition 4.6, H_{T_n} is a semi-group of self maps of C which is n.c. at T_n, $T_n \in H_{T_n}$, and $H \subset H_{T_n}$ for $n \in \mathbb{N}$.

Now fix n. By hypothesis, for each pair $x,y \in C$ there exist $I,J \in H(\subset H_{T_n})$ and $u,v \in \{x,y\}$ such that

$$\|Tx - Ty\| \le \|Iu - Jv\|, \tag{4.13}$$

so

$$\|T_n x - T_n y\| \le k_n \|Iu - Jv\|, \tag{4.14}$$

by (4.11). Therefore, since T_n is continuous and $k_n \in (0,1)$, Corollary 3.9 applied to T_n and H_{T_n} (C compact implies that C is bounded and complete) implies that there exists a unique $x_n \in C$ such that

$$x_n = T_n(x_n) = I(x_n) \quad \forall I \in H_{T_n}. \tag{4.15}$$

Thus we have a sequence $\{x_n\}$ in C which satisfies (4.15). Since C is compact, $\{x_n\}$ has a subsequence $\{x_{i_n}\}$ which converges to some $a \in C$. Equations (4.11) and (4.15) thus imply that

$$a = \lim_{n\to\infty} x_{i_n} = \lim_{n\to\infty} k_{i_n} T x_{i_n} + \lim_{n\to\infty} (1 - k_{i_n}) q = \lim_{n\to\infty} I x_{i_n}. \tag{4.16}$$

But T is continuous, so (4.16) implies that $a = Ta$, and $a = Ia$ for all continuous I. □

Remark 4.8. We see that Theorem 4.7 does indeed extend Theorem 3 in [7] if we observe that the family $\mathcal{F}$ in Theorem 3 [7]. is a family of sets which is a subset of C_g. We can let

$$H = \{\text{maps } h : C \longrightarrow C \mid h \text{ is affine, } h \in C_g\}. \tag{4.17}$$

Then H is a semi-group and $\mathcal{F} \subset H$.

5. Conclusion.

We conclude with further evidence of the generality and applicability of the concept of being nearly commutative at a function g. The theorem below generalizes Theorem 4.2 in [5] by replacing the semi-group C_{gf} with a more general semi-group H.

Theorem 5.1. *Let f and g be commuting self maps of a compact metric space (X,d) such that gf is continuous. If H is a semi-group of self maps of X which is n.c. at gf, and*

$$fx \ne gy \Longrightarrow d(fx,gy) < \delta(H(x,y)), \tag{5.1}$$

then there exists a unique point $a \in X$ such that $a = fa = ga = ha$ for all $h \in H$.

We leave the proof of Theorem 5.1 to the interested reader.

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