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# A KÄHLER EINSTEIN STRUCTURE ON THE TANGENT BUNDLE OF A SPACE FORM

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Abstract. We obtain a Kähler Einstein structure on the tangent bundle of a Riemannian manifold of constant negative curvature. Moreover, the holomorphic sectional curvature of this Kähler Einstein structure is constant. Similar results are obtained for a tube around zero section in the tangent bundle, in the case of the Riemannian manifolds of constant positive curvature.

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**1. Introduction.** The tangent bundle $TM$ of a Riemannian manifold $(M,g)$ can be organized as an almost Kählerian manifold (see [3, 10, 18]) by using the Sasaki metric and an almost complex structure defined by the splitting of the tangent bundle to $TM$ into the vertical and horizontal distributions $VTM$ and $HTM$—(the last one being determined by the Levi Civita connection on $M$)—see also [8, 17, 19]. However, this structure is Kähler only in the case where the base manifold is locally Euclidean. The Sasaki metric is not a "good" metric in the sense of [1] since its Ricci curvature is not constant, that is, the Sasaki metric is not, generally, Einstein.

In the present paper, we are interested in finding a Kähler Einstein structure on the tangent bundle of a space form, following an idea from [2] (see also [1]). We have changed the metric $G$ on the tangent bundle (so that it is no longer a Sasaki metric) by using a certain tensor field on $TM$ obtained in the following way. Denote by $\tau : TM \to M$ the canonical projection of the tangent bundle on the base manifold. Let $y$ be an element of $TM$, denote by

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y,y) = \frac{1}{2}g_{ik}(x)y^iy^k \tag{1.1}$$

the value of the energy density in $y$ and let $g_y \in T^*TM$ be the cotangent vector obtained from $y$ by the "musical" isomorphism between $TM$ and $T^*M$ defined by $g$ (the "lowering" of indices). Identify the tensor field $g$ with its pullback by $\tau$ on $TM$. Then, we may consider the following symmetric tensor field of type $(0,2)$ on $TM$,

$$\tilde{G} = u(t)g + v(t)g_y \otimes g_y, \tag{1.2}$$

where $u,v : [0,\infty) \to \mathbb{R}$ are smooth real-valued functions depending on $t$ only. Of course, we assume that the functions $u$ and $v$ fulfil the conditions under which the bilinear form defined by $\tilde{G}$ is positive definite.

We tried to find expressions for the functions $u,v$ in order to obtain an Einstein metric on $TM$ defined by using the tensor field $\tilde{G}$. During our work, we used also an almost complex structure $J$ on $TM$, related to the considered metric, such that we obtain, in fact, a Kähler Einstein structure on $TM$ in the case where $(M,g)$ has constant (negative) curvature. As a matter of fact, we obtain the existence of a Kähler Einstein structure even in the case where $(M,g)$ has positive constant curvature, but only on a tube around the zero section in $TM$ (Theorem 4.2). The surprising fact was that we have obtained on $TM$ a structure of Kähler manifold with constant holomorphic sectional curvature (Theorem 5.1).

The manifolds, tensor fields, and geometric objects we consider in this paper, are assumed to be differentiable of class $C^\infty$ (i.e., smooth). We use the computations in local coordinates in a fixed local chart, but many results from this paper may be expressed in an invariant form. The well-known summation convention is used throughout this paper, the range for the indices $i,j,k,l,h,s,r$ being always $\{1,\dots,n\}$ (see [4, 5, 6, 7, 14, 15]). We denote by $\Gamma(TM)$ the module of smooth vector fields on $TM$.

**2. Almost Kähler structures on the tangent bundle.** Let $(M,g)$ be a smooth $n$-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that $TM$ has a structure of $2n$-dimensional smooth manifold induced from the smooth manifold structure of $M$. A local chart $(U,\varphi) = (U,x^1,\dots,x^n)$ on $M$ induces a local chart $(\tau^{-1}(U),\Phi) = (\tau^{-1}(U),x^1,\dots,x^n,y^1,\dots,y^n)$ on $TM$, where the local coordinates $x^i,y^i$, $i = 1,\dots,n$ are defined as follows. The first $n$ local coordinates are the local coordinates in the local chart $(U,\varphi)$ of the base point of a tangent vector from $\tau^{-1}(U)$, that is, $x^i = x^i\circ\tau$, $i = 1,\dots,n$, by an abuse of notation. The last $n$ local coordinates $y^i$, $i = 1,\dots,n$ are the vector space coordinates of the same tangent vector, with respect to the natural local basis in the corresponding tangent space, defined by the local chart $(U,\varphi)$.

This special structure of $TM$ allows us to introduce the notion of $M$-tensor field on it (see [9] for detailed explanations). An $M$-tensor field of type $(p,q)$ on $TM$ is defined by sets of functions

$$T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x,y), \quad i_1,\dots,i_p,j_1,\dots,j_q = 1,\dots,n \tag{2.1}$$

assigned to induced local charts $(\tau^{-1}(U),\Phi)$ on $TM$, such that the change rule is that of the components of a tensor field of type $(p,q)$ on the base manifold, when a change of local charts on the base manifold is performed. Remark that any ordinary tensor field on the base manifold may be thought of as an $M$-tensor field on $TM$, having the same type and with the components in the induced local chart on $TM$ (depending only on the base point of the tangent vector), equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold. In the case of a covariant tensor field on the base manifold $M$, the corresponding $M$-tensor field on the tangent bundle $TM$ may be thought of as the pullback of the initial tensor field defined on the base manifold, by the smooth submersion $\tau : TM \to M$. Thus, the components $g_{ik}$ of the metric $g$ on $M$ may be thought of as the components defining an $M$-tensor field of type $(0,2)$ on $TM$. The components $y^i$ define an $M$-tensor field of type $(1,0)$ on $TM$.

The tangent bundle $TM$ of a Riemannian manifold $(M,g)$ can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metric on $TM$ defined by $g$ (see [3, 17]) and the complete lift type pseudo-Riemannian metric defined by $g$ (see [11, 12, 18, 19]). Recall that the Levi Civita connection $\dot\nabla$ of $g$ defines a direct sum decomposition

$$TTM = VTM \oplus HTM \tag{2.2}$$

of the tangent bundle to $TM$ into the vertical distribution $VTM = \ker \tau_*$ and the horizontal distribution $HTM$. The set of vector fields $(\partial/\partial y^1, \ldots, \partial/\partial y^n)$ defines a local frame field for $VTM$ and for $HTM$ we have the local frame field $(\delta/\delta x^1, \ldots, \delta/\delta x^n)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0}\frac{\partial}{\partial y^h}, \qquad \Gamma^h_{i0} = \Gamma^h_{ik} y^k \tag{2.3}$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric $g$.

The distributions $VTM$ and $HTM$ are isomorphic to each other and it is possible to derive an almost complex structure on $TM$ which, together with the Sasaki metric, determines a structure of almost Kählerian manifold on $TM$ (see [3]). Consider now the energy density (kinetic energy)

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2} g_{ik}(x) y^i y^k \tag{2.4}$$

defined on $TM$ by the Riemannian metric $g$ of $M$, where $g_{ik}$ are the components of $g$ in the local chart $(U, \varphi)$. Let $u, v : [0, \infty) \to \mathbb{R}$ be two real smooth functions. Consider the following symmetric $M$-tensor field of type (0,2) on $TM$, defined by the components, (see [13, 16]),

$$G_{ij} = u(t) g_{ij} + v(t) g_{0i} g_{0j}, \tag{2.5}$$

where $g_{0i} = g_{hi} y^h$. The matrix $(G_{ij})$ is symmetric and its inverse (when it exists) has the entries

$$H^{kl} = \frac{1}{u} g^{kl} + w y^k y^l, \tag{2.6}$$

where $g^{kl}$ are the components of the inverse of the matrix $(g_{ij})$ and

$$w = w(t) = -\frac{v}{u(u+2tv)}. \tag{2.7}$$

The conditions under which the matrix $(G_{ij})$ is positive definite, hence nondegenerate, can be obtained easily by studying the property of the expression $G_{ij} z^i z^j$, $z^1, \ldots, z^n \in \mathbb{R}$ to be positive. These conditions are

$$u > 0, \quad u + 2tv > 0, \quad \forall t \geq 0. \tag{2.8}$$

The components $H^{kl}(x, y)$ define a symmetric $M$-tensor field of type $(2,0)$ on $TM$. We use also the components $H_{ij}(x, y)$ of a symmetric $M$-tensor field of type $(0,2)$

obtained from the components $H^{kl}$ by "lowering" the indices

$$H_{ij} = g_{ik}H^{kl}g_{lj} = \frac{1}{u}g_{ij} + w g_{0i}g_{0j}, \tag{2.9}$$

as well as the following $M$-tensor fields on $TM$ obtained by usual algebraic tensor operations

$$\begin{aligned} G^{kl} &= g^{ki}G_{ij}g^{jl} = ug^{kl} + v y^k y^l, \\ G^i_k &= G^{ih}g_{hk} = G_{kh}g^{hi} = u\delta^i_k + v y^i g_{0k}, \\ H^i_k &= H^{ih}g_{hk} = H_{kh}g^{hi} = \frac{1}{u}\delta^i_k + w y^i g_{0k}. \end{aligned} \tag{2.10}$$

Remark that the matrix $(H^i_k)$ is the inverse of the matrix $(G^i_k)$.

The following Riemannian metric may be considered on $TM$

$$G = G_{ij}dx^i dx^j + H_{ij}\dot{\nabla} y^i \dot{\nabla} y^j, \tag{2.11}$$

where $\dot{\nabla} y^i = dy^i + \Gamma^i_{j0}dx^j$ is the absolute differential of $y^i$ with respect to the Levi Civita connection $\dot{\nabla}$ of $g$. Equivalently, we have

$$\begin{gathered} G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G_{ij}, \qquad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = H_{ij}, \\ G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = 0. \end{gathered} \tag{2.12}$$

Remark that $HTM$ and $VTM$ are orthogonal to each other with respect to $G$ but the Riemannian metrics induced from $G$ on $HTM$ and $VTM$ are not the same, so the considered metric $G$ on $TM$ is no longer a metric of Sasaki type. Remark also that the system of 1-forms $(dx^1,\dots,dx^n,\dot{\nabla} y^1,\dots,\dot{\nabla} y^n)$ defines a local frame of $T^*TM$, dual to the local frame $(\delta/\delta x^1,\dots,\delta/\delta x^n,\partial/\partial y^1,\dots,\partial/\partial y^n)$ adapted to the direct sum decomposition (2.2).

An almost complex structure $J$ may be defined on $TM$ by

$$J\frac{\delta}{\delta x^i} = G^k_i \frac{\partial}{\partial y^k}; \qquad J\frac{\partial}{\partial y^i} = -H^k_i \frac{\delta}{\delta x^k}. \tag{2.13}$$

Then we obtain the following theorem.

**Theorem 2.1.** *$(TM,G,J)$ is an almost Kählerian manifold.*

**Proof.** First of all, we may check easily that $J^2 = -I$, by using the local expression (2.13) of $J$, due to the property of the matrix $(H^i_k)$ to be the inverse of the matrix $(G^i_k)$. Then, we have

$$\begin{aligned} G\left(J\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}\right) &= G_{ik}g^{ka}G_{jh}g^{hb}G\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) \\ &= G_{ik}g^{ka}G_{jh}g^{hb}g_{ac}G^{cd}g_{db} = G_{ij} = G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right). \end{aligned} \tag{2.14}$$

The relations

$$G\left(J\frac{\partial}{\partial y^i}, J\frac{\partial}{\partial y^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right), \qquad G\left(J\frac{\partial}{\partial y^i}, J\frac{\delta}{\delta x^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0 \tag{2.15}$$

may be obtained in a similar way, thus $G$ is almost Hermitian with respect to $J$. The associated 2-form $\Omega$, defined by

$$\Omega(X,Y) = G(X,JY), \quad X,Y \in \Gamma(TM), \tag{2.16}$$

is given by

$$\Omega\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right) = \Omega\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right) = 0, \qquad \Omega\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right) = g_{ij}. \tag{2.17}$$

Hence, we have

$$\Omega = g_{ij}\dot{\nabla} y^i \wedge dx^j \tag{2.18}$$

□

and $\Omega$ is closed since it does coincide with the 2-form associated to the Sasaki metric on $TM$ (see [3]).

## 3. A Kähler structure on $TM$.

In this section, we study the integrability of the almost complex structure defined by $J$ on $TM$. To do this, we need the following well-known formulae for the brackets of the vector fields $\partial/\partial y^i, \delta/\delta x^i$, $i = 1,\dots,n$,

$$\left[\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right] = 0; \qquad \left[\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right] = -\Gamma_{ij}^h\frac{\partial}{\partial y^h}; \qquad \left[\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right] = -R_{0ij}^h\frac{\partial}{\partial y^h}, \tag{3.1}$$

where $R_{0ij}^h = R_{kij}^h y^k$ and $R_{kij}^h$ are the local coordinate components of the curvature tensor field of $\dot{\nabla}$ on $M$.

**Theorem 3.1.** *The almost complex structure $J$ on $TM$ is integrable if $(M,g)$ has constant sectional curvature $c$ and the function $v$ is given by*

$$v = \frac{c-uu'}{2tu'-u}. \tag{3.2}$$

Of course we have to study the conditions under which $u,v$ fulfil the conditions $u > 0$, $u+2tv > 0$, for all $t \geq 0$.

**Proof.** First of all, the following formulae can be checked by straightforward computation:

$$\begin{aligned}\dot{\nabla}_i G_{jk} &= \frac{\delta}{\delta x^i}G_{jk} - \Gamma_{ij}^h G_{hk} - \Gamma_{ik}^h G_{jh} = 0,\\ \dot{\nabla}_i G_k^j &= \frac{\delta}{\delta x^i}G_k^j + \Gamma_{ih}^j G_k^h - \Gamma_{ik}^h G_h^j = 0.\end{aligned} \tag{3.3}$$

In a similar way there are obtained the formulae $\dot{\nabla}_i H_{jk} = 0$, $\dot{\nabla}_i H_k^j = 0$. Then, by using the definition of the Nijenhuis tensor field $N_J$ of $J$, that is,

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y], \quad \forall X,Y \in \Gamma(TM), \tag{3.4}$$

we have

$$\begin{aligned}N_J\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right) &= \left(G_i^h\frac{\partial G_j^k}{\partial y^h} - G_j^h\frac{\partial G_i^k}{\partial y^h} + R_{0ij}^k\right)\frac{\partial}{\partial y^k}\\ &\quad - \left(\frac{\delta}{\delta x^j}G_i^k - \frac{\delta}{\delta x^i}G_j^k + G_i^h\Gamma_{hj}^k - G_j^h\Gamma_{ih}^k\right)H_k^l\frac{\delta}{\delta x^l}.\end{aligned} \tag{3.5}$$

The coefficient of $H_k^l(\delta/\delta x^l)$ is just $-\dot\nabla_j G_i^k + \dot\nabla_i G_j^k = 0$ so, we have to study the vanishing of the coefficient of $\partial/\partial y^k$. By using the expression (2.10) of $G_i^k$, we get

$$R_{0ij}^h + G_i^k \frac{\partial}{\partial y^k} G_j^h - G_j^k \frac{\partial}{\partial y^k} G_i^h = (uu' + 2tu'v - uv)(g_{0i}\delta_j^h - g_{0j}\delta_i^h) + R_{0ij}^h = 0. \quad (3.6)$$

Differentiating with respect to $y^k$, taking $y = 0$ and using Schur theorem, it follows that the curvature tensor field of $\dot\nabla$ (in the case where $M$ is connected and $\dim M > 2$) must have the expression

$$R_{hij}^k = c(\delta_i^k g_{hj} - \delta_j^k g_{hi}), \quad (3.7)$$

where $c$ is a constant. Then we obtain the expression (3.2) of $v$.

Next, it follows by a straightforward computation that $N_J(\partial/\partial y^i, \delta/\delta x^j) = 0$, $N_J(\partial/\partial y^i, \partial/\partial y^j) = 0$, whenever $N_J(\delta/\delta x^i, \delta/\delta x^j) = 0$.

Hence, $(M,g)$ must have constant sectional curvature $c$, $v$ is given by (3.2) and then we obtain easily the expression of the function $w$,

$$w = w(t) = \frac{uu' - c}{u(2tc - u^2)}. \quad (3.8)$$

□

**4. A Kähler Einstein structure on $TM$.** In this section, we study the property of $(TM,G)$ to be Einstein. We find the expression of the Levi Civita connection $\nabla$ of the metric $G$ on $TM$, then we get the expression of the curvature tensor field of $\nabla$. Then, by computing the corresponding traces, we find the components of the Ricci tensor field of $\nabla$. Asking for the Ricci tensor field to be proportional with the metric $G$, we find a second-order ordinary differential equation which must be fulfilled by the function $u$. Fortunately, we have been able to find the general solution of this differential equation by elementary methods. For a special value of one of the integration constants we obtained the property of $G$ to be Einstein. At the same time we have been able to study the conditions under which $u > 0$, $u + 2tv > 0$, for all $t \geq 0$.

Recall that the Levi Civita connection $\nabla$ on a Riemannian manifold $(M,g)$ is obtained from the formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) \\ &\quad - g([X,Z],Y) - g([Y,Z],X), \quad \forall X,Y,Z \in \chi(M) \end{aligned} \quad (4.1)$$

and is characterized by the conditions

$$\nabla G = 0, \qquad T = 0, \quad (4.2)$$

where $T$ is the torsion tensor of $\nabla$ (see [4, 5, 6, 7]).

**Theorem 4.1** (see [16]). *The Levi Civita connection $\nabla$ of $G$ has the following expression in the local adapted frame $(\partial/\partial y^1, \dots, \partial/\partial y^n, \delta/\delta x^1, \dots, \delta/\delta x^n)$,*

$$\begin{aligned} &\nabla_{\partial/\partial y^i} \frac{\partial}{\partial y^j} = Q_{ij}^h \frac{\partial}{\partial y^h}, \qquad \nabla_{\delta/\delta x^i} \frac{\partial}{\partial y^j} = \Gamma_{ij}^h \frac{\partial}{\partial y^h} + P_{ji}^h \frac{\delta}{\delta x^h}, \\ &\nabla_{\partial/\partial y^i} \frac{\delta}{\delta x^j} = P_{ij}^h \frac{\delta}{\delta x^h}, \qquad \nabla_{\delta/\delta x^i} \frac{\delta}{\delta x^j} = \Gamma_{ij}^h \frac{\delta}{\delta x^h} + S_{ij}^h \frac{\partial}{\partial y^h}, \end{aligned} \quad (4.3)$$

*where the M-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ are given by*

$$
\begin{aligned}
P^h_{ij} &= \frac{1}{2}\left(\frac{\partial G_{jk}}{\partial y^i} + H_{il}R^l_{0jk}\right)H^{kh}, \\
Q^h_{ij} &= \frac{1}{2}G^{hk}\left(\frac{\partial H_{jk}}{\partial y^i} + \frac{\partial H_{ik}}{\partial y^j} - \frac{\partial H_{ij}}{\partial y^k}\right), \\
S^h_{ij} &= \frac{1}{2}\left(-R^h_{0ij} - \frac{\partial G_{ij}}{\partial y^k}G^{kh}\right).
\end{aligned}
\tag{4.4}
$$

Taking into account the expressions (2.5), (2.9) of $G_{ij}$ and $H_{ij}$ and by using the formulae (3.2), (3.8), and (2.7) we may obtain the following expressions:

$$
\begin{aligned}
P^h_{ij} &= \frac{u'}{2u}\delta^h_j g_{0i} + \frac{uv-c}{2u^2}\delta^h_i g_{0j} - \frac{(c+uv)w}{2v}g_{ij}y^h \\
&\quad + \frac{vw(uv-c)+uw(u'v-uv')}{2uv}g_{0i}g_{0j}y^h, \\
Q^h_{ij} &= -\frac{u'}{2u}(\delta^h_j g_{0i} + \delta^h_i g_{0j}) - \frac{v(u'+2u^2w)}{2u^3w}g_{ij}y^h - \frac{v(2u'w+uw')}{2u^2w}g_{0i}g_{0j}y^h, \\
S^h_{ij} &= \frac{c-uv}{2}\delta^h_j g_{0i} - \frac{c+uv}{2}\delta^h_i g_{0j} + \frac{u'v}{2uw}g_{ij}y^h + \frac{v(v'-2uvw)}{2uw}g_{0i}g_{0j}y^h.
\end{aligned}
\tag{4.5}
$$

Remark that in the obtained formulae we have used the formulae (2.7) in order to replace the energy density $t$, such that it is not involved explicitly. Of course, we can replace $v, v', w, w'$ as functions of $u, u', u''$, and $t$ but we obtain much more complicated expressions for $P^h_{ij}, Q^h_{ij}, S^h_{ij}$.

The curvature tensor field $K$ of the connection $\nabla$ is defined by the well-known formula

$$
K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X,Y,Z \in \Gamma(TM). \tag{4.6}
$$

By using the local frame $(\delta/\delta x^i, \partial/\partial y^i)$, $i = 1,\dots,n$, we obtain, after a standard straightforward computation,

$$
\begin{aligned}
K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= XXX^h_{kij}\frac{\delta}{\delta x^j}, & K\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= XXY^h_{kij}\frac{\partial}{\partial y^h}, \\
K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} &= YYX^h_{kij}\frac{\delta}{\delta x^h}, & K\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} &= YYY^h_{kij}\frac{\partial}{\partial y^h}, \\
K\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= YXX^h_{kij}\frac{\partial}{\partial y^h}, & K\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= YXY^h_{kij}\frac{\delta}{\delta x^h},
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
XXX^h_{kij} &= R^h_{kij} + P^h_{lk}R^l_{0ij} + P^h_{li}S^l_{jk} - P^h_{lj}S^l_{ik},\\
XXY^h_{kij} &= R^h_{kij} + Q^h_{lk}R^l_{0ij} + S^h_{il}P^l_{kj} - S^h_{jl}P^l_{ki},\\
YYX^h_{kij} &= \frac{\partial}{\partial y^i}P^h_{jk} - \frac{\partial}{\partial y^j}P^h_{ik} + P^h_{il}P^l_{jk} - P^h_{jl}P^l_{ik},\\
YYY^h_{kij} &= \frac{\partial}{\partial y^i}Q^h_{jk} - \frac{\partial}{\partial y^j}Q^h_{ik} + Q^h_{il}Q^l_{jk} - Q^h_{jl}Q^l_{ik},\\
YXX^h_{kij} &= \frac{\partial}{\partial y^i}S^h_{jk} + Q^h_{il}S^l_{jk} - S^h_{jl}P^l_{ik},\\
YXY^h_{kij} &= \frac{\partial}{\partial y^i}P^h_{kj} + P^h_{il}P^l_{kj} - P^h_{lj}Q^l_{ik}.
\end{aligned}
\tag{4.8}
$$

Remark that, as a first step, the formulae for the local expressions of $K$ also contained some other terms involving the Christoffel symbols $\Gamma^k_{ij}$. However, after some computations similar to that made for obtaining the formulae (3.3) we have been able to show that those other terms are zero.

Now, we have to replace the expressions (4.5) of the $M$-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ and the expressions (3.2), (3.8), and (2.7) of the functions $v, w$ and of their derivatives in order to obtain the components from (4.8) as functions of $u, u', u'', u^{(3)}$ only. The obtained expressions are quite complicated and, at this stage, we decided to use the mathematica package RICCI in order to do the necessary tensor calculations. It has been useful to consider $c, t, u, v, w, u', v', w', u'', v'', w'', u^{(3)}$ as constants, the tangent vector $y$ as a first-order tensor, the components $G_{ij}, H_{ij}$ as second-order tensors and so on, on the Riemannian manifold $M$, the associated indices being $h, i, j, k, l, r, s$. It was not convenient to think of $u, v, w$ as functions of $t$ since RICCI did not make some useful factorizations after the command TensorSimplify.

The components of the Ricci tensor field of the connection $\nabla$, defined as $\operatorname{Ric}(Y,Z) = \operatorname{trace}(X \to K(X,Y)Z)$, $X, Y, Z \in \Gamma(TM)$ are obtained as follows:

$$
\begin{aligned}
\operatorname{Ric}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) &= \operatorname{Ric}XX_{jk} = XXX^h_{khj} + YXX^h_{khj},\\
\operatorname{Ric}\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j}\right) &= \operatorname{Ric}YY_{jk} = YYY^h_{khj} - YXY^h_{kjh},\\
\operatorname{Ric}\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right) &= \operatorname{Ric}\left(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}\right) = 0.
\end{aligned}
\tag{4.9}
$$

The expressions of $\operatorname{Ric}XX_{jk}$, $\operatorname{Ric}YY_{jk}$ are quite complicated. In order to present a summary description of these expressions, we introduce the function

$$
a = n(u-2tu')(2cu-2ctu'-u^2u') + 2(2ct-u^2)(tuu''+uu'-tu'^2). \tag{4.10}
$$

Then

$$
\begin{aligned}
\operatorname{Ric}XX_{jk} &= \frac{a}{2(u-2tu')^2}g_{jk} + \alpha(t,u,u',u'',u^{(3)})g_{0j}g_{0k},\\
\operatorname{Ric}YY_{jk} &= \frac{a}{2u^2(u-2tu')^2}g_{jk} + \beta(t,u,u',u'',u^{(3)})g_{0j}g_{0k},
\end{aligned}
\tag{4.11}
$$

where $\alpha, \beta$ are rational expressions in $t, u, u', u'', u^{(3)}$.

To study the conditions under which $(TM,G)$ is Einstein, we consider the differences

$$\begin{aligned}\operatorname{Diff} XX_{jk} &= \operatorname{Ric} XX_{jk} - \frac{a}{2u(u-2tu')^2} G_{jk},\\ \operatorname{Diff} YY_{jk} &= \operatorname{Ric} YY_{jk} - \frac{a}{2u(u-2tu')^2} H_{jk}.\end{aligned} \tag{4.12}$$

Then, we obtain

$$\begin{aligned}
\operatorname{Diff} XX_{jk} = {}& \frac{1}{2u^2(u-2tu')^4}\\
&\times [n(u^2-2ct)(2tu'-u)(u^2u''-2tu'^3+2uu'^2)\\
&\quad -8c^2tu^3u'+4cu^5u'+16c^2t^2u^2u'^2+4ctu^4u'^2-6u^6u'^2-24c^2t^3uu'^3\\
&\quad -8ct^2u^3u'^3+10tu^5u'^3+16c^2t^4u'^4-4t^2u^4u'^4-24c^2t^2u^3u''\\
&\quad +20ctu^5u''-4u^7u''+16c^2t^3u^2u'u''-4tu^6u'u''-16ct^3u^3u'^2u''\\
&\quad +8t^2u^5u'^2u''-32c^2t^4u^2u''^2+32ct^3u^4u''^2-8t^2u^6u''^2-8c^2t^3u^3u^{(3)}\\
&\quad +8ct^2u^5u^{(3)}-2tu^7u^{(3)}+16c^2t^4u^2u'u^{(3)}-16ct^3u^4u'u^{(3)}\\
&\quad +4t^2u^6u'u^{(3)}]g_{0j}g_{0k},\\
\operatorname{Diff} YY_{jk} = {}& \frac{1}{2u^2(u^2-2ct)(u-2tu')^2}\\
&\times [n(u^2-2ct)(2tu'-u)(u^2u''-2tu'^3+2uu'^2)\\
&\quad +4cu^3u'-8ctu^2u'^2-6u^4u'^2+12ct^2uu'^3+10tu^3u'^3-8ct^3u'^4\\
&\quad -4t^2u^2u'^4+12ctu^3u''-4u^5u''-8ct^2u^2u'u''-4tu^4u'u''+8t^2u^3u'^2u''\\
&\quad +16ct^3u^2u''^2-8t^2u^4u''^2+4ct^2u^3u^{(3)}-2tu^5u^{(3)}-8ct^3u^2u'u^{(3)}\\
&\quad +4t^2u^2u'u^{(3)}]g_{0j}g_{0k}.
\end{aligned} \tag{4.13}$$

Our task is to find a positive function $u(t)$ such that $\operatorname{Diff} XX_{jk} = \operatorname{Diff} YY_{jk} = 0$. It was hopeless to try to find directly a general common solution of the system of the third-order ordinary differential equations obtained by imposing the conditions $\operatorname{Diff} XX_{jk} = \operatorname{Diff} YY_{jk} = 0$. First, it was quite obvious that the function $u =$ constant is a solution of the obtained system. The case where $u = 1$ has been discussed by the author in [13]. The obtained Kähler Einstein structure on $TM$ in the case where $c < 0$ or on a tube around the zero section in $TM$ in the case where $c > 0$ is even locally symmetric. Another case which can be considered is that where $u^2 = 2ct$. Remark that in this case we have also $u - 2tu' = 0$. It is a singular case and it has been studied separately by the author and N. Papaghiuc in [16]. It has been obtained when we tried (inspired by an idea of Calabi) to get a Kähler structure on $TM$ starting with a Lagrangian depending on $t$ only. The main result in [16] is that the bundle $T_0M$ of the nonzero tangent vectors of a Riemannian manifold $(M,g)$ of constant positive curvature has a Kähler structure. This structure is neither Einstein nor locally symmetric.

If we try to find another function $u$ for which $\operatorname{Diff} XX_{jk} = \operatorname{Diff} YY_{jk} = 0$ and which does not depend on the dimension $n$ of $M$, then we have to find the positive solutions of the second-order differential equation

$$u^2u'' - 2tu'^3 + 2uu'^2 = 0, \tag{4.14}$$

obtained from the vanishing of the coefficients of $n$ in $\operatorname{Diff} XX_{jk}, \operatorname{Diff} YY_{jk}$. Remark that if $u$ is a solution of this equation, then $-u$ is a solution too. We have got the general solution of the (4.14) by transforming it in a second-order differential equation fulfilled by the inverse function $t = t(u)$ of the function $u$. We have obtained a second order Euler differential equation with the general solution

$$t = C_1u^2 + C_2u, \tag{4.15}$$

where $C_1$ and $C_2$ are constants. Then the general solution of (4.14) may be written as

$$u = \sqrt{A^2 + Bt} + A, \tag{4.16}$$

where $A$ and $B$ are constants. The solution $u = -\sqrt{A^2 + Bt} + A$ has the property $u(0) = 0$ and should be excluded. However, the corresponding case can be treated as a singular case. We can take $B \neq 0$ since in the contrary case we obtain the solution = constant and this situation is discussed elsewhere. If $B > 0$, then we have the positive solution $u = A + \sqrt{A^2 + Bt}$ for every $A \in \mathbb{R}$. If $B < 0$, then $u$ is real only if $0 \leq t < -A^2/B$ and we must have $A > 0$. If we replace the expression (4.16) of $u$ in $\operatorname{Diff} XX_{jk}$ and $\operatorname{Diff} YY_{jk}$ we obtain quite complicate expressions but it is quite obvious that we have $\operatorname{Diff} XX_{jk} = \operatorname{Diff} YY_{jk} = 0$ if and only if $B = -2c$. Then $u = \sqrt{A^2 - 2ct} + A$ and, by using the formulae (3.2) and (3.8), we obtain

$$v(t) = \frac{1}{2t}\left(A - \frac{4ct}{A} - \sqrt{A^2 - 2ct}\right), \qquad w(t) = \frac{-A^3 + 3Act + (A^2 - 2ct)^{3/2}}{4ct^2(A^2 - 2ct)}. \tag{4.17}$$

Remark that $v(0)$ is defined and we must have $A \neq 0$. From the found expressions of $u$ and $v$, we arrived at the following situations when the symmetric matrix $(G_{jk})$ is positive.

(1) $c < 0,\ A > 0$ and we have

$$u = A + \sqrt{A^2 - 2ct} > 0, \quad v = \frac{1}{2t}\left(A - \frac{4ct}{A} - \sqrt{A^2 - 2ct}\right) > 0, \quad t \geq 0. \tag{4.18}$$

(2) $c > 0,\ A > 0$ and we have

$$u = A + \sqrt{A^2 - 2ct}, \quad v = \frac{1}{2t}\left(A - \frac{4ct}{A} - \sqrt{A^2 - 2ct}\right), \quad 0 \leq t < \frac{A^2}{2c}. \tag{4.19}$$

Hence, we may state our main result.

**Theorem 4.2.** (1) *Assume that $(M,g)$ has constant negative curvature $c$. Then $(TM, G, J)$, with $u, v$ given by (4.18), where $A > 0$, is a Kähler Einstein manifold.*

(2) *Assume that $(M,g)$ has constant positive curvature $c$. Then the tube around the zero section in $TM$, defined by the condition*

$$\|y\|^2 = g_{jk}y^jy^k = 2t < \frac{A^2}{c}, \tag{4.20}$$

*has a structure of Kähler Einstein manifold, if the functions $u, v$ are given by (4.19).*

**Remark 4.3.** Another singular case is obtained when $C_1 = 0$ in the general solution of the Euler equation. In this case we have $u = At,\ A > 0$, hence $u - tu' = 0$. This case can be discussed separately.

**5. The holomorphic sectional curvature of** $(TM, G, J)$**.** In this section, we obtain the components of the curvature tensor field $K$ of $\nabla$ in the case where $u, v$ are given by (4.18), with $c < 0,\ A > 0$. Similar computations may be done in the remaining cases. Replacing $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ in formulae (4.8) with their expressions in (4.5), where $u, u', u'', u^{(3)}, v, v', v'', w, w', w''$ are computed by using (4.18), we obtain the following quite simple expressions of the components (of course, we have used RICCI to do the corresponding tensor calculations)

$$\begin{gathered} XXX^h_{kij} = \frac{c}{2A}(\delta^h_i G_j k - \delta^h_j G_{ik}), \qquad XXY^h_{kij} = \frac{c}{2A}(g_{jk}G^h_i - g_{ik}G^h_j), \\ YYX^h_{kij} = \frac{c}{2A}(g_{jk}H^h_i - g_{ik}H^h_j), \qquad YYY^h_{kij} = \frac{c}{2A}(\delta^h_i H_{jk} - \delta^h_j H_{ik}), \\ YXX^h_{kij} = \frac{c}{2A}(\delta^h_i Gjk + g_{ik}G^h_j + 2g_{ij}G^H_k), \\ YXY^h_{kij} = -\frac{c}{2A}(\delta^h_j H_{ik} + g_{jk}H^h_i + 2g_{ij}H^h_k). \end{gathered} \tag{5.1}$$

Recall that a Kähler manifold $(M, g, J)$ has holomorphic constant sectional curvature $k$ if its curvature tensor field $\mathbb{R}$ is given by

$$R(X,Y)Z = \frac{k}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}, \quad X,Y,Z \in \chi(M). \tag{5.2}$$

Comparing expressions (5.1) of the components of $K$ with those obtained from (5.2), when we take $M \to TM$, $g \to G$ and for the vector fields $X, Y, Z$ involved in (5.2) we take the elements of the local frame $(\delta/\delta x^i, \partial/\partial y^i)$, $i = 1, \dots, n$, we obtain a quite interesting result.

**Theorem 5.1.** *If* $A > 0$, $c < 0$, *and* $(TM, G, J)$ *has the Kähler Einstein structure defined by the expressions (4.18) of* $u$ *and* $v$, *then* $(TM, G, J)$ *is a complex space form with negative constant holomorphic curvature* $2c/A$.

Similar results are obtained in the cases where $u, v$ are given by (4.19).

The components of the Ricci tensor field Ric of $\nabla$ are

$$\operatorname{Ric} XX_{jk} = \frac{(n+1)c}{A}G_{jk}, \qquad \operatorname{Ric} YY_{jk} = \frac{(n+1)c}{A}H_{jk}, \qquad \operatorname{Ric} XY_{jk} = \operatorname{Ric} YX_{jk} = 0, \tag{5.3}$$

thus

$$\operatorname{Ric} = \frac{(n+1)c}{A}G. \tag{5.4}$$

**Acknowledgements.** Some components of the curvature tensor field of the found Riemannian metric have quite complicated expressions. We have used the mathematica package RICCI for doing tensor calculations, elaborated by John M. Lee in order to work with such complicate expressions. Thus, some of these expressions are

not written down in this paper but it is indicated how they can be obtained by using RICCI. The author is grateful to Victor Fecioru from Brisbane, Australia, for helping him with some advice, software and hardware, necessary in the computations made by using RICCI.

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