H-FUNCTION WITH COMPLEX PARAMETERS II: EVALUATION

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ABSTRACT. Sufficient conditions for computation of the H-functions with complex parameters by means of residues are derived and some examples are given.

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1. Introduction. Let

$$k(s) = \frac{\prod_{j=1}^{P} \Gamma(p_j + \alpha_j s)}{\prod_{j=1}^{Q} \Gamma(q_j + \beta_j s)}. \tag{1.1}$$

Here P and Q are nonnegative integers, p_j $(1 \le j \le P)$, q_j $(1 \le j \le Q)$ are complex numbers, and α_j $(1 \le j \le P)$, β_j $(1 \le j \le Q)$ are nonzero complex numbers. Throughout the present paper an empty product is interpreted as unity and an empty sum as zero.

Let c be a real number such that $c \ne \mathrm{Re}(-p_j/\alpha_j)$, for all j with $\mathrm{Re}\,\alpha_j = 0$, and Λ_c denote a contour from $c - i\infty$ to $c + i\infty$ not passing through any pole of $k(s)$ and such that

(1) if $\mathrm{Re}\,\alpha_j > 0$, then all the points

$$s = -\frac{m}{\alpha_j} - \frac{p_j}{\alpha_j}, \quad m = 0, 1, 2, \ldots, \tag{1.2}$$

are to the left of Λ_c.

(2) If $\mathrm{Re}\,\alpha_j < 0$, then all the points (1.2) are to the right of Λ_c.

(3) If $\mathrm{Re}\,\alpha_j = 0$, and $\mathrm{Re}(-p_j/\alpha_j) < c$, then all the points (1.2) are to the left of Λ_c.

(4) If $\mathrm{Re}\,\alpha_j = 0$, and $\mathrm{Re}(-p_j/\alpha_j) > c$, then all the points (1.2) are to the right of Λ_c.

The authors in [1] introduced the notion of H-function with complex parameters as a Mellin-Barnes type integral [2] as follows,

$$H_Q^P\left(z, c \,\middle|\, \begin{matrix} (p, \alpha)_P \\ (q, \beta)_Q \end{matrix}\right) = \frac{1}{2\pi i} \int_{\Lambda_c} k(s) z^{-s} ds. \tag{1.3}$$

Here, and elsewhere in this paper, $(p, \alpha)_P = (p_1, \alpha_1), (p_2, \alpha_2), \ldots, (p_P, \alpha_P)$, and similarly for $(q, \beta)_Q$.

When $(\alpha)_P$ and $(\beta)_Q$ are integer vectors, the H-function with complex parameters reduces to the Meijer G-function (see [4, 11, 12, 13, 14, 15, 16]), and when $(\alpha)_P$ and $(\beta)_Q$ are real vectors, it reduces to the Fox H-function [5, 19]. The G- and H-functions play an important role in statistics and physical sciences (see [7, 8, 9, 10, 19]).

In [1], the authors derived necessary and sufficient conditions, under which the integral (1.3) defining the H-function converges absolutely. The approach in [1] was based on the following asymptotic expansion of $|k(s)|$.

LEMMA 1.1. *For* $\operatorname{Re} s \geq 0$, *and as* $|s| \to \infty$,

$$
\begin{aligned}
|k(s)| \simeq \ &Y(s)|s|^{\Delta_0} e^{T(\ln|s|)\operatorname{Re} s} e^{-\mu(\ln|s|)\operatorname{Im} s} e^{(\operatorname{Im} s)(\mu - M - T\operatorname{Arg} s)} \\
&\times e^{(\operatorname{Re} s)[X - L - T - \mu \operatorname{Arg} s + (\pi/2)\operatorname{sign}(\operatorname{Im} s)(\sum_{\operatorname{Re}\alpha_j=0}\operatorname{Im}\alpha_j - \sum_{\operatorname{Re}\beta_j=0}\operatorname{Im}\beta_j)]} \\
&\times \frac{\prod_{j=1}^{Q} |(1/2)(1+S(\beta_j,s)) + (1-S(\beta_j,s))\sin\pi(q_j+\beta_j s)|}{\prod_{j=1}^{P} |(1/2)(1+S(\alpha_j,s)) + (1-S(\alpha_j,s))\sin\pi(p_j+\alpha_j s)|},
\end{aligned}
\tag{1.4}
$$

where

$$
\Delta_0 = -\frac{1}{2}(P-Q) + \sum_{j=1}^{P}\operatorname{Re} p_j - \sum_{j=1}^{Q}\operatorname{Re} q_j,
$$

$$
\mu = \sum_{j=1}^{P}\operatorname{Im}\alpha_j - \sum_{j=1}^{Q}\operatorname{Im}\beta_j, \qquad T = \sum_{j=1}^{P}\operatorname{Re}\alpha_j - \sum_{j=1}^{Q}\operatorname{Re}\beta_j,
$$

$$
\begin{aligned}
M = \ &\sum_{j=1}^{P}\left[\operatorname{Arg}(\alpha_j \operatorname{sign}(\operatorname{Re}\alpha_j))\operatorname{Re}\alpha_j + (\ln|\alpha_j|)\operatorname{Im}\alpha_j\right] \\
&- \sum_{j=1}^{Q}\left[\operatorname{Arg}(\beta_j \operatorname{sign}(\operatorname{Re}\beta_j))\operatorname{Re}\beta_j + (\ln|\beta_j|)\operatorname{Im}\beta_j\right],
\end{aligned}
$$

$$
L = \sum_{j=1}^{P}\left[(\operatorname{Arg}\alpha_j)\operatorname{Im}\alpha_j - (\ln|\alpha_j|)\operatorname{Re}\alpha_j\right] - \sum_{j=1}^{Q}\left[(\operatorname{Arg}\beta_j)\operatorname{Im}\beta_j - (\ln|\beta_j|)\operatorname{Re}\beta_j\right],
\tag{1.5}
$$

$$
X = \pi\left(\sum_{\operatorname{Re}\alpha_j<0}|\operatorname{Im}\alpha_j| - \sum_{\operatorname{Re}\beta_j<0}|\operatorname{Im}\beta_j|\right) + \frac{\pi}{2}\left(\sum_{\operatorname{Re}\alpha_j=0}|\operatorname{Im}\alpha_j| - \sum_{\operatorname{Re}\beta_j=0}|\operatorname{Im}\beta_j|\right),
$$

$$
S(\alpha,s) = \begin{cases} -(\operatorname{sign}\operatorname{Im}\alpha)\operatorname{sign}\operatorname{Im} s, & \operatorname{Re}\alpha = 0, \\ \operatorname{sign}\operatorname{Re}\alpha, & \operatorname{Re}\alpha \neq 0, \end{cases}
$$

$$
Y(s) = (2\pi)^{(P-Q)/2}\frac{\prod_{j=1}^{P}|\alpha_j|^{-(1/2)+\operatorname{Re} p_j}}{\prod_{j=1}^{Q}|\beta_j|^{-(1/2)+\operatorname{Re} q_j}}e^{-m(s)},
$$

where

$$
\begin{aligned}
m(s) = \ &-\frac{\pi}{2}(\operatorname{sign}\operatorname{Im} s)\left(\sum_{\operatorname{Re}\alpha_j=0}\operatorname{Im} p_j - \sum_{\operatorname{Re}\beta_j=0}\operatorname{Im} q_j\right) + \left(\sum_{j=1}^{P}\operatorname{Im} p_j - \sum_{j=1}^{Q}\operatorname{Im} q_j\right)\operatorname{Arg} s \\
&+ \sum_{\operatorname{Re}\alpha_j\neq0}\operatorname{Arg}(\alpha_j \operatorname{sign}\operatorname{Re}\alpha_j)\operatorname{Im} p_j - \sum_{\operatorname{Re}\beta_j\neq0}\operatorname{Arg}(\beta_j \operatorname{sign}\operatorname{Re}\beta_j)\operatorname{Im} q_j.
\end{aligned}
\tag{1.6}
$$

Here we write $f(z) \simeq g(z)$, as $|z| \to \infty$, $\alpha \leq \arg z \leq \beta$, if $f(z) = g(z)[1 + o(1)]$, as $|z| \to \infty$, within the sector $\alpha \leq \arg z \leq \beta$.

The following theorem gives the conditions, derived in [1], under which the integral (1.3) defining the *H*-function converges absolutely.

THEOREM 1.2. *If*

$$N = \sum_{j=1}^{P} |\operatorname{Re} \alpha_j| - \sum_{j=1}^{Q} |\operatorname{Re} \beta_j|, \tag{1.7}$$

and M, T, and μ are as in Lemma 1.1, then the integral (1.3) converges absolutely if and only if,

(1) *$\mu = 0$, $N > 0$, and for $|\arg z - M| < (\pi/2)N$, or*

(2) *$\mu = 0$, $N \geq 0$, $\arg z = M \pm (\pi/2)N$, and $\Delta_c = \Delta_0 + cT < -1$.*

These conditions, naturally, reduce to those of the classical case where the α_j's and β_j's are real. Also in [1] some well-known integrals that are not special cases of the classical Fox *H*-function were expressed as *H*-function with complex parameters.

The present paper is a continuation of the work in [1]. In this paper, we determine sufficient conditions that enable one to compute the *H*-function with complex parameters (1.3) as a sum of residues at the right or the left poles of $k(s)z^{-s}$. It should be mentioned that Braaksma [3] was the first to apply the residue theorem to evaluate the integral (1.3) in the special case when $\alpha_j = \pm 1$ and $\beta_j = \pm 1$. See also [17, 18], where the residue theorem was used to compute the integral (1.3), with real α_j's and β_j's, for two other contours. It is striking that under some conditions, the *H*-function with complex parameters can be computed, in certain annulus of the complex plane, by the residues at the right poles as well as the residue at the left poles. This phenomenon has no parallel when the vectors $(\alpha_j)_P$ and $(\beta_j)_Q$ are real vectors. We apply our result to compute the *H*-function (1.3) in some particular cases.

2. Evaluation by means of residues.
We need the following remark.

REMARK 2.1. If w is a complex number, that is, not a zero of $\sin z$, and z_0 is the zero of $\sin z$ closest to w, then

$$|\sin w| \geq \min\left\{\frac{1}{\pi}|w - z_0|, \frac{1}{4}\right\} e^{|\operatorname{Im} w|}. \tag{2.1}$$

To see this, observe that $|\sin x| \geq (2/\pi)|x|$, for all $-\pi/2 \leq x \leq \pi/2$. Since z_0 is the zero of $\sin z$ closest to w, then necessarily, $|\operatorname{Re}(w - z_0)| \leq \pi/2$, and moreover,

$$|\sin w|^2 = |\sin(w - z_0)|^2 = \sin^2 \operatorname{Re}(w - z_0) + \sinh^2 \operatorname{Im}(w - z_0)$$
$$\geq \left(\frac{2}{\pi}\right)^2 \operatorname{Re}^2(w - z_0) + \operatorname{Im}^2(w - z_0) \geq \frac{4}{\pi^2}|w - z_0|^2. \tag{2.2}$$

If $|\operatorname{Im} w| > \ln 2$,

$$|\sin w| \geq |\sinh \operatorname{Im} w| \geq \frac{1}{2}(e^{|\operatorname{Im} w|} - 1) = \frac{1}{2}e^{|\operatorname{Im} w|}(1 - e^{-|\operatorname{Im} w|}) > \frac{1}{4}e^{|\operatorname{Im} w|}. \tag{2.3}$$

If $|\operatorname{Im} w| \leq \ln 2$, then (2.2) implies

$$\frac{1}{\pi}|w - z_0|e^{|\operatorname{Im} w|} \leq \frac{2}{\pi}|w - z_0| \leq |\sin w|. \tag{2.4}$$

Combining (2.3) and (2.4) yields (2.1).

Let Ω_c denote a contour from $c - i\infty$ to $c + i\infty$ not passing through any pole of the function $k(s)$, defined by (1.1). Denote by PO, $(PO)_r$, and $(PO)_l$ the set of poles of $k(s)$, the set of poles of $k(s)$ that are to the right of Ω_c, and the set of poles of $k(s)$ that are to the left of Ω_c, respectively.

THEOREM 2.2. *Let $\int_{\Omega_c} k(s)z^{-s}ds$ converge absolutely, that is, the conditions of Theorem 1.2 are satisfied.*
(a) *If $T \leq 0$, then*

$$\int_{\Omega_c} k(s)z^{-s}ds = -2\pi i \sum_{s_j \in (PO)_r} z^{-s_j} \operatorname{Res}\left(k(s), s_j\right); \tag{2.5}$$

and in case $T = 0$, z must also satisfy $|z| > \exp(-L + 2\pi \sum_{\operatorname{Re}\alpha_j < 0} |\operatorname{Im}\alpha_j|)$.
(b) *If $T \geq 0$, then*

$$\int_{\Omega_c} k(s)z^{-s}ds = 2\pi i \sum_{s_j \in (PO)_l} z^{-s_j} \operatorname{Res}\left(k(s), s_j\right); \tag{2.6}$$

and in case $T = 0$, z must also satisfy $|z| < \exp(-L + \pi(\sum_{j=1}^{P} |\operatorname{Im}\alpha_j| - \sum_{j=1}^{Q} |\operatorname{Im}\beta_j|) - 2\pi \sum_{\operatorname{Re}\alpha_j > 0} |\operatorname{Im}\alpha_j|)$.

PROOF. Making the change of variable $s \to s - c$, one sees that

$$\int_{\Omega_c} k(s)z^{-s}ds = z^{-c} \int_{\Omega_0} k_c(s)z^{-s}ds, \qquad k_c(s) = \frac{\prod_{j=1}^{P}\Gamma(p_j + c\alpha_j + \alpha_j s)}{\prod_{j=1}^{Q}\Gamma(q_j + c\beta_j + \beta_j s)}, \tag{2.7}$$

where the contour of integration Ω_0 is the horizontal shifting of the contour Ω_c by c units. The poles of $k_c(s)$ are precisely the horizontal shifting of the poles of $k(s)$ by c units, and have the same position with respect to Ω_0 as the corresponding poles of $k(s)$ with respect to Ω_c; and moreover, if s_0 is a pole of $k(s)$, then $\operatorname{Res}_{s=s_0} k(s) = \operatorname{Res}_{s=s_0-c} k_c(s)$. Thus, without a loss of generality we may assume that $c = 0$.

We construct a family of circles with centers at the origin and whose radii R_r approaching infinity. Moreover, we insure that the family stays a positive distance away from the set PO. The poles of $k(s)$ are lying on finitely many half-lines, and equidistant on each of these half-lines. Let $l_1, l_2, \ldots, l_J$ be those half-lines. Note that $J \leq P$. Let z_i and y_i be the initial point and the distance between consecutive poles on l_i, respectively, $i = 1, 2, \ldots, J$. For each positive integer r, the annulus $r - 1 \leq |s| \leq r$, contains finitely many poles of $k(s)$ that are on l_i, say $M_i(r)$, $i = 1, 2, \ldots, J$. If $Z_1^i(r)$ and $Z_2^i(r)$ are the points of intersections of l_i with the circles $|s| = r - 1$ and $|s| = r$, respectively, then

$$(M_i(r) - 1)y_i \leq |Z_1^i(r) - Z_2^i(r)|. \tag{2.8}$$

Now

$$Z_1^i(r) = z_i + r_1 e^{i\theta_i}, \qquad Z_2^i(r) = z_i + r_2 e^{i\theta_i}, \tag{2.9}$$

for some r_1 and r_2, and since

$$r - 1 - |z_i| = |Z_1^i(r)| - |z_i| \leq r_1, \qquad r_2 \leq |Z_2^i(r)| + |z_i| = r + |z_i|, \tag{2.10}$$

it follows that

$$|Z_1^i(r) - Z_2^i(r)| = r_2 - r_1 \le r + |z_i| - (r - 1 - |z_i|) = 1 + 2|z_i|. \tag{2.11}$$

Therefore,

$$\sum_{i=1}^{J} M_i(r) \le \sum_{i=1}^{J} \left(1 + \frac{1}{y_i}|Z_1^i(r) - Z_2^i(r)|\right) \le J + \sum_{i=1}^{J} \frac{1}{y_i}(1 + 2|z_i|) = v, \tag{2.12}$$

which means that the total number of poles in the annulus $r - 1 \le |s| \le r$ is bounded by the constant v which is independent of r.

Let $s_1, s_2, \dots, s_{J_r}$ be the poles of $k(s)$ on or within the annulus $r - 1 \le |s| \le r$. Then $J_r \le v$. Assume further that $r - 1 = s_0 \le |s_1| \le |s_2| \le \cdots \le |s_{J_r}| \le s_{J_r+1} = r$, and $s_{i_0} \in \{s_0, s_1, \dots, s_{J_r}\}$ such that

$$|s_{i_0+1}| - |s_{i_0}| = \max_{0 \le i \le J_r}\{|s_{i+1}| - |s_i|\}. \tag{2.13}$$

Clearly,

$$|s_{i_0+1}| - |s_{i_0}| \ge \frac{1}{v+1}. \tag{2.14}$$

Let $R_r = (1/2)(|s_{i_0+1}| + |s_{i_0}|)$. If s is such that $|s| = R_r$, then

$$|w - s| \ge |R_r - |w|| \ge \frac{1}{2}(|s_{i_0+1}| - |s_{i_0}|) \ge \frac{1}{2(v+1)} \quad \forall w \in \text{PO}, \tag{2.15}$$

and hence, the promised family of circles may be taken as all circles centered at the origin and whose radii are R_r, $r \ge 1$.

Let

$$I_{C_r} = \frac{1}{2\pi i}\int_{C_r} k(s)z^{-s}ds. \tag{2.16}$$

The integral is taken in a clockwise direction around the contour C_r, consisting of the contours C_r^1 and C_r^2, where C_r^1 is a large circular arc, with center at the origin and radius R_r, lying to the right of Ω_0 and originating from and ending on Ω_0; and $C_r^2 = \{s \in \Omega_0 : |s| \le R_r\}$. It is clear from the definition of R_r that C_r does not pass through any pole of $k(s)$. Now we split I_{C_r} up into a sum of two integrals

$$I_{C_r} = I_{C_r^1} + I_{C_r^2}. \tag{2.17}$$

The integral I_{C_r} is equal to the negative of the sum of all the residues of $k(s)z^{-s}$ at its poles within the contour C_r (due to the negative orientation of the contour C_r), that will cover $(\text{PO})_r$ as $r \to \infty$, and since

$$\frac{1}{2\pi i}\int_{\Omega_0} k(s)z^{-s}ds = \lim_{r \to \infty} I_{C_r^2}, \tag{2.18}$$

Theorem 2.2(a) would be proved if we show that $\lim_{r \to \infty} I_{C_r^1} = 0$. Since

$$|I_{C_r^1}| \le \frac{1}{2}\max_{s \in C_r^1}|sk(s)z^{-s}| \quad \text{(for large } r\text{)}, \qquad d(\text{PO}, C_r^1) \ge \frac{1}{2(1+v)}, \tag{2.19}$$

it follows that $\lim_{r\to\infty} I_{C_r^1} = 0$, if we can show that $\lim_{|s|\to\infty} |sk(s)z^{-s}| = 0$, for $\mathrm{Re}(s) \geq 0$, $d(s,\mathrm{PO}) \geq 1/(2(1+\nu))$. Here $d(\mathbb{A}, \mathbb{B})$ denotes the distance between the two sets $\mathbb{A}$ and $\mathbb{B}$. For this purpose we use the asymptotic expansion of $|k(s)|$ given in Lemma 1.1.

We deduce that

$$
\begin{aligned}
f(s) &:= \frac{\prod_{j=1}^{Q} |(1/2)(1+S(\beta_j,s)) + (1-S(\beta_j,s))\sin\pi(q_j+\beta_j s)|}{\prod_{j=1}^{P} |(1/2)(1+S(\alpha_j,s)) + (1-S(\alpha_j,s))\sin\pi(p_j+\alpha_j s)|} \\
&= \frac{\prod_{\mathrm{Re}\,\beta_j<0} |2\sin\pi(q_j+\beta_j s)|}{\prod_{\mathrm{Re}\,\alpha_j<0} |2\sin\pi(p_j+\alpha_j s)|} \\
&\quad \times \left[\frac{1}{2}(1+\mathrm{sign}\,\mathrm{Im}\,s) \frac{\prod_{\mathrm{Re}\,\beta_j=0,\mathrm{Im}\,\beta_j>0} |2\sin\pi(q_j+\beta_j s)|}{\prod_{\mathrm{Re}\,\alpha_j=0,\mathrm{Im}\,\alpha_j>0} |2\sin\pi(p_j+\alpha_j s)|} \right. \\
&\quad \left. + \frac{1}{2}(1-\mathrm{sign}\,\mathrm{Im}\,s) \frac{\prod_{\mathrm{Re}\,\beta_j=0,\mathrm{Im}\,\beta_j<0} |2\sin\pi(q_j+\beta_j s)|}{\prod_{\mathrm{Re}\,\alpha_j=0,\mathrm{Im}\,\alpha_j<0} |2\sin\pi(p_j+\alpha_j s)|} \right].
\end{aligned}
\tag{2.20}
$$

Our first objective is to obtain an upper estimate of $f(s)$ for all s such that $\mathrm{Re}\,s \geq 0$, and $d(s,\mathrm{PO}) \geq 1/(2(1+\nu))$. Then $\sin\pi(p_j+\alpha_j s) \neq 0$, and according to (2.1)

$$
\begin{aligned}
e^{-\pi|\mathrm{Im}(p_j+\alpha_j s)|} |\sin\pi(p_j+\alpha_j s)| &\geq \min\left\{ |\alpha_j| d(s,\mathrm{PO}), \frac{1}{4} \right\} \\
&\geq \min\left\{ \frac{\min_{j=1,2,\dots,P}\{|\alpha_j|\}}{2(1+\nu)}, \frac{1}{4} \right\}.
\end{aligned}
\tag{2.21}
$$

Since

$$
\begin{aligned}
&|\mathrm{Im}(\beta_j s)| - |\mathrm{Im}(\alpha_j s)| \\
&\leq (\mathrm{Re}\,s)|\mathrm{Im}\,\beta_j| + |\mathrm{Im}\,s||\mathrm{Re}\,\beta_j| - (|\mathrm{Im}\,s||\mathrm{Re}\,\alpha_j| - (\mathrm{Re}\,s)|\mathrm{Im}\,\alpha_j|) \\
&= (|\mathrm{Im}\,\beta_j| + |\mathrm{Im}\,\alpha_j|)\mathrm{Re}\,s + (|\mathrm{Re}\,\beta_j| - |\mathrm{Re}\,\alpha_j|)|\mathrm{Im}\,s|,
\end{aligned}
\tag{2.22}
$$

we see that, through the use of (2.21) and the fact that $|\sin z| \leq e^{|\mathrm{Im}\,z|}$, for all complex z,

$$
\begin{aligned}
\frac{\prod_{\mathrm{Re}\,\beta_j<0} |2\sin\pi(q_j+\beta_j s)|}{\prod_{\mathrm{Re}\,\alpha_j<0} |2\sin\pi(p_j+\alpha_j s)|} &\leq C \exp\left(\pi(\mathrm{Re}\,s)\left[\sum_{\mathrm{Re}\,\beta_j<0} |\mathrm{Im}\,\beta_j| + \sum_{\mathrm{Re}\,\alpha_j<0} |\mathrm{Im}\,\alpha_j| \right] \right. \\
&\quad \left. + \pi|\mathrm{Im}\,s|\left[\sum_{\mathrm{Re}\,\beta_j<0} |\mathrm{Re}\,\beta_j| - \sum_{\mathrm{Re}\,\alpha_j<0} |\mathrm{Re}\,\alpha_j| \right] \right).
\end{aligned}
\tag{2.23}
$$

Here and throughout the proof, C denotes a universal positive constant, that may be distinct in different instances. Since $\mathrm{Im}(q_j+\beta_j s) = \mathrm{Im}\,q_j + (\mathrm{Im}\,\beta_j)\mathrm{Re}\,s$, if $\mathrm{Re}\,\beta_j = 0$, we see that

$$
\begin{aligned}
&\frac{\prod_{\mathrm{Re}\,\beta_j=0,\mathrm{Im}\,\beta_j>0} |2\sin\pi(q_j+\beta_j s)|}{\prod_{\mathrm{Re}\,\alpha_j=0,\mathrm{Im}\,\alpha_j>0} |2\sin\pi(p_j+\alpha_j s)|} \\
&\leq C\left(\exp\pi(\mathrm{Re}\,s)\left[\sum_{\mathrm{Re}\,\beta_j=0,\mathrm{Im}\,\beta_j>0} |\mathrm{Im}\,\beta_j| - \sum_{\mathrm{Re}\,\alpha_j=0,\mathrm{Im}\,\alpha_j>0} |\mathrm{Im}\,\alpha_j| \right] \right),
\end{aligned}
\tag{2.24}
$$

$$\frac{\prod_{\operatorname{Re}\beta_j=0,\,\operatorname{Im}\beta_j<0}|2\sin\pi(q_j+\beta_j s)|}{\prod_{\operatorname{Re}\alpha_j=0,\,\operatorname{Im}\alpha_j<0}|2\sin\pi(p_j+\alpha_j s)|}$$

$$\leq C\exp\left(\pi(\operatorname{Re}s)\left[\sum_{\operatorname{Re}\beta_j=0,\,\operatorname{Im}\beta_j<0}|\operatorname{Im}\beta_j|-\sum_{\operatorname{Re}\alpha_j=0,\,\operatorname{Im}\alpha_j<0}|\operatorname{Im}\alpha_j|\right]\right). \qquad (2.25)$$

Combining (2.24) and (2.25) gives

$$\begin{aligned}
&\frac{1}{2}(1+\operatorname{sign}\operatorname{Im}s)\frac{\prod_{\operatorname{Re}\beta_j=0,\,\operatorname{Im}\beta_j>0}|2\sin\pi(q_j+\beta_j s)|}{\prod_{\operatorname{Re}\alpha_j=0,\,\operatorname{Im}\alpha_j>0}|2\sin\pi(p_j+\alpha_j s)|}\\
&\qquad+\frac{1}{2}(1-\operatorname{sign}\operatorname{Im}s)\frac{\prod_{\operatorname{Re}\beta_j=0,\,\operatorname{Im}\beta_j<0}|2\sin\pi(q_j+\beta_j s)|}{\prod_{\operatorname{Re}\alpha_j=0,\,\operatorname{Im}\alpha_j<0}|2\sin\pi(p_j+\alpha_j s)|}\\
&\leq C\exp\left((\operatorname{Re}s)\left[-\frac{\pi}{2}\left(\sum_{\operatorname{Re}\alpha_j=0}|\operatorname{Im}\alpha_j|-\sum_{\operatorname{Re}\beta_j=0}|\operatorname{Im}\beta_j|\right)\right.\right.\\
&\qquad\qquad\left.\left.-\frac{\pi}{2}(\operatorname{sign}\operatorname{Im}s)\left(\sum_{\operatorname{Re}\alpha_j=0}\operatorname{Im}\alpha_j-\sum_{\operatorname{Re}\beta_j=0}\operatorname{Im}\beta_j\right)\right]\right).
\end{aligned} \qquad (2.26)$$

Using this estimate in the definition of $f(s)$, and observing that

$$\begin{aligned}
&-X+2\pi\sum_{\operatorname{Re}\alpha_j<0}|\operatorname{Im}\alpha_j|\\
&\quad=\pi\left(\sum_{\operatorname{Re}\alpha_j<0}|\operatorname{Im}\alpha_j|+\sum_{\operatorname{Re}\beta_j<0}|\operatorname{Im}\beta_j|\right)-\frac{\pi}{2}\left(\sum_{\operatorname{Re}\alpha_j=0}|\operatorname{Im}\alpha_j|-\sum_{\operatorname{Re}\beta_j=0}|\operatorname{Im}\beta_j|\right),\\
&N=T-2\sum_{\operatorname{Re}\beta_j<0}|\operatorname{Re}\beta_j|-\sum_{\operatorname{Re}\alpha_j<0}|\operatorname{Re}\alpha_j|,
\end{aligned} \qquad (2.27)$$

we obtain

$$\begin{aligned}
f(s)\leq C\exp\left(\frac{\pi}{2}(T-N)|\operatorname{Im}s|+(\operatorname{Re}s)\left[-\frac{\pi}{2}(\operatorname{sign}\operatorname{Im}s)\left(\sum_{\operatorname{Re}\alpha_j=0}\operatorname{Im}\alpha_j-\sum_{\operatorname{Re}\beta_j=0}\operatorname{Im}\beta_j\right)\right.\right.\\
\left.\left.-X+2\pi\sum_{\operatorname{Re}\alpha_j<0}|\operatorname{Im}\alpha_j|\right]\right).
\end{aligned} \qquad (2.28)$$

Since if $z\neq 0$, $|z^{-s}|=e^{-(\operatorname{Re}s)\ln|z|+(\operatorname{Im}s)\arg z}$, and $|Y(s)|\leq C$, then (2.28) and Lemma 1.1 yield

$$\begin{aligned}
|sk(s)z^{-s}|\leq C|s|^{\Delta_0+1}\times\exp\left((\operatorname{Im}s)(\arg z-M-T\operatorname{Arg}s)+\frac{\pi}{2}(T-N)|\operatorname{Im}s|\right)\\
\times\exp\left((\operatorname{Re}s)\left[-L-T+T\ln|s|-\ln|z|+2\pi\sum_{\operatorname{Re}\alpha_j<0}|\operatorname{Im}\alpha_j|\right]\right),
\end{aligned} \qquad (2.29)$$

for $\operatorname{Re}s\geq 0$, and $d(s,\mathrm{PO})\geq 1/(2(1+v))$.

Let

$$w(s) = -T\left(\frac{\pi}{2} - |\operatorname{Arg} s|\right), \qquad \Theta(s) = \frac{\pi}{2}N - (\operatorname{sign} \operatorname{Im} s)(\arg z - M),$$
$$V = -L - T - \ln|z| + 2\pi \sum_{\operatorname{Re} \alpha_j < 0} |\operatorname{Im} \alpha_j|, \tag{2.30}$$

and note that

$$\Theta(s) \geq b = \min\left\{\frac{\pi}{2}N \pm (\arg z - M)\right\} \geq 0, \tag{2.31}$$

since by assumption, the conditions of Theorem 1.2 are satisfied. Moreover,

$$(\operatorname{Im} s)(\arg z - M - T \operatorname{Arg} s) + \frac{\pi}{2}(T - N)|\operatorname{Im} s| = -|\operatorname{Im} s|[w(s) + \Theta(s)], \tag{2.32}$$

and (2.29) can be rewritten in the form

$$\left|sk(s)z^{-s}\right| \leq C|s|^{\Delta_0+1} e^{-|\operatorname{Im} s|[w(s)+\Theta(s)]} e^{(\operatorname{Re} s)[V+T\ln|s|]}. \tag{2.33}$$

Armed with (2.33), we now proceed to establish part (a) of the theorem. Let

$$R_1 = \left\{s : d(s, \mathrm{PO}) \geq \frac{1}{2(1+v)}, \ \frac{\pi}{4} \leq |\operatorname{Arg} s| \leq \frac{\pi}{2}\right\},$$
$$R_2 = \left\{s : d(s, \mathrm{PO}) \geq \frac{1}{2(1+v)}, \ |\operatorname{Arg} s| < \frac{\pi}{4}\right\}, \tag{2.34}$$

and notice that,

$$|\operatorname{Im} s| \geq \frac{|s|}{\sqrt{2}}, \quad s \in R_1, \tag{2.35}$$

$$\operatorname{Re} s > \frac{|s|}{\sqrt{2}}, \quad s \in R_2. \tag{2.36}$$

(a1) Assume that $T < 0$.

We consider two subcases:

(i) $|\arg z - M| < (\pi/2)N$.

In this case $b > 0$. Since $T < 0$, $(V + T\ln|s|) \to -\infty$, as $|s| \to \infty$, and hence, the function $e^{(\operatorname{Re} s)[V+T\ln|s|]}$ is bounded for $\operatorname{Re} s \geq 0$. On the other hand, $w(s) = |T|(\pi/2 - |\operatorname{Arg} s|) \geq 0$, $\Theta(s) \geq b > 0$, and for all s in R_1, $|\operatorname{Im} s| \geq (1/\sqrt{2})|s|$. Thus,

$$\left|sk(s)z^{-s}\right| \leq C|s|^{\Delta_0+1} e^{-(b/\sqrt{2})|s|}, \quad s \in R_1. \tag{2.37}$$

The assumption $T < 0$ implies $\lim_{|s|\to\infty}[V + T\ln|s|] = -\infty$, and consequently, $V + T\ln|s| < -1$ as $|s| \to \infty$. Since $\operatorname{Re} s > (|s|/\sqrt{2})$, $s \in R_2$, it follows that

$$\left|sk(s)z^{-s}\right| \leq C|s|^{\Delta_0+1} e^{(\operatorname{Re} s)[V+T\ln|s|]} \leq C|s|^{\Delta_0+1} e^{-(1/\sqrt{2})|s|}, \quad s \in R_2. \tag{2.38}$$

From (2.37) and (2.38) it immediately follows that $\lim_{|s|\to\infty} |sk(s)z^{-s}| = 0$, for $\operatorname{Re} s \geq 0$ and $d(s, \mathrm{PO}) \geq 1/(2(1+v))$.

(ii) $|\arg z - M| = (\pi/2)N$.

In this case, necessarily $\Delta_0 < -1$ because the convergence conditions of Theorem 1.2 are satisfied. Thus, $|s|^{\Delta_0+1} \to 0$, as $|s| \to \infty$. Since the functions $e^{-|\operatorname{Im} s|[w(s)+\Theta(s)]}$,

and $e^{(\operatorname{Re} s)[V + T \ln|s|]}$ are bounded for all $\operatorname{Re} s \geq 0$, it follows from (2.33) that $\lim_{|s| \to \infty} |sk(s)z^{-s}| = 0$, for $\operatorname{Re} s \geq 0$ and $d(s, \text{PO}) \geq 1/(2(1+v))$.

(a2) Assume $T = 0$, and z satisfies $|z| > \exp(-L + 2\pi \sum_{\operatorname{Re} \alpha_j < 0} |\operatorname{Im} \alpha_j|)$, in addition to the conditions of Theorem 1.2.

In this case $w(s) = 0$ for all s, and we again consider two subcases:

(i) $|\arg z - M| < (\pi/2)N$.

As before, $\Theta(s) \geq b > 0$. The condition on z implies that $V < 0$, and thus

$$[V + T \ln|s|] \operatorname{Re} s = V \operatorname{Re} s \leq 0, \quad \operatorname{Re} s \geq 0. \tag{2.39}$$

It follows from this, (2.33), (2.35), and (2.36) that

$$|sk(s)z^{-s}| \leq C|s|^{\Delta_0 + 1} e^{-(b/\sqrt{2})|s|}, \quad s \in R_1, \tag{2.40}$$

$$|sk(s)z^{-s}| \leq C|s|^{\Delta_0 + 1} e^{-(|V|/\sqrt{2})|s|}, \quad s \in R_2. \tag{2.41}$$

It immediately follows from (2.40) and (2.41) that $\lim_{|s| \to \infty} |sk(s)z^{-s}| = 0$, for $\operatorname{Re} s \geq 0$ and $d(s, \text{PO}) \geq 1/(2(1+v))$.

(ii) $|\arg z - M| = (\pi/2)N$.

In this case, necessarily $\Delta_0 < -1$ because the convergence conditions of Theorem 1.2 are satisfied. Thus, $|s|^{\Delta_0 + 1} \to 0$, as $|s| \to \infty$, and since the functions $e^{-\Theta(s)|\operatorname{Im} s|}$, and $e^{V \operatorname{Re} s}$ are bounded for all $\operatorname{Re} s \geq 0$, it follows from (2.33) that $|sk(s)z^{-s}| \to 0$, as $|s| \to \infty$, for $\operatorname{Re} s \geq 0$ and $d(s, \text{PO}) \geq 1/(2(1+v))$. This completes the proof of Theorem 2.2(a).

Now we proceed to prove Theorem 2.2(b).

Let $\alpha'_j = -\alpha_j$, $j = 1, 2, \ldots, P$; $\beta'_j = -\beta_j$, $j = 1, 2, \ldots, Q$, and define

$$k_1(s) = \frac{\prod_{j=1}^{P} \Gamma(p_j + \alpha'_j s)}{\prod_{j=1}^{Q} \Gamma(q_j + \beta'_j s)}. \tag{2.42}$$

Then under the change of variables $s \to -s$,

$$\int_{\Omega_c} k(s)z^{-s} ds = -\int_{\Omega'_{-c}} k(-s)\left(\frac{1}{z}\right)^{-s} ds = \int_{-\Omega'_{-c}} k_1(s)\left(\frac{1}{z}\right)^{-s} ds, \tag{2.43}$$

where Ω'_{-c} is the reflection of Ω_c about the origin. Observe that $-\Omega'_{-c}$ is directed from $-c - i\infty$ to $-c + i\infty$. We want to apply part (a) to $k_1(s)$ and the contour $-\Omega'_{-c}$. For this purpose, let T', M', L', and N' denote the constants associated with $k_1(s)$ that are computed as in Lemma 1.1 and Theorem 1.2, respectively. It is easy to see that

$$T' = -T, \qquad N' = N, \qquad M' = -M. \tag{2.44}$$

However,

$$L' = \sum_{j=1}^{P} \left[(\operatorname{Arg} \alpha'_j) \operatorname{Im} \alpha'_j - (\ln|\alpha'_j|) \operatorname{Re} \alpha'_j \right] - \sum_{j=1}^{Q} \left[(\operatorname{Arg} \beta'_j) \operatorname{Im} \beta'_j - (\ln|\beta'_j|) \operatorname{Re} \beta'_j \right]$$

$$= -L + \pi \left[\sum_{j=1}^{P} |\operatorname{Im} \alpha_j| - \sum_{j=1}^{Q} |\operatorname{Im} \beta_j| \right].$$

$$\tag{2.45}$$

Hence,

$$\left| M' - \arg\left(\frac{1}{z}\right) \right| = |M - \arg(z)|, \quad \text{for any } z, \tag{2.46}$$

and $T \geq 0$ if and only if $T' \leq 0$. It is clear that $s \in (\text{PO})_l$ if and only if $-s$ is a pole of $k_1(s)$, that is, to the right of $-\Omega'_{-c}$. Furthermore, if s_0 is a pole of $k_1(s)$ of order m, then there is a $\rho > 0$ such that

$$k_1(s) = f(s) + \sum_{j=1}^{m} \frac{a_{-j}}{(s - s_0)^j}, \quad \text{for } |s - s_0| < \rho, \tag{2.47}$$

where $a_{-m} \neq 0$ and f is analytic at s_0. Hence

$$k(s) = f(-s) + \sum_{j=1}^{m} (-1)^j \frac{a_{-j}}{(s + s_0)^j}, \quad \text{for } |s + s_0| < \rho, \tag{2.48}$$

and therefore

$$\text{Res}\left(k(s), -s_0\right) = -\text{Res}\left(k_1(s), s_0\right). \tag{2.49}$$

Thus, if $T > 0$, then

$$\int_{\Omega_c} k(s) z^{-s} ds = \int_{-\Omega'_{-c}} k_1(s) \left(\frac{1}{z}\right)^{-s} ds = -2\pi i \sum_{s_j \in (\text{PO})_l} \left(\frac{1}{z}\right)^{-s_j} \text{Res}\left(k_1(s), -s_j\right)$$

$$= 2\pi i \sum_{s_j \in (\text{PO})_l} z^{-s_j} \text{Res}\left(k(s), s_j\right). \tag{2.50}$$

In case $T = 0$, then $T' = 0$, and z must also satisfy

$$\frac{1}{|z|} > \exp\left(-L' + 2\pi \sum_{\text{Re}\,\alpha'_j < 0} |\text{Im}\,\alpha'_j|\right) \tag{2.51}$$

or, equivalently,

$$|z| < \exp\left(-L + \pi \left(\sum_{j=1}^{P} |\text{Im}\,\alpha_j| - \sum_{j=1}^{Q} |\text{Im}\,\beta_j|\right) - 2\pi \sum_{\text{Re}\,\alpha_j > 0} |\text{Im}\,\alpha_j|\right). \tag{2.52}$$

This completes the proof of Theorem 2.2.   □

REMARK 2.3. If $\mu = 0$, $N > 0$, and $T \neq 0$, then the H-function with complex parameters is analytic in the sector $|\arg z - M| < (\pi/2)N$. This is the case since the series of the residues at the left poles, if $T > 0$, and the series of the residues at the right poles, if $T < 0$, represent analytic functions.

REMARK 2.4. If $\sum_{\text{Re}\,\alpha_j = 0} |\text{Im}\,\alpha_j| > \sum_{j=1}^{Q} |\text{Im}\,\beta_j| + \sum_{\text{Re}\,\alpha_j \neq 0} |\text{Im}\,\alpha_j|$, then

$$r_1 = \exp\left(-L + 2\pi \sum_{\text{Re}\,\alpha_j < 0} |\text{Im}\,\alpha_j|\right)$$

$$< \exp\left(-L + \pi \left(\sum_{j=1}^{P} |\text{Im}\,\alpha_j| - \sum_{j=1}^{Q} |\text{Im}\,\beta_j|\right) - 2\pi \sum_{\text{Re}\,\alpha_j > 0} |\text{Im}\,\alpha_j|\right) = r_2. \tag{2.53}$$

Thus, if $T = 0$, then the *H*-function with complex parameters can be computed for any z in the annulus $r_1 < |z| < r_2$ by the residues at the right poles as well as the residues at the left poles. This situation has no analogy in the real case, that is, when $(\alpha)_P$ and $(\beta)_Q$ are real vectors.

3. Some special cases of the *H*-function. In general, if w and $\alpha \neq 0$ are complex numbers, then

$$\operatorname*{Res}_{s=-(m+w)/\alpha} \Gamma(w + \alpha s) = \frac{(-1)^m}{\alpha(m!)}, \quad m = 1, 2, \ldots. \tag{3.1}$$

EXAMPLE 3.1. Let $a_1, a_2, \ldots, a_{P+1}$; $b_1, b_2, \ldots, b_P$ be complex numbers such that $\sum_{j=1}^{P+1} \operatorname{Re} a_j < \sum_{j=1}^{P} \operatorname{Re} b_j$. Then

$$\begin{aligned}
_{P+1}F_P\big((a)_{P+1}; (b)_P; (-1)^P z\big) &= i \operatorname{sign} \operatorname{Im}(z) \prod_{j=1}^{P+1} \Gamma(1 - a_j) \prod_{j=1}^{P} \Gamma(b_j) \\
&\quad \times H_{2P+1}^1 \left(z^i, c \left| \begin{matrix} (0, i) \\ (1 - a_j, i)_{P+1}, (b_j, -i)_P \end{matrix} \right. \right),
\end{aligned} \tag{3.2}$$

$|z| = 1$, where $c > 0$ if $\operatorname{Im} z \geq 0$, $c < 0$ if $\operatorname{Im} z < 0$, and $_pF_q(z)$ is the generalized hypergeometric function (see [2, 4, 18]).

In fact for the function

$$\begin{aligned}
H_{2P+1}^1 &\left(x, c \left| \begin{matrix} (0, i) \\ (1 - a_j, i)_{P+1}, (b_j, -i)_P \end{matrix} \right. \right) \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(is)}{\prod_{j=1}^{P+1} \Gamma(1 - a_j + is) \prod_{j=1}^{P} \Gamma(b_j - is)} x^{-s} ds.
\end{aligned} \tag{3.3}$$

it is easy to see that $N = 0$, $M = 0$, and

$$\Delta_c = P - \sum_{j=1}^{P+1} \operatorname{Re}(1 - a_j) - \sum_{j=1}^{P} \operatorname{Re} b_j = -1 + \sum_{j=1}^{P+1} \operatorname{Re} a_j - \sum_{j=1}^{P} \operatorname{Re} b_j < -1, \tag{3.4}$$

so the conditions of Theorem 1.2 are satisfied.

Since $L = 0$ and $T = 0$, we can compute the value of the function using the residues at the poles $s = im$, $m = 0, 1, 2, \ldots$.

If $c > 0$, then the poles are left poles. Hence, computing with left poles, for $0 < x \leq 1$,

$$\begin{aligned}
H_{2P+1}^1 &\left(x, c \left| \begin{matrix} (0, i) \\ (1 - a_j, i)_{P+1}, (b_j, -i)_P \end{matrix} \right. \right) \\
&= -i \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{x^{-im}}{\prod_{j=1}^{P+1} \Gamma(1 - a_j - m) \prod_{j=1}^{P} \Gamma(b_j + m)} \\
&= -i \sum_{m=0}^{\infty} \frac{(-1)^{mP}}{m!} \frac{\prod_{j=1}^{P+1} (a_j)_m}{\prod_{j=1}^{P+1} \Gamma(1 - a_j) \prod_{j=1}^{P} \Gamma(b_j) \prod_{j=1}^{P} (b_j)_m} x^{-im} \\
&= -\frac{i {}_{P+1}F_P\big((a)_{P+1}; (b)_P; (-1)^P x^{-i}\big)}{\prod_{j=1}^{P+1} \Gamma(1 - a_j) \prod_{j=1}^{P} \Gamma(b_j)}.
\end{aligned} \tag{3.5}$$

Let $z = x^{-i}$, then $|z| = 1$, $x = z^i$, and we obtain (3.2) for $\text{Im}(z) \geq 0$.

If $c < 0$, then the poles are right poles. Hence, computing with right poles, for $x > 1$,

$$H^1_{2P+1}\left(x, c \,\middle|\, \begin{matrix} (0, i) \\ (1 - a_j, i)_{P+1}, (b_j, -i)_P \end{matrix}\right) = \frac{i_{P+1}F_P((a)_{P+1}; (b)_P; (-1)^P x^{-i})}{\prod_{j=1}^{P+1} \Gamma(1 - a_j) \prod_{j=1}^{P} \Gamma(b_j)}. \tag{3.6}$$

Let $z = x^{-i}$, then $|z| = 1$, $x = z^i$, and we obtain (3.2) for $\text{Im}(z) < 0$.

REMARK 3.2. It seems to the authors that (3.2) provides the first integral representation for the generalized hypergeometric geometric function $_{P+1}F_P(z)$ on the unit circle.

EXAMPLE 3.3. If $\text{Re}\, p \neq 0$, $1 + \text{Re}\, p < \text{Re}\, q$, and $\text{Im}\, p > 0$, then

$$H^1_1\left(x, 0 \,\middle|\, \begin{matrix} (p, i) \\ (q, i) \end{matrix}\right) = \begin{cases} 0, & \text{if } x \geq 1, \\ -ix^{-ip}(1 - x^{-i})^{q-p-1}, & \text{if } 0 < x < 1. \end{cases} \tag{3.7}$$

Since $N = 0 = M$ and $\Delta_0 = \text{Re}\, p - \text{Re}\, q < -1$, the H-function exists for $x > 0$. Since $T = 0 = L$, the function can be computed by the residues at the right poles for $x > 1$, and by the residues at the left poles for $0 < x < 1$. The poles are at $s = im + ip$, $m = 0, 1, 2, \ldots$. Since $\text{Im}\, p > 0$, all the poles are on the left, and hence

$$H^1_1\left(x, 0 \,\middle|\, \begin{matrix} (p, i) \\ (q, i) \end{matrix}\right) = 0, \quad \text{if } x > 1, \tag{3.8}$$

and for $0 < x < 1$,

$$\begin{aligned} H^1_1\left(x, 0 \,\middle|\, \begin{matrix} (p, i) \\ (q, i) \end{matrix}\right) &= \sum_{m=0}^{\infty} \left[\text{Res}_{s=i(m+p)} \Gamma(p + is)\right] \frac{1}{\Gamma(q - p - m)} x^{-im-ip} \\ &= -i \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(q - p - m)} x^{-im-ip} = -ix^{-ip}(1 - x^{-i})^{q-p-1}. \end{aligned} \tag{3.9}$$

EXAMPLE 3.4. If p and q are complex numbers such that $\text{Re}\, p, \text{Re}\, q \neq 0$, and $\text{Re}\, p + \text{Re}\, q < 0$, then

$$H^2_0\left(x, 0 \,\middle|\, (p, i), (q, -i)\right) = \begin{cases} 0, & \text{if } \text{Im}(p)\text{Im}(q) < 0, \\ \dfrac{-i(\text{sgn}\,\text{Im}\, p)\Gamma(p + q)}{x^{-iq}(1 + x^i)^{p+q}}, & \text{if } \text{Im}(p)\text{Im}(q) > 0, \end{cases} \tag{3.10}$$

for all $x > 0$.

It is easy to check that $M = N = 0$, and $\Delta_0 = -1 + \text{Re}\, p + \text{Re}\, q < -1$. Hence,

$$H^2_0\left(x, 0 \,\middle|\, (p, i), (q, -i)\right) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(p + is)\Gamma(q - is) x^{-s} ds \tag{3.11}$$

exists for all $x > 0$.

Since $T = 0$ and $L = \pi$, the function can be computed using the residues at the right poles of $\Gamma(p + is)\Gamma(q - is)x^{-s}$ for $x > e^{-\pi}$ and the residues at the left poles for

$0 < x < e^{\pi}$. Thus in the interval $(e^{-\pi}, e^{\pi})$, the function can be computed using the residues at the right as well as at the left poles.

The poles of $\Gamma(p + is)$ are at $s = im + ip$, while those of $\Gamma(q - is)$ are at $s = -im - iq$, and

$$\text{Res}_{s=i(m+p)} \Gamma(p + is) = -i\frac{(-1)^m}{m!}, \qquad \text{Res}_{s=-i(m+q)} \Gamma(q - is) = i\frac{(-1)^m}{m!}. \qquad (3.12)$$

(i) The case $\text{Im}\, p > 0$, $\text{Im}\, q < 0$.

In this case all the poles are left poles and hence

$$H_0^2\left(x, 0 \left| \begin{matrix} (p, i), (q, -i) \end{matrix} \right.\right) = 0, \quad \text{for } x > e^{-\pi}, \qquad (3.13)$$

and for $0 < x < e^{\pi}$,

$$H_0^2\left(x, 0 \left| \begin{matrix} (p, i), (q, -i) \end{matrix} \right.\right)$$

$$= \sum_{m=0}^{\infty} \left[\text{Res}_{s=i(m+p)} \Gamma(p + is) \right] \Gamma(p + q + m) x^{-im-ip}$$

$$+ \sum_{m=0}^{\infty} \left[\text{Res}_{s=-i(m+q)} \Gamma(q - is) \right] \Gamma(p + q + m) x^{im+iq}$$

$$= -i\Gamma(p + q) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (p + q)_m x^{-im-ip} + i\Gamma(p + q) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (p + q)_m x^{im+iq}$$

$$= -i\Gamma(p + q) x^{-ip} (1 + x^{-i})^{-p-q} + i\Gamma(p + q) x^{iq} (1 + x^i)^{-p-q} = 0. \qquad (3.14)$$

(ii) The case $\text{Im}\, p < 0$, $\text{Im}\, q > 0$.

In this case all the poles are right poles and hence,

$$H_0^2\left(x, 0 \left| \begin{matrix} (p, i), (q, -i) \end{matrix} \right.\right) = 0, \quad \text{for } 0 < x < e^{\pi}, \qquad (3.15)$$

and for $x > e^{-\pi}$, the exact computation as above shows that,

$$H_0^2\left(x, 0 \left| \begin{matrix} (p, i), (q, -i) \end{matrix} \right.\right) = 0, \quad \text{for } x > e^{-\pi}. \qquad (3.16)$$

(iii) The case $\text{Im}\, p < 0$, $\text{Im}\, q < 0$.

In this case, the poles at $s = im + ip$ are right poles, and the poles at $s = -im - iq$ are left poles. Thus, for $x > e^{-\pi}$,

$$H_0^2\left(x, 0 \left| \begin{matrix} (p, i), (q, -i) \end{matrix} \right.\right)$$

$$= - \sum_{m=0}^{\infty} \left[\text{Res}_{s=i(m+p)} \Gamma(p + is) \right] \Gamma(p + q + m) x^{-im-ip}$$

$$= i\Gamma(p + q) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (p + q)_m x^{-im-ip}$$

$$= i\Gamma(p + q) x^{-ip} (1 + x^{-i})^{-p-q} = \frac{i\Gamma(p + q)}{x^{-iq} (1 + x^i)^{p+q}}, \qquad (3.17)$$

and for $0 < x < e^\pi$,

$$H_0^2\left(x,0\,\middle|\,{(p,i),(q,-i)}\right)$$

$$= \sum_{m=0}^{\infty}\left[\operatorname{Res}_{s=-i(m+q)}\Gamma(q-is)\right]\Gamma(p+q+m)x^{im+iq} \tag{3.18}$$

$$= i\Gamma(p+q)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(p+q)_m x^{im+iq} = i\Gamma(p+q)x^{iq}(1+x^i)^{-p-q}.$$

Therefore,

$$H_0^2\left(x,0\,\middle|\,{(p,i),(q,-i)}\right) = i\Gamma(p+q)x^{iq}(1+x^i)^{-p-q}, \quad \forall x > 0. \tag{3.19}$$

(iv) The case $\operatorname{Im} p > 0$, $\operatorname{Im} q > 0$.

In this case, the poles at $s = im + ip$ are left poles, and the poles at $s = -im - iq$ are right poles. Thus, for $0 < x < e^\pi$,

$$H_0^2\left(x,0\,\middle|\,{(p,i),(q,-i)}\right)$$

$$= \sum_{m=0}^{\infty}\left[\operatorname{Res}_{s=i(m+p)}\Gamma(p+is)\right]\Gamma(p+q+m)x^{-im-ip} \tag{3.20}$$

$$= -i\Gamma(p+q)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(p+q)_m x^{-im-ip}$$

$$= -i\Gamma(p+q)x^{-ip}(1+x^{-i})^{-p-q} = -i\Gamma(p+q)x^{iq}(1+x^i)^{-p-q},$$

and for $x > e^{-\pi}$,

$$H_0^2\left(x,0\,\middle|\,{(p,i),(q,-i)}\right)$$

$$= -\sum_{m=0}^{\infty}\left[\operatorname{Res}_{s=-i(m+q)}\Gamma(q-is)\right]\Gamma(p+q+m)x^{im+iq} \tag{3.21}$$

$$= -i\Gamma(p+q)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(p+q)_m x^{im+iq} = -i\Gamma(p+q)x^{iq}(1+x^i)^{-p-q}.$$

Therefore,

$$H_0^2\left(x,0\,\middle|\,{(p,i),(q,-i)}\right) = -i\Gamma(p+q)x^{iq}(1+x^i)^{-p-q}, \quad \forall x > 0. \tag{3.22}$$

REMARK 3.5. The special case $x = 1$ of the previous example is evaluated in [6, formula 6.411].

EXAMPLE 3.6. If a is a complex number, then

$$H_2^1\left(z,c\,\middle|\,{(0,1)\atop(ia,-i),(1-ia,i)}\right) = \frac{1}{2\pi}\left(e^{\pi a - e^\pi z} - e^{-\pi a - e^{-\pi}z}\right), \quad c > 0,\ |\arg z| < \frac{\pi}{2}. \tag{3.23}$$

It is easy to check that $N = 1$, $M = 0$, and the conditions for existence are satisfied for $|\arg z| < \pi/2$. Since $T = 1 > 0$, we can compute the function using the left poles at $s = -m$, $m = 0, 1, 2, \ldots$. Thus,

$$
\begin{aligned}
H_2^1\left(z, c \,\middle|\, \begin{matrix} (0,1) \\ (ia,-i),(1-ia,i) \end{matrix}\right) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\Gamma(ia-is)\Gamma(1-ia+is)} z^{-s} ds \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \cdot \frac{1}{\Gamma(ia+im)\Gamma(1-ia-im)} z^m \\
&= \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \sin\pi i(a+m)}{m!} z^m \\
&= \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \sinh\pi(a+m)}{m!} z^m \\
&= \frac{e^{a\pi}}{2\pi} \sum_{m=0}^{\infty} \frac{(-e^\pi z)^m}{m!} - \frac{e^{-a\pi}}{2\pi} \sum_{m=0}^{\infty} \frac{(-e^{-\pi} z)^m}{m!} \\
&= \frac{1}{2\pi} \left(e^{\pi a - e^\pi z} - e^{-\pi a - e^{-\pi} z}\right).
\end{aligned}
\tag{3.24}
$$

EXAMPLE 3.7. If $c > 0$ and a is a complex number, $\mathrm{Re}\, a > c$, then

$$
H_0^3\left(z, c \,\middle|\, \begin{matrix} (0,1),(ia,-i),(1-ia,i) \end{matrix}\right) = \pi \sum_{m=0}^{\infty} \frac{(-z)^m}{m!\sinh\pi(a+m)}, \quad |\arg z| < \frac{\pi}{2}. \tag{3.25}
$$

Now, $N = 1$, $M = 0$, and the conditions of convergence are satisfied for $|\arg z| < \pi/2$. Since $T = 1 > 0$, we can compute the value of the function using the residues at the left poles of the function $\Gamma(ia-is)\Gamma(1-ia+is)\Gamma(s)z^{-s}$. Because $\mathrm{Re}\, a > c > 0$, the only left poles we have are the poles $s = -m$ of $\Gamma(s)$. Hence,

$$
\begin{aligned}
H_0^3\left(z, c \,\middle|\, \begin{matrix} (0,1),(ia,-i),(1-ia,i) \end{matrix}\right) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(ia-is)\Gamma(1-ia+is)\Gamma(s)z^{-s} ds \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma(ia+im)\Gamma(1-ia-im) z^m \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \cdot \frac{\pi}{\sin\pi i(a+m)} z^m \\
&= \pi \sum_{m=0}^{\infty} \frac{(-z)^m}{m!\sinh\pi(a+m)}.
\end{aligned}
\tag{3.26}
$$

EXAMPLE 3.8. Let a and b be complex numbers such that $1 + \mathrm{Re}\, a < \mathrm{Re}\, b$, and $\mathrm{Im}\, a > -1/2$, then

$$
H_1^1\left(x, \frac{1}{2} \,\middle|\, \begin{matrix} (a,i) \\ (b,i) \end{matrix}\right) = \begin{cases} -\dfrac{i}{\Gamma(b-a)} x^{-ia} (1-x^{-i})^{b-a-1}, & \text{if } 0 < x < 1, \\ 0, & \text{if } x > 1. \end{cases} \tag{3.27}
$$

It is easy to see that $N = 0 = M$, $\Delta_{1/2} = \mathrm{Re}\, a - \mathrm{Re}\, b < -1$, and the conditions of Theorem 1.2 are fulfilled. Moreover, $T = 0$, $L = 0$, and the poles are at $s = ia + im$,

$m = 0, 1, 2, \ldots$. Since $\text{Im}\, a > -1/2$, there are no right poles, and therefore

$$H_1^1\left(x, \frac{1}{2}\,\bigg|\,\begin{matrix}(a, i)\\(b, i)\end{matrix}\right) = 0, \quad \text{for } x > 1. \tag{3.28}$$

For $0 < x < 1$,

$$H_1^1\left(x, \frac{1}{2}\,\bigg|\,\begin{matrix}(a, i)\\(b, i)\end{matrix}\right) = -i \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(b - a - m)} x^{-ia - im} = -\frac{i}{\Gamma(b - a)} x^{-ia}\left(1 - x^{-i}\right)^{b - a - 1}. \tag{3.29}$$

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