IJMMS 25:3 (2001) 145–152
PII. S0161171201004902
http://ijmms.hindawi.com

COMMON FIXED POINT THEOREMS FOR COMMUTING k-UNIFORMLY LIPSCHITZIAN MAPPINGS

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(Received 15 January 2000 and in revised form 3 April 2000)

Abstract. We give a common fixed point existence theorem for any sequence of commuting k-uniformly Lipschitzian mappings (eventually, for $k = 1$ for any sequence of commuting nonexpansive mappings) defined on a bounded and complete metric space (X,d) with uniform normal structure. After that we deduce, by using the Kulesza and Lim (1996), that this result can be generalized to any family of commuting k-uniformly Lipschitzian mappings.

2000 Mathematics Subject Classification. Primary 47H10.

1. Introduction. In classical theorems concerning the existence of fixed points for family of mappings, such as the Kakutani theorem [4] and its well-known generalization due to Ryll-Nardzewski [13], the mappings of the family are usually assumed to be linear, or at least to be weakly continuous and affine [11]. In the nonlinear theory, a stronger geometric structure is required. In particular for a family of nonexpansive mappings, Khamsi proved in [7] that any family of nonexpansive mappings defined on a metric space (X,d) with compact and normal convexity structure $\mathcal{F}$, has a common fixed point. In his proof, Khamsi investigated the concept of 1-local retract. In this paper, we prove that any sequential family of k-uniformly Lipschitzian mappings defined on a bounded metric space with a uniform normal convexity structure $\mathcal{F}$ with constant β, which contains all closed ball of (X,d), has a common fixed point provided that $k^2\beta < 1$. Recall that any nonexpansive mapping defined on a bounded complete metric space with uniform normal structure with constant β has a nonempty fixed point set (Khamsi [6]). For more details on fixed point theory for nonexpansive and k-uniformly Lipschitzian mappings in metric spaces we refer the reader to [1, 2, 3].

2. Definitions and preliminaries. In this work, (X,d) will be a metric space. We use $B(x,r)$ to denote the closed ball centered at $x \in X$ with radius $r > 0$. For a subset A of X, we write

$$\begin{aligned} r_x(A) &= \sup_{y\in A} d(x,y), \qquad r(A) = \inf_{x\in A} r_x(A), \\ \delta(A) &= \sup_{x\in A} r_x(A), \qquad \operatorname{cov}(A) = \cap_{B\in\mathcal{F}} B, \end{aligned} \tag{2.1}$$

where $\mathcal{F}$ is the family of closed balls containing A. A subset A of X is said to be admissible if and only if $A = \mathrm{cov}(A)$. In other words, A is admissible if it is an intersection of a family of closed balls centered in X.

Definition 2.1. Let $\mathcal{F}$ be a nonempty family of a subset of X. We say that $\mathcal{F}$ defines a convexity structure on X if and only if it is stable by intersection.

In this work, we always assume that $\mathcal{F}$ contains the balls. Also we denote by $\mathcal{A}(X)$ the smallest convexity structure on X.

Definition 2.2. We say that $\mathcal{F}$ has the property (R) if and only if any decreasing sequence $(X_n)_n$ of nonempty bounded closed subsets of X with $X_n \in \mathcal{F}$ has a nonempty intersection.

Definition 2.3. (i) We say that X has uniform normal structure if and only if $r(A) \le \beta\delta(A)$ for some $0 < \beta < 1$ and for every $A \in \mathcal{F}$.

(ii) We say that $\mathcal{F}$ is normal if and only if $r(A) < \delta(A)$ for every $A \in \mathcal{F}$.

Let us recall that a self mapping $T : X \to X$ is said to be k-uniformly Lipschitzian if there exists a $k > 0$ such that

$$d(T^i x, T^i y) \le k d(x,y) \tag{2.2}$$

for every $i \in \mathbb{N}$ and every x, y in X. A 1-uniformly Lipschitzian map is called nonexpansive. For such class of mappings we recall the following most important result.

Theorem 2.4 (see [6]). *Let (X,d) be a complete bounded metric space. Assume that X has uniform normal structure. Then any nonexpansive mapping defined on X has a fixed point.*

In [7], Khamsi gave the definition and a characterization of a 1-local retract subset of a metric space.

Definition 2.5. A subset A is said to be a k-local retract if for any family $(B_i)_i$ of closed balls centered in A such that $\cap_{i\in I} B(x_i, r_i) \neq \emptyset$, we have $A \cap_{i\in I} B(x_i, k r_i) \neq \emptyset$.

It is immediate that uniform normal structure is not hereditary. However, for 1-local retract subsets we have the following lemma.

Lemma 2.6. *Let (X,d) be a metric space. Suppose that $\mathcal{A}(X)$ is a uniform normal convexity structure with constant $\beta < 1$. If Y is a 1-local retract subset of X, then $\mathcal{A}(Y)$ is a uniform normal structure with the same constant β.*

The proof is based on the next lemma.

Lemma 2.7 (see [7]). *Let (X,d) be a metric space and A a nonempty bounded subset of X. Then*

(1) $\mathrm{cov}(A) = \cap_{x\in X} B(x, r_x(A))$.

(2) $r_x(A) = r_x(\mathrm{cov}A)$ *for every x in X.*

(3) $\delta(A) = \delta(\mathrm{cov}A)$.

(4) $r(\mathrm{cov}A) \le r(A)$.

(5) *If (X,d) has the (n,∞) property and is convex, then* $\delta(A)/2 \le r(\mathrm{cov}A) \le ((n-1)/n)\,\delta(A)$.

Recall that (X,d) is said to have the (n,∞) property if for any family $(B_i)_{i\in I}$ of closed balls of X such that $\cap_{i\in J}B_i \neq \emptyset$ for any finite subfamily J of I with $\operatorname{card}(J)$ less than n, we have $\cap_{i\in I}B_i \neq \emptyset$.

A metric space (X,d) is said to be convex if for all x,y in X and $\alpha \in [0,1]$ there exists a $z \in X$ such that

$$d(z,x) = \alpha d(x,y), \qquad d(z,y) = (1-\alpha)d(x,y). \tag{2.3}$$

Proof of Lemma 2.6. We assume that A is not a singleton. By (4) of Lemma 2.7, we have $r(\operatorname{cov} A) \leq r(A)$. Since $A \in \mathcal{A}(Y)$, then $A = \cap_{i\in I}B(x_i,r_i)\cap Y$ with $x_i \in Y$. Hence $\operatorname{cov} A \subset \cap_{i\in I}B(x_i,r_i)$. Let $z \in \operatorname{cov}(A)$ and define $r = r_z(A)$, then $z \in B = \cap_{x\in A}B(x,r)\cap \cap_{i\in I}B(x_i,r_i)$ is in $\mathcal{A}(X)$. Since Y is a 1-local retract of X then $B\cap Y \neq \emptyset$. Let $w \in B\cap Y$, so $w \in A = \cap_{i\in I}B(x_i,r_i)\cap Y$ and $w \in \cap_{x\in A}B(x,r)$. We deduce that $r_w(A) \leq r$. Hence $r(A) \leq r = r_z(A)$.

Since z is arbitrary in $\operatorname{cov}(A)$ we obtain from (2.1) that $r(A) \leq r(\operatorname{cov}(A))$. But $\operatorname{cov}(A) \in \mathcal{A}(X)$ which is uniform normal, then

$$r(A) \leq r(\operatorname{cov}(A)) \leq \beta\delta(\operatorname{cov}(A)) = \beta\delta(A) \tag{2.4}$$

from property (4) of Lemma 2.7. □

3. Fixed points for k-uniformly Lipschitzian mappings.

In the next theorem, we obtain fixed point theorem for k-uniformly Lipschitzian mapping by utilizing the existence theorem of nonexpansive mapping [7]. To our knowledge this connection has not been utilized. Moreover, Theorem 3.1 contains the result of Theorem 2.4.

Theorem 3.1. *Let (X,d) be a complete bounded metric space. Assume that X has a uniform normal structure with constant $\beta < 1$. Then any k-uniformly Lipschitzian mapping $T : X \to X$ has a fixed point if $k^2\beta < 1$.*

Proof. First we need the following two lemmas.

Lemma 3.2. *Under the same hypothesis as Theorem 3.1, and for $T : X \to X$ k-uniformly Lipschitzian, let*

$$d'(x,y) = \sup_{i=0,1,\dots} d(T^i x, T^i y). \tag{3.1}$$

Then

(1) *(X,d') is a bounded complete metric space.*

(2) *T is d'-nonexpansive, that is,*

$$d'(Tx,Ty) \leq d'(x,y) \quad \forall x,y \in X. \tag{3.2}$$

Lemma 3.3. *Under the same hypothesis as Theorem 3.1, and for $T : X \to X$ k-uniformly Lipschitzian, the family of all admissible subsets of (X,d') is a uniform normal convexity structure with constant c $(c \leq k^2\beta)$.*

Proof of Lemma 3.2. (1-1) d' is a metric on X. Indeed

(1-1-a) For every x, y in X, we have $d'(x,y) = 0$ is equivalent to $d(T^i x, T^i y) = 0$ for every $i = 0, 1, 2, \ldots$.

Specifically for $i = 0$, it implies that $d(x,y) = 0$. Since d is a metric on X, then $x = y$.

(1-1-b) For every $i = 0, 1, 2, \ldots$, and every x, y, z in X, we have

$$d(T^i x, T^i y) \leq d(T^i x, T^i z) + d(T^i z, T^i y), \tag{3.3}$$

since d is a metric on X.

By passing to the supremum on $i \in \mathbb{N}$, we obtain that

$$d'(x,y) \leq d'(x,z) + d'(z,y). \tag{3.4}$$

(1-1-c) It is immediate that $d'(x,y) = d'(y,x)$ for all x, y in X.

(1-2) Since T is k-uniformly Lipschitzian on X, and by definition of d', we have the inequality

$$d(x,y) \leq d'(x,y) \leq k d(x,y) \tag{3.5}$$

for all x, y in X. It follows from this inequality that (X, d') is a bounded complete metric space since (X, d) is.

(2) For every x, y in X, we have

$$\begin{aligned} d'(Tx, Ty) &= \sup\{d(T^{i+1}x, T^{i+1}y) \mid i = 0, 1, 2, \ldots\} \\ &\leq \sup\{d(T^i x, T^i y) \mid i = 0, 1, 2, \ldots\} \\ &= d'(x,y). \end{aligned} \tag{3.6}$$

□

Proof of Lemma 3.3. Let A be an admissible subset for d', then

$$A = \cap_{x \in X} B'(x, r'_x(A)) \subset \operatorname{cov}(A) = \cap_{x \in X} B(x, r_x(A)). \tag{3.7}$$

On the other hand, it follows from the definition of d' that

$$d(z,y) \leq d'(z,y) \leq k d(z,y) \quad \forall z, y \in X. \tag{3.8}$$

Hence

$$r'_z(A) \leq k r_z(A) \quad \forall z \in X. \tag{3.9}$$

By passing in (3.9) to the infimum on $z \in \cap_{x \in X} B'(x, r'_x(A))$, we get

$$\inf_{z \in \cap_{x \in X} B'(x, r'_x(A))} r'_z(A) \leq k \inf_{z \in \cap_{x \in X} B'(x, r'_x(A))} r_z(A), \tag{3.10}$$

which implies that

$$\begin{aligned} r'(A) &= \{\inf r'_z(A) \mid z \in A = \cap_{x \in X} B'(x, r'_x(A))\} \\ &\leq k\{\inf r_z(A) \mid z \in \cap_{x \in X} B'(x, r'_x(A))\} \\ &\leq k \inf\left\{\sup_{x \in A} d(z,x) \mid d(z,x) \leq \frac{r_x(A)}{k}\right\} \\ &\leq k \inf\left\{k \sup_{x \in A} d(z,x) \mid d(z,x) \leq r_x(A)\right\} \end{aligned} \tag{3.11}$$

since

$$\cap_{x\in X} B\left(x, \frac{r_x(A)}{k}\right) \subset \cap_{x\in X} B'(x, r'_x(A)). \tag{3.12}$$

Therefore

$$r'(A) \le k^2 r(\operatorname{cov}(A)) \le k^2 \beta\delta(\operatorname{cov}(A)) = k^2\beta\delta(A) \le k^2\beta\delta'(A). \tag{3.13}$$

□

Proof of Theorem 3.1. It follows immediately from Theorem 2.4, property (2) of Lemma 3.2, and Lemma 3.3. □

By Theorem 3.1, we have $\operatorname{Fix}(T) \neq \emptyset$ for every k-uniformly Lipschitzian mapping T defined on a bounded complete metric space (X,d) with uniform normal convexity structure $\mathcal{F}$ with constant $\beta < 1/k^2$. Moreover, $\operatorname{Fix}(T)$ is a k-local retract of X, that is, for every closed ball $B(x_i, r_i)_{i\in I}$, we have

$$\cap_{i\in I} B(x_i, r_i) \neq \emptyset \quad \text{implies} \quad \cap_{i\in I} B(x_i, kr_i) \cap \operatorname{Fix}(T) \neq \emptyset. \tag{3.14}$$

Now we are able to show the following.

Theorem 3.4. *Let $T_n : X \to X$; $n = 0, 1, 2, \dots$ be a family of commuting k-uniformly Lipschitzian mappings. Suppose that X has a uniform normal convexity structure $\mathcal{F}$ with constant $\beta < 1/k^2$. Then $\cap_{n\in N} \operatorname{Fix}(T_n) \neq \emptyset$ and is a k-local retract of X.*

Proof of Theorem 3.4. The first part of the theorem follows immediately from Theorem 3.1. For the second part, let $(B_i)_{i\in I}$ be a family of closed balls centered in $\cap_{n\in N} \operatorname{Fix}(T_n)$ such that $(B_i)_{i\in I} \neq \emptyset$.

We have

$$B_d(x_i, r_i) \subset B_{d'}(x_i, kr_i) \subset B_d(x_i, kr_i). \tag{3.15}$$

Hence

$$\cap_{i\in I} B_{d'}(x_i, kr_i) \neq \emptyset, \tag{3.16}$$

and since $(T_n)_n$ are nonexpansive mappings on (X, d'), it follows from Theorem 3.1 that

$$\cap_{n\in N} \operatorname{Fix}(T_n) \cap \cap_{i\in I} B_{d'}(x_i, kr_i) \neq \emptyset, \tag{3.17}$$

which implies that

$$\begin{aligned} \emptyset \neq \cap_{n\in N} \operatorname{Fix}(T_n) \cap \cap_{i\in I} B_{d'}(x_i, kr_i) \\ \neq \emptyset \subset \cap_{n\in N} \operatorname{Fix}(T_n) \cap \cap_{i\in I} B_d(x_i, kr_i) \neq \emptyset. \end{aligned} \tag{3.18}$$

□

The problem of whether the conclusion of Theorem 3.4 holds for any commuting family $(T_i)_{i\in I}$ of k-uniformly Lipschitzian mappings ($k > 1$) was open for several years. However, by using the result of Lim and Kulesza [8] in which they show that weak compactness and weak countably compactness are equivalent, if the metric space has normal structure, we prove the following.

Theorem 3.5. *Let (X,d) be a bounded complete metric space with a uniform normal convexity structure $(\beta < 1)$. Then any commuting family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzian mappings has a common fixed point provided that $k^2\beta < 1$.*

Proof. Since (X,d') has uniform normal structure with constant c $(c < k^2\beta)$, then by the well-known theorem of Khamsi [6], $\mathcal{A}(X_{d'})$ is countably compact.

Hence by the Lim and Kulesza result, it follows that $\mathcal{A}(X_{d'})$ is in fact compact. On the other hand, since each T_i, $i \in I$ is d'-nonexpansive (Lemma 3.3), it follows that the result of Theorem 3.4 is a direct consequence of Khamsi's theorem in which he shows that any commuting family of nonexpansive mappings defined on a bounded metric space for which $\mathcal{A}(X_{d'})$ is compact and normal, has a common fixed point. □

We remark that the result of Theorem 3.5 was deduced from Lim and Kulesza theorem and the uniform convexity of (X,d') (Lemma 3.3); but the problem of whether the compactness and normality of (X,d) imply the compactness and normality of (X,d') is still open.

4. Applications.

It was proved by Nachbin [10] and Kelley [5] that all Banach spaces which have the $(2,\infty)$ property are those of form $C(E)$, where E is a compact Stonian, for example l_∞ and L_∞. Then by Theorem 3.5 and property (5) of Lemma 2.7, we have the following.

Corollary 4.1. *The unit balls of l_∞, L_∞, and $C(E)$, where E is a compact Stonian have the common fixed point property for every commuting family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzian mappings provided that $k < \sqrt{2}$.*

Lindenstrauss [9] has proved that l_1 has a $(3,\infty)$ property.

Corollary 4.2. *The unit ball of l_1 has the common fixed point property for every commuting family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzian mappings provided that $k < \sqrt{3/2}$.*

Also, we deduce from Theorem 3.5 and property (5) of Lemma 2.7, the following corollary.

Corollary 4.3. *If (X,d) is a Banach space with the (n,∞) property, and if $k < \sqrt{n/(n-1)}$, then its unit ball has the common fixed point property for every commuting family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzian mappings.*

More recently, Prus [12] has proved that all Banach spaces L_p $(1 < p < +\infty)$ have uniform normal structure with constant $\beta = (\min(2^{1/p}, 2^{1/q}))^{-1}$, where $q = p(p-1)^{-1}$ is the conjugate of p.

Hence, we have the following.

Corollary 4.4. *The unit balls of L_p have the common fixed point property for every commuting family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzian mappings provided that $k < \sqrt{\min(2^{1/p}, 2^{1/q})}$.*

Now we recall the definition of the most geometrical characterization of l_∞, L_∞, and $C(E)$, where E is a compact Stonian.

Definition 4.5. A metric space (X,d) is said to be hyperconvex if and only if any family $\{B(x_i,r_i),\ i\in I\}$ of closed balls of (X,d) such that

$$d(x_i,x_j)\le r_i+r_j \tag{4.1}$$

for every $i,j\in I$, has a nonempty intersection.

Remarks. (1) Every hyperconvex metric space is complete, and if A is an admissible subset of (X,d), then also (A,d) is a hyperconvex metric space (see [2]).

(2) Every hyperconvex space is convex. Indeed:

For all x,y in X and for any $\alpha\in[0,1]$, let u,v in X. We have

$$\alpha[d(x,u)+d(x,v)]+(1-\alpha)[d(y,u)+d(y,v)]\ge d(u,v). \tag{4.2}$$

The hyperconvexity of (X,d) implies that

$$\cap_{u\in X}B(u,\alpha d(x,u)+(1-\alpha)d(y,u))\neq\emptyset. \tag{4.3}$$

Hence, for every x,y in X and for every $\alpha\in[0,1]$, there exists a $z\in X$ such that

$$z\in\cap_{u\in X}B(u,\alpha d(x,u)+(1-\alpha)d(y,u)); \tag{4.4}$$

that is,

$$d(u,z)\le\alpha d(x,u)+(1-\alpha)d(y,u)\quad\forall u\in X. \tag{4.5}$$

Therefore

$$d(x,z)=(1-\alpha)d(x,y),\qquad d(y,z)=\alpha d(x,y). \tag{4.6}$$

Also by Theorem 3.5 and property (5) of Lemma 2.7, we obtain the following theorem.

Theorem 4.6. *Let (X,d) be a bounded hyperconvex metric space. Then any family of commuting k-uniformly Lipschitzian mappings defined on X has a common fixed point if $k<\sqrt{2}$.*

Proof. (X,d) is a bounded hyperconvex metric space. Then from the above remarks, it is complete. Let us prove that (X,d) has the $(2,\infty)$ property. Indeed:

Let $\{B(x_i,r_i),i\in I\}$ be a family of closed balls of (X,d), such that

$$B(x_i,r_i)\cap B(x_j,r_j)\neq\emptyset\quad\forall i,j\in I\ (i\neq j). \tag{4.7}$$

Then we have

$$d(x_i,x_j)\le d(x_i,x)+d(x_j,x)\le r_i+r_j, \tag{4.8}$$

where $x\in B(x_i,r_i)\cap B(x_j,r_j)$.

The hyperconvexity of (X,d) implies that $\cap_{i\in I}B(x_i,r_i)\neq\emptyset$. Then (X,d) is a convex metric space with the $(2,\infty)$ property. Therefore, by property (5) of Lemma 2.7, $\mathcal{A}(X)$ is a uniform convexity structure with constant $\beta=1/2$. Hence, Theorem 3.5 completes the proof. □

References

[1] K. Goebel and W. A. Kirk, *A fixed point theorem for transformations whose iterates have uniform Lipschitz constant*, Studia Math. **47** (1973), 135–140. MR 49#1242. Zbl 265.47044.

[2] ______, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990. MR 92c:47070. Zbl 708.47031.

[3] K. Goebel, W. A. Kirk, and R. L. Thele, *Uniformly Lipschitzian families of transformations in Banach spaces*, Canad. J. Math. **26** (1974), 1245–1256. MR 50#10919. Zbl 285.47039.

[4] S. Kakutani, *Two fixed point theorems concerning bicompact convex sets*, Proc. Imp. Acad. Tokyo **14** (1938), 242–245.

[5] J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952), 323–326. MR 13,659e. Zbl 046.12002.

[6] M. A. Khamsi, *On metric spaces with uniform normal structure*, Proc. Amer. Math. Soc. **106** (1989), no. 3, 723–726. MR 90c:54028. Zbl 671.47052.

[7] ______, *One-local retract and common fixed point for commuting mappings in metric spaces*, Nonlinear Anal. **27** (1996), no. 11, 1307–1313. MR 97f:54051. Zbl 882.54039.

[8] J. Kulesza and T. C. Lim, *On weak compactness and countable weak compactness in fixed point theory*, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3345–3349. MR 97a:54048. Zbl 865.47044.

[9] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. (1964), no. 48, 112 p. MR 31#3828. Zbl 141.12001.

[10] L. Nachbin, *A theorem of the Hahn-Banach type for linear transformations*, Trans. Amer. Math. Soc. **68** (1950), 28–46. MR 11,369a. Zbl 035.35402.

[11] I. Namioka and E. Asplund, *A geometric proof of Ryll-Nardzewski's fixed point theorem*, Bull. Amer. Math. Soc. **73** (1967), 443–445. MR 35#799. Zbl 177.40404.

[12] S. Prus, *On Bynum's fixed point theorem*, Atti Sem. Mat. Fis. Univ. Modena **38** (1990), no. 2, 535–545. MR 91k:47146. Zbl 724.46020.

[13] C. Ryll-Nardzewski, *On fixed points of semigroups of endomorphisms of linear spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. II: Contributions to Probability Theory, Part 1, pp. 55–61, Univ. California Press, Berkeley, Calif., 1967. MR 35#5977. Zbl 189.17501.

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