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SINGULAR POINTS AND LIE ROTATED VECTOR FIELDS

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ABSTRACT. This paper gives the definition of Lie rotated vector fields in the plane and the conditions of movement of singular points on Lie rotated vector fields with variable parameters.

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1. Introduction. Many engineering problems are usually run into a class of nonlinear equations that contain variable parameters. In order to study whole orbits or whole phase diagrams of vector fields that contain parameters, it is a complicated and interesting problem how the whole orbit or whole phase diagram change as parameter is changed. It is extremely complicated for general containing parameter vector fields to change in the plane, but for some special containing parameter rotated vector fields, their change has regular rule as parameter is changed. These are many results in this respects [3, 4, 5, 6, 7].

In Section 2, we present the basic definitions of Lie rotated vector fields. We define Lie rotated vector fields using one parameter group approach. In accordance with the strict definition of rotated vector field, the singular points of $X(\mu)$ must be kept fixed, but in this paper, the singular points of $X(\mu)$ can be moved as parameter μ is changed. In Section 3, we discuss the motion of singular points on Lie rotated vector fields. In the section, we require the singular points of $X(\mu)$ to be strictly moved as parameter μ is changed, and permit the moved singular points to disappear or decompose, which do not coincide with the singular points of original vector field. We give some conditions and properties corresponding to the vector field Y. In this paper, we give some examples to illustrate the concept and notion of Lie rotated vector fields.

2. Lie rotated vector fields. We consider vector fields on the plane $x = (x_1, x_2) \in \mathbb{R}^2$,

$$X = \big(X_1(x), X_2(x)\big), \qquad Y = \big(Y_1(x), Y_2(x)\big). \tag{2.1}$$

For the vector fields (2.1), we define

$$X \wedge Y = X_1Y_2 - X_2Y_1, \qquad \langle X, Y\rangle = X_1Y_1 + X_2Y_2. \tag{2.2}$$

If X and Y are vector fields, then $[X, Y]$ is a vector field which is operated by Lie bracket, i.e.,

$$[X,Y] = (Z_1, Z_2), \tag{2.3}$$

where Z_1 and Z_2 are expressed as,

$$Z_1 = \langle X, \nabla Y_1\rangle - \langle Y, \nabla X_1\rangle, \qquad Z_2 = \langle X, \nabla Y_2\rangle - \langle Y, \nabla X_2\rangle, \tag{2.4}$$

respectively, where ∇ is gradient operator.

Let the plane vector fields $X(\mu) = (X_1(x,\mu), X_2(x,\mu))$ be defined by the following differential equations:

$$\frac{dx_1}{dt} = X_1(x,\mu), \qquad \frac{dx_2}{dt} = X_2(x,\mu), \tag{2.5}$$

where X_1 and X_2 are functions of x and parameter $\mu \in I \subset \mathbb{R}$, and the singular points are isolated.

Definition 2.1. Let the plane vector field $X(\mu)$ be determined by (2.5), where $X_1, X_2 \in C^3(\mathbb{R}^2 \times I, \mathbb{R})$, $I = \{\mu \mid |\mu| < \delta\}$ is a real interval, δ is a given positive number. If vector field Y exists which is defined by the following differential equations:

$$\frac{dx_1}{dt} = Y_1(x), \qquad \frac{dx_2}{dt} = Y_2(x), \tag{2.6}$$

where Y_1 and $Y_2 \in C^3(\mathbb{R}^2, \mathbb{R})$. At all ordinary points of $X(0)$, such that the following relation holds

$$L(0) \overset{\text{def}}{=} X(0) \wedge \{X'_\mu(0) + [X(0), Y]\} > 0 \ (< 0), \tag{2.7}$$

where $X'_\mu(0)$ is the derivative of the vector field $X(\mu)$ at $\mu = 0$, then $X(\mu)$, $\mu \in I$, is called Lie rotated vector fields.

Remark 2.2. If the vector field $X(\mu)$ is defined on $D \times I$, where $D \subset \mathbb{R}^2$, such that $X(0)$ satisfies relation (2.7) at all ordinary points of $X(0)$ on D, then $X(\mu)$, $\mu \in I$, is called Lie rotated vector fields on D.

Lemma 2.3. *Let ψ^s be a one parameter transform group which is produced by C^1 vector field Y, $s \in \mathbb{R}$, and let X be C^1 vector field. If s is fixed, and $\varphi_p(t)$ is an integral curve of X through the point p, $\varphi_p(0) = p$, then $\psi^s \circ \varphi_p(t)$ is an integral curve of $\psi^s_* X$ through the point $\psi^s(p)$. If $X|_p = 0$, then $(\psi^s_* X)|_{\psi^s(p)} = 0$.*

Proof. The proof follows from [1] and [2]. In fact, if $\varphi_p(t)$ is an integral curve of X through the point p, then

$$(\psi^s \circ \varphi_p(t))\big|_{t=0} = \psi^s(p) \tag{2.8}$$

and

$$\begin{aligned}(\psi^s \circ \varphi_p(t))_{*t} \cdot \left(\frac{d}{dt}\Big|_t\right) &= \psi^s_{*\varphi_p(t)} \circ \varphi_{p(t)*_t} \cdot \left(\frac{d}{dt}\Big|_t\right) \\ &= \psi^s_{*\varphi_p(t)} \cdot X_{\varphi_p(t)} = (\psi^s_* X)_{\psi^s \circ \varphi_p(t)}.\end{aligned} \tag{2.9}$$

It follows that $\psi^s \circ \varphi_p(t)$ is an integral curve of $\psi^s_* X$ through the point $\psi^s(p)$.

Next, due to

$$\psi^s_* X|_q = D\psi^s(\psi^{-s}(q)) \cdot X(\psi^{-s}(q)), \quad q \in \mathbb{R}^2. \tag{2.10}$$

Set $q = \psi^s(p)$, note that we already suppose $X|_p = 0$, again note that ψ^s is a one parameter transform which is produced by Y, then

$$\psi^s_* X|_{\psi^s(p)} = D\psi^s(p) \cdot X(p) = D\psi^s(p) \cdot X|_p = 0, \tag{2.11}$$

i.e., $\psi^s(p)$ is a singular point of $\psi^s_* X$. □

Lemma 2.4. *Let ψ^s be a one parameter transform group which is produced by C^1 vector field Y, $s \in \mathbb{R}$, fix s, then the index of isolated singularity of C^1 vector field X is not changed under the ψ^s transform.*

Proof. In fact, by the condition of the lemma, it is known that ψ^s is a differentiable homeomorphism, then the lemma follows from [8, Theorem 4.2].

Next, if $X(\mu)$ is a Lie rotated vector field, then Y is a corresponding vector field which satisfies (2.7), and ψ^s is a one parameter transform group which is produced by Y, $s \in \mathbb{R}$. □

Lemma 2.5. *Let $X(\mu)$ be a Lie rotated vector field, for all $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon)$, such that when $|\mu| < \delta$, $\psi^s_* X(\mu)$ constitutes a rotated vector field.*

Proof. Let the singular points of $\psi^s_* X(\mu)$, $\mu \neq 0$, on the plane $\mathbb{R}^2$ be $p_{\mu_1}, \dots, p_{\mu_k}$ and the singular points of $X(0)$ on the plane $\mathbb{R}^2$ are $p_1, \dots, p_m$, $\forall \varepsilon > 0$, $0 < \varepsilon \ll 1$, let $S_\varepsilon(p_{\mu_i})$ or $S_\varepsilon(p_j)$ $(1 \le i \le k,\ 1 \le j \le m)$ be open neighborhood p_{μ_i} $(1 \le i \le k)$ and p_j $(1 \le j \le m)$, and radius ε, such that $\overline{S_\varepsilon(p)} \cap \overline{S_\varepsilon(q)} = \emptyset$, where p and $q \in \{p_{\mu_i}\} \cup \{p_j\}$ $(1 \le i \le k,\ 1 \le j \le m)$, $p \neq q$. Let ψ^s be a one parameter transform group which is produced by C^1 vector field Y, $s \in \mathbb{R}$. By the limit definition of Lie bracket, we have

$$\begin{aligned} \psi^s_* X(\mu) &= \psi^0_* X(\mu) + \frac{s}{1!} \frac{d}{dt} \Big|_{s=0} \psi^s_* X(\mu) + \frac{s^2}{2!} \frac{d^2}{dt^2} \Big|_{s=0} \psi^s_* X(\mu) + \cdots \\ &= X(\mu) + \frac{s}{1!} [X(\mu), Y] + \frac{s^2}{2!} [[X(\mu), Y], Y] + \cdots. \end{aligned} \tag{2.12}$$

Next, we notice that $X(\mu)$ can be unfolded as

$$X(\mu) = X(0) + \frac{\mu}{1!} X'_\mu(0) + \frac{\mu^2}{2!} X''_\mu(0) + \cdots, \tag{2.13}$$

since

$$[X(\mu), Y] = [X(0), Y] + \frac{\mu}{1!} [X'_\mu(0), Y] + \frac{\mu^2}{2!} [X''_\mu(0), Y] + \cdots. \tag{2.14}$$

Let $s = \mu$, it follows from (2.12), (2.13), and (2.14) that

$$\begin{aligned} \psi^\mu_* X(\mu) &= X(0) + \mu \{X'_\mu(0) + [X(0), Y]\} \\ &\quad + \frac{1}{2} \mu^2 \{X''_\mu(0) + 2[X'_\mu(0), Y] + [[X(0), Y], Y]\} + \cdots. \end{aligned} \tag{2.15}$$

At the ordinary points of $\mathbb{R}^2\backslash\{\bigcup_{i=1}^k S_\varepsilon(p_{\mu_i})\}\bigcup\{\bigcup_{j=1}^m S_\varepsilon(p_j)\}$, for given $\varepsilon > 0$, we sooner or later can find $\delta_1 = \delta_1(\varepsilon) > 0$, such that when $|\mu| < \delta_1$, we have

$$\begin{aligned}\psi_*^\mu X(\mu)\wedge\frac{\partial}{\partial\mu}\{\psi_*^\mu X(\mu)\} &= X(0)\wedge\{X_\mu'(0)+[X(0),Y]\}\\ &\quad+\mu X(0)\wedge\{X_\mu''(0)+2[X_\mu'(0),Y]+[[X(0),Y],Y]\}+\cdots\\ &= L(0)+O(\mu)>0\ (<0)\end{aligned} \tag{2.16}$$

and let $\vartheta(\mu)$ be the crossing angle of $\psi_*^\mu X(\mu)$ and the x_1 axis, for given $\varepsilon > 0$, we sooner or later can find $\delta_2 = \delta_2(\varepsilon)$, such that when $|\mu| < \delta_2$, at the ordinary points of $\mathbb{R}^2\setminus\{\bigcup_{i=1}^k S_\varepsilon(p_{\mu_i})\}\bigcup\{\bigcup_{j=1}^m S_\varepsilon(p_j)\}$ ($\psi_*^\mu X(\mu)$ is $X(0)$ when $\mu = 0$, $\vartheta(0)$ is the crossing angle of $X(0)$ and the x_1 axis), so

$$0 < |\vartheta(\mu)-\vartheta(0)| < \pi. \tag{2.17}$$

Take $\delta = \min\{\delta_1,\delta_2\}$, then when $|\mu| < \delta$, $\psi_*^\mu X(\mu)$ constitutes a rotated vector field. □

Remark 2.6. In accordance with the strict definition of rotated vector field, the singular points must be kept fixed, but the singular points of $\psi_*^\mu X(\mu)$ in Lemma 2.5 can be moved as parameter μ is changed. In the unmistakable circumstance, when $|\mu| < \delta$, we call $\psi_*^\mu X(\mu)$ a rotated vector.

In the above lemma, δ needs not be a quite small positive number, i.e., $0 < \delta \ll 1$ need not be set up. For the sake of distinctness, we cite an example to illustrate this equation.

Example 2.7. Let $X(\mu) = (x_2, -x_1+\mu x_2)$, if we take $Y = (-x_2/2, 0)$, then at all the ordinary points of $X(0)$, we have

$$X(0) = \frac{1}{2(x_1^2+x_2^2)} > 0, \tag{2.18}$$

that is, $X(\mu)$ is a Lie rotated vector field.

Now we consider the range of μ, because

$$\psi_*^\mu X(\mu) = \left(\frac{1}{2}\mu x_1+\left(1-\frac{1}{4}\mu^2\right)x_2, -x_1+\frac{1}{2}\mu x_2\right) \tag{2.19}$$

so

$$\psi_*^\mu X(\mu)\wedge\frac{\partial}{\partial\mu}\{\psi_*^\mu X(\mu)\} = \frac{1}{2}(x_1^2+x_2^2)-\frac{1}{2}\mu x_1x_2+\frac{1}{8}\mu^2x_2^2. \tag{2.20}$$

Formula (2.16) is compared with formula (2.20), we can find that $O(\mu)$ in formula (2.16) is replaced by $O(\mu)$ in formula (2.20),

$$O(\mu) = -\frac{1}{2}\mu x_1x_2+\frac{1}{8}\mu^2x_2^2 \tag{2.21}$$

yet go a step further calculating, we have

$$\psi_*^\mu X(\mu) \wedge \frac{\partial}{\partial\mu}\{\psi_*^\mu X(\mu)\} = \frac{1}{2}x_2^2 + \frac{1}{8}(\mu x_2 - 2x_1)^2 \tag{2.22}$$

which is larger than zero at the ordinary points of $X(0)$ and $\psi_*^\mu X(\mu)$ for all $\mu \in \mathbb{R}$, but the range of μ that satisfies formula (2.17) is $|\mu| < 4$, thus we take $\delta = 4$, when $|\mu| < \delta = 4$, $\psi_*^\mu X(\mu)$ constitutes rotated vector field.

3. The motion of singular points. Let $X(\mu)$ be a Lie rotated vector field, we require the singular points of $X(\mu)$ to be strictly moved as parameter μ is changed, and permit the singular points that have been moved disappear or decompose, but require the singular points that have been decomposed to be at most limited in number, which do not coincide with the singular points of the original vector field.

If p is a singular point of $X(\mu)$, we name $J_\mu(p)$ for index of singular point p of $X(\mu)$, under the same circumstances, $J_0(p_0)$ for index of singular point p_0 of $X(\mu)$, $J_{\mu *}(q)$ for index of singular point q of $\psi_*^\mu X(\mu)$ $(\mu \neq 0)$.

Theorem 3.1. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$, and let $Y|_{p_0} = 0$. If the singular point p_0 of $X(0)$ disappears or decomposes as p_i $(1 \le i \le k)$ in $X(\mu)$ $(\mu \neq 0)$, then $J_0(p_0) = 0$, and $J_\mu(p_i) = 0$ $(\mu \neq 0,\ 1 \le i \le k)$.*

Proof. First of all, we prove that $J_0(p_0) = 0$. In fact, because of $X(\mu)|_{p_0} \neq 0$ $(\mu \neq 0)$, utilize Lemma 2.3 and condition $Y|_{p_0} = 0$, we know that $\psi_*^\mu X(\mu)|_{p_0} \neq 0$ $(\mu \neq 0)$, it follows from Lemma 2.5, for given $\delta > 0$, when $|\mu| < \delta$, $\psi_*^\mu X(\mu)$ constitutes a rotated vector field. Take $\eta > 0$ as quite small positive number, such that $\overline{S_\eta(p_0)}$ does not contain the singular points of $\psi_*^\mu X(\mu)$ $(\mu \neq 0)$, and only contains the isolate singular point p_0 of $X(0)$. It is easy to know that $J_{\mu *}(p_0) = 0$ about $\partial S_\eta(p_0)$. By (2.17) of Lemma 2.5, it follows that $J_0(p_0) = 0$ when $|\mu| < \delta$.

Using the same method, we prove $J_{\mu *}(\psi^\mu(p_i)) = 0$ $(\mu \neq 0,\ 1 \le i \le k)$ and by Lemma 2.4, we find $J_\mu(p_i) = 0$ $(\mu \neq 0,\ 1 \le i \le k)$. □

Corollary 3.2. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$, if $Y|_{p_0} \neq 0$, and moved the singular points $p_i \neq \psi^{-\mu}(p_0)$ $(\mu \neq 0,\ 1 \le i \le k)$, then $J_0(p_0) = 0$ and $J_\mu(p_i) = 0$ $(\mu \neq 0,\ 1 \le i \le k)$.*

Proof. Since $X(0)|_{p_0} = 0$, let the singular point of $X(\mu)$ $(\mu \neq 0)$ disappears or decomposes into $p_1,\dots,p_k$ points which do not coincide with singular point p_0 of $X(0)$, i.e., $X(\mu)|_{p_i} = 0$ $(1 \le i \le k)$, yet because of $X(\mu)|_{p_0} \neq 0$ $(\mu \neq 0)$ and $Y|_{p_0} \neq 0$. By Lemma 2.3, we have $\psi_*^\mu X(\mu)|_{\psi^\mu(p_0)} \neq 0$ and $\psi_*^\mu X(\mu)|_{\psi^\mu(p_i)} = 0$, but by condition $\psi^\mu(p_i) \neq p_0$, we know that $\psi_*^\mu X(\mu)|_{p_0} \neq 0$, as in the proof of Theorem 3.1, we can prove that $J_0(p_0) = 0$ and $J_\mu(p_i) = 0$ $(\mu \neq 0,\ 1 \le i \le k)$. □

Corollary 3.3. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$, if $Y|_{p_0} \neq 0$, but for some i_0 $(1 \le i_0 \le k)$, set up $\psi^\mu(p_{i_0}) = p_0$ $(\mu \neq 0)$, then $J_0(p_0) = j_\mu(p_{i_0})$, $J_\mu(p_i) = 0$ $(\mu \neq 0,\ 1 \le i \le k$ and $i \neq i_0)$.*

Example 3.4. Let $X(\mu) = (x_2^2, -x_1 + \mu)$, and let

$$Y = (3x_1 - \alpha x_2, 2x_2) \tag{3.1}$$

when $|\mu| < \delta$, we take $\alpha > 0$ and $\alpha \ll 1$, on the range of $D = \{(x_1, x_2) \mid x_2 < \alpha^{-1}\} \subset \mathbb{R}^2$, at all ordinary points $\in D$ of $X(0)$, set up

$$L(0) = \alpha x_1^2 + x_2^2 - \alpha x_3^2 > 0, \tag{3.2}$$

that is, $X(\mu)$ constitutes a Lie rotated vector field on D, the singular points of $X(\mu)$ are strictly moved as parameter μ is changed. We note that $Y|_{p_0} = 0$, $p_0 = (0,0)$ is singular point of $X(0)$, by Theorem 3.1, we can find that $J_0(p_0) = 0$ and $J_\mu(p_i) = 0$ $(\mu \neq 0)$, where $p_i = (\mu, 0)$.

Theorem 3.5. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$, p_0 is elementary.*

(1) *If $Y|_{p_0} = 0$, then p_0 cannot be moved as parameter μ is changed.*

(2) *If $Y|_{p_0} \neq 0$, then p_0 can be moved as parameter μ is changed, and the moved point is the singular point $\psi^{-\mu}(p_0)$ of $X(\mu)$ $(\mu \neq 0)$.*

Proof. (1) We note $J_0(p_0) = \pm 1 \neq 0$, it is proved immediately from Theorem 3.1.

(2) First of all, we prove that p_0 is indeed moved as μ is changed, suppose that it is not real, i.e., p_0 is not moved as μ is changed, then that $X(\mu)|_{p_0} = 0$ $(\mu \neq 0)$, by Lemma 2.3, we know that $\psi^\mu_* X(\mu)|_{\psi^\mu(p_0)} = 0$. Because p_0 is isolate singular point of $X(0)$, we take $\bar{\delta} > 0$ and ample small $\eta > 0$, it follows that $\psi^\mu(p_0) \notin \overline{S_\eta(p_0)}$. When $0 < |\mu| < \bar{\delta} < \delta$, then for $\partial S_\eta(p_0)$, we have $J_{\mu *}(p_0) = 0$ (since $\psi^\mu_* X(\mu)|_{p_0} \neq 0$), where $\mu \neq 0$. But $J_0(p_0) = \pm 1 \neq 0$, this is a contradiction from Lemma 2.5. Thus we have proved p_0 is indeed moved as μ is changed, and by Corollaries 3.2 and 3.3, it follows that p_0 is moved as the singular point $\psi^{-\mu}(p_0)$ of $X(\mu)$ $(\mu \neq 0)$ when μ is changed. □

Lemma 3.6. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$, and there is an elliptic region at the singular point p_0.*

(1) *If $Y|_{p_0} = 0$, then the singular point p_0 cannot be moved when parameter $\mu \neq 0$.*

(2) *If $Y|_{p_0} \neq 0$, then when parameter $\mu \neq 0$, singular point p_0 is moved, and p_0 be moved as singular point $\psi^{-\mu}(p_0)$ of $X(\mu)$.*

Proof. (1) We already know that $Y|_{p_0}$, suppose the original equation is not real, then when $\mu \neq 0$, singular point p_0 is moved, thus we let p_0 moved as the singular point p_μ of $X(\mu)$, $X(\mu)|_{p_\mu} = 0$, $p_\mu \neq p_0$, $\mu \neq 0$. From Lemma 2.3, we know that $\psi^\mu_* X(\mu)|_{\psi^\mu(p_\mu)} = 0$, and by $Y|_{p_0} = 0$, we know that $\psi^\mu(p_\mu) \neq p_0$ $(\mu \neq 0)$. Let Ω be an elliptic region at the singular point p_0 of $X(0)$, for arbitrary fixed μ $(0 < |\mu| < \delta)$, it is sure to have some elliptic trajectory r of $X(0)$, which does not contain the point of $\psi^\mu(p_\mu)$ on r and in r. Since r has single direction, and there is no singular point of $\psi^\mu_* X(\mu)$ on r and in r. By Lemma 2.5, we can know that positive half trajectory or negative half trajectory of $\psi^\mu_* X(\mu)$ which pass through the point p will wander about without a home to go to, where p is any point which passes the inner region of r, this is a contradiction.

(2) Now we know $Y|_{p_0} \neq 0$, yet use reduction to absurdity. Suppose, when $\mu \neq 0$, singular point p_0 is not moved, i.e., establish $X(\mu)|_{p_0} = 0$, namely we have $\psi^\mu_* X(\mu)|_{\psi^\mu(p_\mu)} = 0$, and $\psi^\mu(p_0) \neq p_0$. The method of the proof is completely alike as part (1), we can prove it is a contradiction. Thus let $\mu \neq 0$, singular point p_0 is moved as singular point p_μ $(p_0 \neq p_\mu)$ of $X(\mu)$, i.e., $\psi^\mu_* X(\mu)|_{\psi^\mu\ (p_\mu)} = 0$ $(\mu \neq 0)$. If $\psi^\mu(p_\mu) \neq p_0$, the

method of the proof is alike as in part (1), yet it is a contradiction, thus only establish $\psi^{\mu}(p_{\mu}) = p_0$, or $p_{\mu} = \psi^{-\mu}(p_0)$. □

Lemma 3.7. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$, and when $\mu \neq 0$, p_0 is moved as the singular point p_{μ} $(p_{\mu} \neq p_0)$ of $X(\mu)$ as μ is changed. If $Y|_{p_0} = 0$, then for singular point p_{μ} (or p_0), at least there are a positive half trajectory and a negative half trajectory of $X(\mu)$ (or $X(0)$) to get into it.*

Proof. We only prove the circumstance of point p_{μ} (the proof is completely alike as the circumstance of point p_0).

From Lemma 3.6, we know that there is no elliptic region which links with the singular point p_0 of $X(0)$, the same do the singular point p_{μ} of $X(\mu)$, and from Theorem 3.1, we know that the index of p_{μ} of $X(\mu)$ is zero. Take p_{μ} as circular center, make the circumference of a circle l with radius rather small, and let that hyperbolic region of point p_{μ} which intersects with the circumference of a circle l has h. By the Bendixson's formula in §6 of Chapter 3 of [8], we can immediately find $h = 2$. □

From Lemmas 3.6 and 3.7, we have the following theorem.

Theorem 3.8. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$, and let $Y|_{p_0} = 0$, then some singulars while can be moved as parameter μ is changed in $X(\mu)$ only contain two hyperbolic regions and their index is zero.*

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