

FIXED POINTS, INTERSECTION THEOREMS, VARIATIONAL INEQUALITIES, AND EQUILIBRIUM THEOREMS

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ABSTRACT. From a fixed point theorem for compact acyclic maps defined on admissible convex sets in the sense of Klee, we first deduce collectively fixed point theorems, intersection theorems for sets with convex sections, and quasi-equilibrium theorems. These quasi-equilibrium theorems are applied to give simple and unified proofs of the known variational inequalities of the Hartman-Stampacchia-Browder type. Moreover, from our new fixed point theorem, we deduce new variational inequalities which can be used to obtain fixed point results for convex-valued maps. Finally, various general economic equilibrium theorems are deduced in the forms of the Nash type, the Tarafdar type, and the Yannelis-Prabhakar type. Our results are stated for not-necessarily locally convex topological vector spaces and for abstract economies with arbitrary number of commodities and agents. Our new results extend a lot of known works with much simpler proofs.

Keywords and phrases. Multimap (map), closed map, compact map, upper semicontinuous (u.s.c.), lower semicontinuous (l.s.c.), acyclic map, quasiconcave, quasiconvex, admissible subset of a topological vector space (t.v.s.), fixed point, convex space, polytope, quasi-equilibrium problem, variational inequality, economic equilibrium theorem, abstract economy, equilibrium point, maximal point.

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1. Introduction. Recently, we obtained a new fixed point theorem for compact multimaps defined on admissible convex subsets in not-necessarily locally convex topological vector spaces (see [51, 53]). Our theorem is one of the most general results and substantially extends a large number of known theorems, including Kakutani's theorem [26] for Euclidean spaces and Himmelberg's theorem [17] for locally convex topological vector spaces. Because these theorems were so useful in various problems in mathematical sciences including economic and game theories, it seems to be quite natural to generalize known results on applications of theorems of Kakutani and Himmelberg in view of our theorem. In this way, we can treat more general topological vector spaces than locally convex ones.

In the first half of the present paper, from a particular form of our fixed point theorem with the aid of some known selection theorems, we deduce new results on collectively fixed points (see Section 3), intersection theorems for sets with convex sections (see Section 4), and quasi-equilibrium problems (see Section 5). These quasi-equilibrium theorems are applied to give simple and unified proofs of the known variational inequalities of the Hartman-Stampacchia-Browder type (see Section 6).

In the second half, we deduce new variational inequalities (see Section 7) which can

be used to obtain fixed point theorems generalizing a large number of historically well-known extensions of the Brouwer or Kakutani theorems. Finally, various general economic equilibrium theorems are deduced. These are the Nash type (see Section 8), the Tarafdar type (see Section 9), and the Yannelis-Prabhakar type (see Section 10). Our results are stated for not-necessarily locally convex topological vector spaces and for abstract economies with arbitrary number of commodities and agents.

Consequently, our new results extend a lot of known works due to von Neumann [71], Nash [41], Fan [11, 12, 13, 14], Ma [37], Idzik [21, 22], Yannelis and Prabhakar [73], Tarafdar [69], Kim and Tan [30], and others, with much simpler proofs.

2. A fixed point theorem for acyclic maps. A *multimap* or *map* $T : X \rightarrow Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^-(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y , a map $T : X \rightarrow Y$ is said to be *closed* if its graph $\text{Gr}(T) = \{(x, y) : x \in X, y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y .

A map $T : X \rightarrow Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B = \emptyset\}$ is a closed subset of X ; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. map T with closed values is closed.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any nonempty convex or star-shaped subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \rightarrow Y$ is said to be *acyclic* if it is u.s.c. with acyclic compact values.

Recall that a real-valued function $g : X \rightarrow \mathbb{R}$ on a topological space X is *lower* (resp., *upper*) *semicontinuous* (l.s.c.) (resp., u.s.c.) if $\{x \in X : g(x) > r\}$ (resp., $\{x \in X : g(x) < r\}$) is open for each $r \in \mathbb{R}$. If X is a convex set in a vector space, then $g : X \rightarrow \mathbb{R}$ is *quasiconcave* (resp., *quasiconvex*) if $\{x \in X : g(x) > r\}$ (resp., $\{x \in X : g(x) < r\}$) is convex for each $r \in \mathbb{R}$.

BERGE'S THEOREM (see [5]). *Let X and Y be topological spaces, $f : X \times Y \rightarrow \mathbb{R}$ a real function, $F : X \rightarrow Y$ a multimap, and*

$$\hat{f}(x) = \sup_{y \in F(x)} f(x, y), \quad G(x) = \{y \in F(x) : f(x, y) = \hat{f}(x)\}, \quad \text{for } x \in X. \quad (2.1)$$

- (a) *If f is u.s.c. and F is u.s.c. with compact values, then \hat{f} is u.s.c.*
- (b) *If f is l.s.c. and F is l.s.c, then \hat{f} is l.s.c.*
- (c) *If f is continuous and F is continuous with compact values, then \hat{f} is continuous and G is u.s.c.*

Throughout this paper, all topological spaces are assumed to be Hausdorff, a t.v.s. is a topological vector space, and co and $\overline{}$ denote the convex hull and closure, respectively.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [31]) provided that, for every compact subset K of X and every neighborhood V of the

origin of E , there exists a continuous map $h : K \rightarrow X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E .

Note that every nonempty convex subset of a locally convex t.v.s. is admissible (see Hukuhara [19] and Nagumo [40]). Other examples of admissible t.v.s. are ℓ^p and $L^p(0, 1)$ for $0 < p < 1$, the space $S(0, 1)$ of equivalence classes of measurable functions on $[0, 1]$, the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F -normable t.v.s. is admissible and that every compact convex locally convex subset of a t.v.s. is admissible. For details, see Klee [31], Hadžić [15], Weber [72], and references therein.

The following particular form of our new fixed point theorem in [51, 53] is the basis of our arguments in this paper. We give its proof for completeness.

THEOREM 2.1. *Let E be a t.v.s. and let X be an admissible convex subset of E . Then any compact acyclic map $F : X \rightarrow X$ has a fixed point $x \in X$; that is, $x \in F(x)$.*

PROOF. Let ν be a fundamental system of neighborhoods of the origin of E . Since F is closed and compact, it is sufficient to show that for any $V \in \nu$, there exists an $x_V \in X$ such that $(x_V + V) \cap F(x_V) \neq \emptyset$.

Since $\overline{F(X)}$ is a compact subset of the admissible subset X , there exists a continuous map $h : \overline{F(X)} \rightarrow X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{F(X)}$ and $h(\overline{F(X)}) \subset L \cap X$. Let $M := h(\overline{F(X)})$. Then M is a compact subset of L and hence $P := \text{co}M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{F(X)} \rightarrow P$ and $F|_P : P \rightarrow \overline{F(X)}$. Since h and $F|_P$ are acyclic maps, it is well known (see [53]) that their composition $h(F|_P)$ has a fixed point $x_V \in P$; that is, $x_V \in hF(x_V)$ and hence $x_V = h(y)$ for some $y \in F(x_V)$. Since $y - h(y) \in V$, we have $y \in h(y) + V = x_V + V$. Therefore, $(x_V + V) \cap F(x_V) \neq \emptyset$. This completes the proof. \square

Note that Theorem 2.1 generalizes many of known fixed point theorems even when E is locally convex, especially, for the case F has convex values (see Kakutani [26], Himmelberg [17], and Park [46, 60]).

COROLLARY 2.2. *Let X be an admissible compact convex subset of a t.v.s. E and $F : X \rightarrow E$ a map with closed convex graph. If $F(X) \supset X$, then F has a fixed point.*

PROOF. Let $G := F^{-1}|_X : X \rightarrow X$ be defined by $G(y) = \{x \in X : y \in F(x)\}$ for $y \in X$. Then the graph $\text{Gr}(G)$ of G is the symmetric part of $\text{Gr}(F) \cap (X \times X)$ and, hence, closed and convex. Therefore, G has closed convex values. Now, by Theorem 2.1, G has a fixed point $x \in G(x)$; that is, $x \in F(x)$. This completes the proof. \square

Note that if F is single-valued, Corollary 2.2 reduces to Penot [61, Proposition 1.7] whenever E is locally convex, and to Park [50, Theorem 7] whenever E^* separates points of E .

3. Collectively fixed points. A convex space is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes* (see Lassonde [34]).

LEMMA 3.1 (see [55]). *Let X be a topological space, Y a convex space, and $S, T : X \rightarrow Y$ maps satisfying*

- (1) *for each $x \in X$, $\text{co}S(x) \subset T(x)$;*
- (2) *for each $y \in Y$, $S^-(y)$ is compactly open in X ; or $X = \bigcup \{\text{Int}S^-(y) : y \in Y\}$.*

Then, for any nonempty compact subset K of X , there exists a continuous function $f : K \rightarrow Y$ such that

- (3) *$f(x) \in T(x)$ for each $x \in K$;*
- (4) *$f(K)$ is contained in a polytope of Y ;*
- (5) *for any compact subset L of X containing K , there exists a continuous extension $\hat{f} : L \rightarrow Y$ of f such that $\hat{f}(x) \in T(x)$ for each $x \in L$ and $\hat{f}(L)$ is contained in a polytope of Y .*

LEMMA 3.2 (see [18]). *Let X be a paracompact space, Y a convex space, and $A : X \rightarrow Y$ a map such that $X = \bigcup \{\text{Int}A^-(y) : y \in Y\}$. Then $\text{co}A : X \rightarrow Y$ has a continuous selection $s : X \rightarrow Y$; that is, $s(x) \in \text{co}A(x)$ for all $x \in X$.*

Let $\{X_i\}_{i \in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j \in I} X_j, \quad X^i = \prod_{j \in I \setminus \{i\}} X_j. \quad (3.1)$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x_j^i denote the j th coordinate of x^i . If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its i th coordinate is x_i and, for $j \neq i$, the j th coordinate is x_j^i . Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x in X^i .

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) = \{y_i \in X_i : [x^i, y_i] \in A\}, \quad A(x_i) = \{y^i \in X^i : [y^i, x_i] \in A\}. \quad (3.2)$$

For a family $\{E_i\}_{i \in I}$ of t.v.s., let $E = \prod_{i \in I} E_i$. Similarly $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$ for subsets X_i and K_i of E_i for $i \in I$.

We begin with the following collectively fixed point theorems.

THEOREM 3.3. *Let $\{X_i\}_{i \in I}$ be a family of convex sets, each in a t.v.s. E_i , K_i a nonempty compact subset of X_i , and $T_i : X \rightarrow K_i$ a closed map with convex values for each $i \in I$. If X is admissible in E , then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

PROOF. Define $T : X \rightarrow K$ by $T(x) = \prod_{i \in I} T_i(x)$ for each $x \in X$. Then $T : X \rightarrow X$ is a compact closed map with convex values. Since X is admissible, by Theorem 2.1, T has a fixed point $\hat{x} \in K$; that is, $\hat{x} \in T(\hat{x})$ and hence $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$. \square

REMARKS. (1) If T_i is u.s.c. for each $i \in I$, then T is also u.s.c. This was first shown by Fan [11, Lemma 3].

(2) If each E_i is locally convex, Theorem 3.3 reduces to Idzik [22, Theorem 5] and Kim et al. [28, Theorem 5].

(3) If I is a singleton, Theorem 3.3 reduces to a particular form of Theorem 2.1 for convex-valued maps.

THEOREM 3.4. *Let $\{X_i\}_{i \in I}$ and $\{K_i\}_{i \in I}$ be the same as in Theorem 3.3. For each $i \in I$, let $S_i, T_i : X \rightarrow K_i$ be maps satisfying*

- (1) *for each $x \in X$, $\text{co}S_i(x) \subset T_i(x)$;*
- (2) *$D = \text{co}K \subset \bigcup_{y \in K_i} \text{Int}S_i^-(y)$.*

If D is admissible in E , then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.

PROOF. Since K is compact in E , $D = \text{co}K$ is σ -compact (see Lassonde [35]) and hence Lindelöf. Since D is regular, we know that D is paracompact. Consider $S_i|_D : D \rightarrow K_i$. Note that $D = \bigcup_{y \in K_i} (\text{Int}S_i^-(y)) \cap D$ by (2) and $\text{Int}_D(S_i|_D)^-(y) = (\text{Int}S_i^-(y)) \cap D$ for $y \in K_i$. Therefore, by Lemma 3.2, $(\text{co}S_i)|_D : D \rightarrow X_i$ has a continuous selection $s_i : D \rightarrow K_i$ such that $s_i(x) \in \text{co}S_i(x) \subset T_i(x)$ for each $x \in D$. Define $s : D \rightarrow K$ by $s(x) = \prod_{i \in I} s_i(x)$ for $x \in D$. Since D is an admissible convex subset of E and $s : D \rightarrow D$ is a continuous compact map, by Theorem 2.1, s has a fixed point $\hat{x} \in K$; that is, $\hat{x} = s(\hat{x})$ and hence $\hat{x}_i = s_i(\hat{x}) \in T_i(\hat{x})$ for each $i \in I$. This completes the proof. \square

EXAMPLES. (1) Yannelis and Prabhakar [73, Theorem 3.2]: I is a singleton, E_i is locally convex, $S_i = T_i$, and X_i is paracompact.

(2) Ding, Kim, and Tan [10, Theorem 2]: each E_i is locally convex.

(3) Husain and Tarafdar [20, Theorem 2.2]: each E_i is locally convex.

If all X_i 's are compact in Theorem 3.4, we do not need the admissibility of X as follows.

THEOREM 3.5. *Let $\{X_i\}_{i \in I}$ be a family of compact convex spaces and, for each $i \in I$, let $S_i, T_i : X \rightarrow X_i$ be maps satisfying*

- (1) *$\text{co}S_i(x) \subset T_i(x)$ for each $x \in X$; and*
- (2) *$X = \bigcup_{y \in X_i} \text{Int}S_i^-(y)$.*

Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$.

PROOF. By Lemma 3.1, for each $i \in I$, T_i has a continuous selection $f_i : X \rightarrow K_i$, where K_i is a polytope in X_i . Note that each K_i is a compact convex subset of a finite dimensional space E_i , which is a locally convex t.v.s. Define a map $f : K \rightarrow K$ by $f(x) = \prod_{i \in I} f_i(x)$ for $x \in K$. Note that f is continuous and that K is admissible as a convex subset of a locally convex t.v.s. E . Therefore, by Theorem 2.1, we have a fixed point $\hat{x} \in X$ of f ; that is, $\hat{x}_i = f_i(\hat{x}) \in T_i(\hat{x})$ for all $i \in I$. This completes the proof. \square

EXAMPLES. (1) If I is a singleton and $S_i = T_i$, then Theorem 3.5 reduces to the well-known Fan-Browder fixed point theorem (see Park [47]).

(2) For the case I is a singleton, Theorem 3.5 was due to Ben-El-Mechaiekh et al. [4, Theorem 1] and Simons [64, Theorem 4.3]. This was extended by many authors (see Park [47]).

4. Intersection theorems for sets with convex sections. The collectively fixed point theorems in Section 3 can be reformulated to generalize various von Neumann type intersection theorems for sets with convex sections as follows.

THEOREM 4.1. *Let $\{X_i\}_{i \in I}$ be a family of convex sets, each in a t.v.s. E_i , K_i a nonempty compact subset of X_i , and A_i a closed subset of X such that $A_i(x^i)$ is a nonempty convex subset of K_i for each $x^i \in X^i$ and each $i \in I$. If X is admissible in E , then $\bigcap_{i \in I} A_i \neq \emptyset$.*

PROOF. We use Theorem 3.3 with $T_i : X \rightarrow K_i$ defined by $T_i(x) = A_i(x^i)$ for $x \in X$. Then, for each $x \in X$, we have

$$\begin{aligned} (x, y) \in \text{Gr}(T_i) &\iff (x_i, x^i) \in X_i \times X^i \text{ and } y \in A_i(x^i) \subset K_i \\ &\iff (x_i, x^i, y) \in X_i \times (A_i \cap (X^i \times K_i)), \end{aligned} \quad (4.1)$$

which implies that $\text{Gr}(T_i)$ is closed in $X \times K_i$. Hence, each T_i is a closed map with nonempty convex values. Therefore, by Theorem 3.3, there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$. Since $\hat{x}_i \in K_i \subset X_i$, we have $\hat{x} = [\hat{x}^i, \hat{x}_i] \in A_i$ for all $i \in I$. This completes the proof. \square

EXAMPLES. (1) von Neumann [71]: $I = \{1, 2\}$ and E_i are Euclidean.

(2) Fan [11, Theorem 2]: E_i are locally convex and $X_i = K_i$ for all $i \in I$. This result was applied in [11] to obtain a minimax theorem generalizing von Neumann's and Ville's.

(3) Idzik [22, Corollary 1]: each E_i is locally convex.

Note that, as the von Neumann intersection theorem is equivalent to Kakutani's fixed point theorem, Theorems 3.3 and 4.1 are easily seen to be equivalent.

From Theorem 3.4, we have the following equivalent form.

THEOREM 4.2. Let $\{X_i\}_{i \in I}$ and $\{K_i\}_{i \in I}$ be the same as in Theorem 4.1. For each $i \in I$, A_i and B_i are subsets of X satisfying

- (1) for each $x^i \in X^i$, $\emptyset = \text{co}B_i(x^i) \subset A_i(x^i) \subset K_i$; and
- (2) for each $y \in K_i$, $B_i(y)$ is open in X^i .

If $\text{co}K$ is admissible in E , then we have $\bigcap_{i \in I} A_i \neq \emptyset$.

PROOF. We apply Theorem 3.4 with $S_i, T_i : X \rightarrow K_i$ given by $S_i(x) = B_i(x^i)$ and $T_i(x) = A_i(x^i)$ for each $x \in X$. Then, for each $i \in I$, we have the following:

- (a) for each $x \in X$, we have $\text{co}S_i(x) \subset T_i(x) \subset K_i$;
- (b) for each $y \in K_i$,

$$x \in S_i^-(y) \iff y \in S_i(x) \iff (x, y) \in \text{Gr}(S_i) \subset X \times K_i \quad (4.2)$$

and, on the other hand,

$$x \in S_i^-(y) \iff y \in S_i(x) = B_i(x^i) \iff (x^i, y) \in B_i. \quad (4.3)$$

Hence,

$$S_i^-(y) = \{x = [x^i, x_i] \in X : x^i \in B_i(y), x_i \in X_i\} = B_i(y) \times X_i. \quad (4.4)$$

Note that $S_i^-(y)$ is open in $X = X^i \times X_i$.

Therefore, by Theorem 3.4, there exists an $\hat{x} \in K$ such that $\hat{x} \in T\hat{x} = \prod_{i \in I} T_i(\hat{x})$; that is, $\hat{x}_i \in T_i(\hat{x}) = A_i(\hat{x}^i)$ for all $i \in I$. Hence, $\hat{x} = [\hat{x}^i, \hat{x}_i] \in \bigcap_{i \in I} A_i \neq \emptyset$. This completes the proof. \square

From Theorem 3.5, we have the following equivalent form.

THEOREM 4.3. *Let $\{X_i\}_{i \in I}$ be a family of compact convex spaces and, for each $i \in I$, let A_i and B_i are subsets of X satisfying the following:*

- (1) *for each $x^i \in X^i$, $\emptyset = \text{co}B_i(x^i) \subset A_i(x^i)$; and*
- (2) *for each $y \in X_i$, $B_i(y)$ is open in X^i .*

Then we have $\bigcap_{i \in I} A_i \neq \emptyset$.

PROOF. Apply Theorem 3.5 instead of Theorem 3.4 and follow the proof of Theorem 4.2. □

EXAMPLES. (1) Fan [12, Théorème 1]: I is finite and $A_i = B_i$ for all $i \in I$.

(2) Fan [13, Theorem 1’]: $I = \{1, 2\}$ and $A_i = B_i$ for all $i \in I$.

From these results, Fan [13] deduced an analytic formulation, fixed point theorems, extension theorems of monotone sets, and extension theorems for invariant vector subspaces.

(3) Ma [37, Theorem 2]: $A_i = B_i$ for all $i \in I$.

(4) Chang [9, Theorem 4.2] first obtained Theorem 4.3 with a different proof. She also obtained a noncompact version of Theorem 4.3 as [9, Theorem 4.3].

5. Quasi equilibrium problems. Theorem 3.3 can be reformulated to the form of a quasi-equilibrium theorem as follows.

THEOREM 5.1. *Let $\{X_i\}_{i \in I}$ be a family of convex sets, each in a t.v.s. E_i , K_i a nonempty compact subset of X_i , $S_i : X \rightarrow K_i$ a closed map, and $f_i, g_i : X = X^i \times X_i \rightarrow \mathbb{R}$ u.s.c. functions for each $i \in I$. Suppose that for each $i \in I$,*

- (1) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*
- (2) *the function M_i , defined on X by*

$$M_i(x) = \max_{y \in S_i(x)} g_i(x^i, y), \tag{5.1}$$

is l.s.c.; and

- (3) *for each $x \in X$, the set*

$$\{y \in S_i(x) : f_i(x^i, y) \geq M_i(x)\}, \tag{5.2}$$

is convex.

If X is admissible in E , then there exists an $\hat{x} \in K$ such that for each $i \in I$,

$$\hat{x}_i \in S_i(\hat{x}), \quad f_i(\hat{x}^i, \hat{x}_i) \geq M_i(\hat{x}). \tag{5.3}$$

PROOF. For each $i \in I$, define a map $T_i : X \rightarrow K_i$ by

$$T_i(x) = \{y \in S_i(x) : f_i(x^i, y) \geq M_i(x)\}, \tag{5.4}$$

for $x \in X$. Note that each $T_i(x)$ is nonempty by (1) since $S_i(x)$ is compact and $g_i(x^i, \cdot)$ is u.s.c. on $S_i(x)$. We show that $\text{Gr}(T_i)$ is closed in $X \times K_i$. In fact, let $(x_\alpha, y_\alpha) \in \text{Gr}(T_i)$ and $(x_\alpha, y_\alpha) \rightarrow (x, y)$. Then

$$\begin{aligned} f_i(x^i, y) &\geq \overline{\lim}_\alpha f_i(x_\alpha^i, y_\alpha) \geq \overline{\lim}_\alpha M_i(x_\alpha) \\ &\geq \underline{\lim}_\alpha M_i(x_\alpha) \geq M_i(x) \end{aligned} \tag{5.5}$$

and, since $\text{Gr}(S_i)$ is closed in $X \times K_i$, $y_\alpha \in S_i(x_\alpha)$ implies $y \in S_i(x)$. Hence $(x, y) \in \text{Gr}(T_i)$. Now we apply Theorem 3.3. Then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$; that is, $\hat{x}_i \in S_i(\hat{x})$ and $f_i(\hat{x}^i, \hat{x}_i) \geq M_i(\hat{x})$. This completes the proof. \square

EXAMPLE. Idzik [21, Theorem 7], Marchi and Martínez-Legas [38, Theorem 5.1, Corollary 5.1]: each E_i is locally convex, $f_i = g_i$ is continuous, and S_i is continuous. In this case, (2) follows from Berge's theorem.

If $f_i = g_i = 0$, then Theorem 5.1 reduces to Theorem 3.3.

We have another quasi-equilibrium theorem equivalent to Theorem 2.1 by following the proof of Theorem 5.1.

THEOREM 5.2. *Let X be an admissible convex subset of a t.v.s. E , $f : X \times X \rightarrow \mathbb{R}$ an u.s.c. function, and $S : X \rightarrow X$ a compact closed map. Suppose that*

(1) *the function M defined on X by*

$$M(x) = \max_{y \in S(x)} f(x, y) \quad \text{for } x \in X, \quad (5.6)$$

is l.s.c.; and

(2) *for each $x \in X$, the set*

$$\{y \in S(x) : f(x, y) = M(x)\} \quad (5.7)$$

is acyclic.

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in S(\hat{x}), \quad f(\hat{x}, \hat{x}) = M(\hat{x}). \quad (5.8)$$

EXAMPLES. (1) If $f(x, y) = 0$ for all $x, y \in X$, then Theorem 5.2 reduces to Theorem 2.1. If f and S are continuous, then condition (1) holds by Berge's theorem.

(2) For a locally convex t.v.s. E , particular forms of Theorem 5.2 were obtained by Takahashi [67, Theorem 4] and Im and Kim [24, Theorem 1]. Those authors applied their results to best approximation problems and optimization problems, respectively (see also Park [48] and Park and Chen [56]).

6. Applications to variational inequalities. Theorem 5.2 can be used to give simple proofs of the variational inequalities of the Hartman-Stampacchia-Browder type as follows:

(i) Hartman and Stampacchia [16, Lemma 3.1]: *let K be a compact convex set in \mathbb{R}^n and $B : K \rightarrow \mathbb{R}^n$ a continuous map. Then there exists a $u_0 \in K$ such that*

$$\langle B(u_0), v - u_0 \rangle \geq 0 \quad \forall v \in K. \quad (6.1)$$

Put $X = K$, $f(x, y) = \langle B(x), -y \rangle$, $S(x) = K$ for $x, y \in K$, and apply Theorem 5.2.

(ii) Browder [6, Theorem 3], [7, Theorem 2]: *let E be a t.v.s. on which its topological dual E^* is equipped with a topology such that the pairing $\langle \cdot, \cdot \rangle : E^* \times E \rightarrow \mathbb{R}$ is continuous. Let K be an admissible compact convex subset of E , and $T : K \rightarrow E^*$ continuous. Then there exists a $u_0 \in K$ such that*

$$\langle T(u_0), v - u_0 \rangle \geq 0 \quad \forall v \in K. \quad (6.2)$$

Apply Theorem 5.2 as in (i).

(iii) Lions and Stampacchia [36], Stampacchia [66], and Mosco [39]: *let V be an inner product space, X a compact convex subset of V , and $a : V \times V \rightarrow \mathbb{R}$ a continuous bilinear form on V . Then for every $v' \in V^*$, there exists a (unique) vector $u \in X$ such that*

$$a(u, u - w) \leq \langle v', u - w \rangle \quad \forall w \in X. \tag{6.3}$$

Put $E = V$, $S(x) = X$ for $x \in X$,

$$f(u, w) = a(u, -w) - \langle v', -w \rangle \quad \forall u, w \in X, \tag{6.4}$$

and apply Theorem 5.2.

(iv) Karamardian [27, Lemma 3.2]: *let X be an admissible compact convex subset of a t.v.s. E , F a topological space, $g : X \rightarrow F$ a function, and $\psi : X \times F \rightarrow \mathbb{R}$ a function. If for each $y \in F$, $\psi(\cdot, y)$ is quasiconvex on X and the function $(u, v) \mapsto \psi(u, g(v))$ is continuous on $X \times X$, then there exists an $\bar{x} \in X$ such that*

$$\psi(\bar{x}, g(\bar{x})) \leq \psi(x, g(\bar{x})) \quad \forall x \in X. \tag{6.5}$$

Put $S(x) = X$, $f(x, y) = -\psi(y, g(x))$ for $x, y \in X$, and apply Theorem 5.2.

Note that Karamardian [27] applied (iv) to obtain a variational inequality (v) below, Fan's best approximation theorem, and a solution of the generalized complementarity theorem [27, Theorem 3.1].

(v) Karamardian [27, Corollary 3.1], Juberg and Karamardian [25, Lemma], Park [45, Corollary 1.3]: *let X be an admissible compact convex subset of a t.v.s. E , F a topological space, and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbb{R}$ a function which is linear in the second variable. Suppose that $g : X \rightarrow F$ is a function such that $(x, y) \mapsto \langle g(x), y \rangle$ is continuous on $X \times E$. Then there exists an $\bar{x} \in X$ such that*

$$\langle g(\bar{x}), y - \bar{x} \rangle \geq 0 \quad \forall y \in X. \tag{6.6}$$

Put $S(x) = X$, $f(x, y) = \langle g(x), -y \rangle$ for $x, y \in X$, and apply Theorem 5.2.

(vi) Parida, Sahoo, and Kumar [43, Theorem 3.1], Behera and Panda [1, Theorem 2.2], Siddiqi, Khaliq, and Ansari [63]: *let X be an admissible compact convex subset of a t.v.s. E on which E^* is equipped with a topology such that the pairing $\langle \cdot, \cdot \rangle : E^* \times E \rightarrow \mathbb{R}$ is continuous, $T : X \rightarrow E^*$ and $\theta : X \times X \rightarrow E$ continuous maps such that*

- (1) $\langle T(y), \theta(y, y) \rangle \geq 0$ for all $y \in X$; and
- (2) for each $y \in X$, the function $\langle Ty, \theta(\cdot, y) \rangle : X \rightarrow \mathbb{R}$ is quasiconvex.

Then there exists an $x_0 \in X$ such that

$$\langle T(x_0), \theta(y, x_0) \rangle \geq 0 \quad \forall y \in X. \tag{6.7}$$

Put $S(x) = X$, $f(x, y) = -\langle T(x), \theta(y, x) \rangle$ for $x, y \in X$, and apply Theorem 5.2.

REMARKS. (1) Note that the statements (ii)-(vi) are more general than the original ones.

(2) In the frame of the KKM theory, some of (i)-(vi) can be obtained without assuming the admissibility. However, in this section, we want to show the applicability of Theorem 2.1.

(3) In [59], using the Idzik fixed point theorem [23], different versions of results of this section were given.

7. More variational inequalities. From Theorem 3.4 we have the following.

THEOREM 7.1. *Let X be a convex subset of a t.v.s. E , K a nonempty compact subset of X , and $S, T : X \rightarrow K$ a map such that*

- (1) *for each $x \in X$, $\text{co}S(x) \subset T(x)$; and*
- (2) *$\{\text{Int}S^-(y)\}_{y \in K}$ covers X .*

If $\text{co}K$ is admissible in E , then T has a fixed point.

REMARK. Theorem 7.1 for $S = T$ is due to Ben-El-Mechaiekh et al. [3, Theorem 3.2] whenever E is locally convex. Ben-El-Mechaiekh [2] raised a question whether the local convexity can be eliminated. Later, Kim and Tan [30], Zhang [74], Chang and Zhang [8] used Theorem 7.1 for $S = T$ for a locally convex t.v.s. to obtain a type of variational inequalities.

From Theorem 7.1, we deduce the following equilibrium theorem.

THEOREM 7.2. *Let X be a convex subset of a t.v.s., $p, q : X \times X \rightarrow \mathbb{R}$ functions, and K a nonempty compact subset of X . Suppose that*

- (1) *$q(x, x) \leq 0$ for all $x \in K$;*
- (2) *for each $y \in K$, $\{x \in X : p(x, y) > 0\}$ is open in X ;*
- (3) *for each $x \in X$, $\{y \in K : q(x, y) > 0\} \supset \text{co}\{y \in K : p(x, y) > 0\}$; and*
- (4) *$p(x, y) \leq 0$ for all $x \in X$ and $y \in X \setminus K$.*

If $\text{co}K$ is admissible, then there exists an $x_0 \in X$ such that

$$p(x_0, y) \leq 0 \quad \forall y \in X. \quad (7.1)$$

PROOF. Suppose that for each $x \in X$, there exists a $y \in K$ such that $p(x, y) > 0$ and hence $q(x, y) > 0$ by (3). Define $S, T : X \rightarrow K$ by

$$S(x) = \{y \in K : p(x, y) > 0\}, \quad T(x) = \{y \in K : q(x, y) > 0\} \quad (7.2)$$

for each $x \in X$. Then

- (i) for each $x \in X$, $\text{co}S(x) \subset T(x)$ by (3); and
- (ii) for each $x \in X$, there exists a $y \in K$ such that $y \in S(x)$ or $x \in S^-(y) = \{x \in X : p(x, y) > 0\}$. Since $S^-(y)$ is open in X by (2), X is covered by $\{\text{Int}S^-(y)\}_{y \in K}$. Therefore, by Theorem 7.1, T has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T(\hat{x})$. Hence, $\hat{x} \in K$ and $q(\hat{x}, \hat{x}) > 0$. This contradicts (1). Therefore, there exists an $x_0 \in X$ such that

$$p(x_0, y) \leq 0 \quad \forall y \in K. \quad (7.3)$$

However, this inequality holds for all $y \in X$ because of (4). □

Let \mathbb{K} be the real field \mathbb{R} or the complex field \mathbb{C} . In order to obtain variational inequalities related to multimaps, we need the following simple consequence of Berge's theorem.

LEMMA 7.3. *Let E be a t.v.s. over \mathbb{K} , X a nonempty subset of E , F a topological space, $T : X \rightarrow F$ an u.s.c. map with compact values, and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbb{K}$ a function such that for each $y \in E$, $(f, x) \mapsto \text{Re}\langle f, x - y \rangle$ is l.s.c. on $F \times X$. Then for each $y \in E$, the function*

$$x \mapsto \inf_{f \in T(x)} \operatorname{Re}\langle f, x - y \rangle \tag{7.4}$$

is l.s.c. on X .

REMARK. Lemma 7.3 was recently given by Park and Chen [57, Lemma 2]. From Lemma 7.3, we deduce the following.

LEMMA 7.4. *Let E be a t.v.s. over \mathbb{K} , F a vector space over \mathbb{K} , and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbb{K}$ a bilinear function. Let X be a nonempty bounded subset of E such that, for each $z \in F$, $y \mapsto \langle z, y \rangle$ is continuous on X . Suppose that F has the $\eta(F, E)$ -topology; that is, the topology of uniform convergence on bounded subsets of E , and $T : X \rightarrow F$ is u.s.c. with compact values. Then for each $y \in E$, the function*

$$x \mapsto \inf_{f \in T(x)} \operatorname{Re}\langle f, x - y \rangle \tag{7.5}$$

is l.s.c. on X .

PROOF. As in Kum [33, Lemma B], the pairing $\langle \cdot, \cdot \rangle : F \times X \rightarrow \mathbb{K}$ is continuous. Therefore, by Lemma 7.3, the proof is completed. \square

REMARKS. (1) Note that if $F = E^*$, the topological dual of E , then $y \mapsto \langle z, y \rangle$ is obviously continuous for each $z \in E^*$.

(2) Particular forms of Lemma 7.4 were due to Browder [6, Lemma 1], Shih and Tan [62, Lemma 1], Kim and Tan [30, Lemma 2], and Chang and Zhang [8, Lemma 2] with proofs more lengthy than ours.

The following is the main result of Section 7.

THEOREM 7.5. *Let X be a bounded convex subset of a t.v.s. E over \mathbb{K} , \mathbb{K} a nonempty compact subset of X , F a vector space over \mathbb{K} with the $\eta(F, E)$ -topology, and $\langle \cdot, \cdot \rangle : F \times E \rightarrow \mathbb{K}$ a bilinear function such that, for each $z \in F$, $y \mapsto \langle z, y \rangle$ is continuous on X . Let $T : X \rightarrow F$ be an u.s.c. map with compact values, and $\alpha : X \times X \rightarrow \mathbb{R}$ a function such that*

- (1) for each $x \in X$, $\alpha(x, x) = 0$, $\alpha(x, \cdot)$ is concave, and $\alpha(\cdot, x)$ is l.s.c.; and
- (2) for each $x \in X$ and $y \in X \setminus K$,

$$\inf_{f \in T(x)} \operatorname{Re}\langle f, x - y \rangle + \alpha(x, y) \leq 0. \tag{7.6}$$

If $\operatorname{co}K$ is admissible, then there exists an $x_0 \in X$ such that

$$\inf_{f \in T(x_0)} \operatorname{Re}\langle f, x_0 - y \rangle + \alpha(x_0, y) \leq 0 \quad \forall y \in X. \tag{7.7}$$

Moreover, the set of all solutions x_0 is a closed subset of X . Further if $T(x_0)$ is convex and $\alpha(x_0, \cdot)$ is linear, then there exists an $f_0 \in T(x_0)$ such that

$$\operatorname{Re}\langle f_0, x_0 - y \rangle + \alpha(x_0, y) \leq 0 \quad \forall y \in X. \tag{7.8}$$

PROOF. We use Theorem 7.2 with $p = q$. Let

$$p(x, y) = \inf_{f \in T(x)} \operatorname{Re}\langle f, x - y \rangle + \alpha(x, y). \tag{7.9}$$

Then we have the following:

- (i) $p(x, x) = 0$ by the property of $\langle \cdot, \cdot \rangle$ and (1).
- (ii) For each $y \in \mathbb{K}$, $\{x \in X : p(x, y) > 0\}$ is open in X since $x \mapsto p(x, y)$ is l.s.c. on X by Lemma 7.4 and (1).
- (iii) For each $x \in X$, $\{y \in K : p(x, y) > 0\}$ is convex in \mathbb{K} . In fact, for any $y_1, y_2 \in \mathbb{K}$ satisfying $p(x, y_1) > 0$ and $p(x, y_2) > 0$, let $y = ty_1 + (1-t)y_2$ for some $t \in (0, 1)$. Then

$$\begin{aligned} p(x, y) &= \inf_{f \in T(x)} \operatorname{Re} \langle f, x - (ty_1 + (1-t)y_2) \rangle + \alpha(x, ty_1 + (1-t)y_2) \\ &\geq t \inf_{f \in T(x)} \operatorname{Re} \langle f, x - y_1 \rangle + (1-t) \inf_{f \in T(x)} \operatorname{Re} \langle f, x - y_2 \rangle + t\alpha(x, y_1) + (1-t)\alpha(x, y_2) \\ &= tp(x, y_1) + (1-t)p(x, y_2) > 0. \end{aligned} \tag{7.10}$$

Note that $y \in K$ by (2).

- (iv) For each $y \in X \setminus K$ and $x \in X$, we have $p(x, y) \leq 0$ by (2).

Therefore by Theorem 7.2, there exists an $x_0 \in X$ such that

$$p(x_0, y) \leq 0 \quad \forall y \in X. \tag{7.11}$$

Moreover, the set of all solutions x_0 is

$$\bigcap_{y \in X} \{x \in X : p(x, y) \leq 0\}, \tag{7.12}$$

which is the intersection of nonempty closed sets by (ii).

To prove the final assertion, suppose that $T(x_0)$ is convex and $\alpha(x_0, \cdot)$ is linear on X . We define a function $g : T(x_0) \times X \rightarrow \mathbb{R}$ by

$$g(f, y) = \operatorname{Re} \langle f, x_0 - y \rangle + \alpha(x_0, y) \quad \text{for } (f, y) \in T(x_0) \times X. \tag{7.13}$$

Then g is linear in $f \in T(x_0)$ and in $y \in X$. Note that for a given $y \in X$, $f \mapsto g(f, y)$ is continuous on F with the $\eta(F, E)$ -topology. Therefore, by the Kneser minimax theorem [32], we have

$$\inf_{f \in T(x_0)} \sup_{y \in X} g(f, y) = \sup_{y \in X} \min_{f \in T(x_0)} g(f, y). \tag{7.14}$$

Since the right-hand side of (7.14) is less than or equal to zero by the first conclusion, we have

$$\inf_{f \in T(x_0)} \sup_{y \in X} g(f, y) \leq 0. \tag{7.15}$$

Since $f \mapsto \sup_{y \in X} g(f, y)$ is l.s.c. and Tx_0 is compact, there exists an $f_0 \in T(x_0)$ such that $\sup_{y \in X} g(f_0, y) \leq 0$. This completes the proof. \square

REMARKS. (1) For a locally convex t.v.s. E , Theorem 7.5 reduces to Chang and Zhang [8, Theorem 1] and Zhang [74, Theorem 3].

(2) In case E is locally convex, $F = E^*$, and $\alpha = 0$, Theorem 7.5 reduces to Kim and Tan [30, Theorem 1], which extends Browder [7, Theorem 6].

(3) Note that in some cases we can choose a topology on F different from $\eta(F, E)$ and the assumption on the boundedness of X can be removed from Theorem 7.5. For example, if we choose the topology $\sigma(F, E)$ (see [57]) or in the case of a normed vector space E and $F = E^*$ (see [30, Theorem 3]), we need not to assume the boundedness of X .

For a subset X of a t.v.s. E , the *inward set* $I_X(x)$ of X at $x \in E$ is defined by

$$I_X(x) = \{x + r(y - x) : r > 0, y \in X\}, \tag{7.16}$$

and $\bar{I}_X(x)$ denotes its closure.

For a locally convex t.v.s. E , $F = E^*$, and $\alpha = 0$, we have the following from Theorem 7.5 [58].

THEOREM 7.6. *Let X be a bounded convex subset of a locally convex t.v.s. E , K a nonempty compact subset of X , and $T : X \rightarrow E^*$ a continuous map, where E^* has the $\eta(E^*, E)$ -topology. Suppose that for each $x \in X$ and $y \in X \setminus K$, we have*

$$\operatorname{Re}\langle T(x), x - y \rangle \leq 0. \tag{7.17}$$

Then there exists an $x_0 \in X$ such that

$$\operatorname{Re}\langle T(x_0), x_0 - y \rangle \leq 0 \quad \forall y \in \bar{I}_X(x_0). \tag{7.18}$$

Moreover, the set of all solutions x_0 is a closed subset of X .

REMARKS. (1) Theorem 7.6 is due to Park and Kang [58, Corollary] and strengthens [30, Corollary 2].

(2) In [58], Theorem 7.6 was used to obtain a far-reaching generalization of fixed point theorems of Kim and Tan [30].

8. The Nash type equilibrium theorems. From the intersection Theorem 4.3, we can deduce the following equivalent form of a generalized Fan type minimax theorem.

THEOREM 8.1. *Let $\{X_i\}_{i \in I}$ be a family of compact convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \rightarrow \mathbb{R}$ functions satisfying*

- (1) $g_i(x) \leq f_i(x)$ for each $x \in X$;
- (2) for each $x^i \in X^i$, $x_i \mapsto f_i(x^i, x_i)$ is quasiconcave on X_i ; and
- (3) for each $x_i \in X_i$, $x^i \mapsto g_i(x^i, x_i)$ is l.s.c. on X^i .

Let $\{t_i\}_{i \in I}$ be a family of real numbers. Then either

- (a) there exists an $i \in I$ and an $x^i \in X^i$ such that

$$g_i(x^i, x_i) \leq t_i \quad \forall x_i \in X_i; \tag{8.1}$$

or (b) there exists an $x \in X$ such that

$$f_i(x) > t_i \quad \forall i \in I. \tag{8.2}$$

PROOF. Suppose that (a) does not hold; that is, for any $i \in I$ and any $x^i \in X^i$, there exists an $x_i \in X_i$ such that $g_i(x^i, x_i) > t_i$. Let

$$A_i = \{x \in X : f_i(x) > t_i\}, \quad B_i = \{x \in X : g_i(x) > t_i\} \tag{8.3}$$

for each $i \in I$. Then

- (4) for each $x^i \in X^i$, $\emptyset = B_i(x^i) \subset A_i(x^i)$;
- (5) for each $x^i \in X^i$, $A_i(x^i)$ is convex; and
- (6) for each $y \in X_i$, $B_i(y)$ is open in X^i .

Therefore, by Theorem 4.3, there exists an $x \in \bigcap_{i \in I} A_i$. This is equivalent to (b). \square

EXAMPLES. (1) Fan [12, Theorem 2], [13, Theorem 3]: I is finite and $f_i = g_i$ for all $i \in I$. From this, Fan [12, 13] deduced Sion's minimax theorem [65], the Tychonoff fixed point theorem, solutions to systems of convex inequalities, extremum problems for matrices, and a theorem of Hardy-Littlewood-Pólya.

- (2) Ma [37, Theorem 3]: $f_i = g_i$ for all $i \in I$.

REMARKS. (1) We obtained Theorem 8.1 from Theorem 4.3. As was pointed out by Fan [12] for his case, we can deduce Theorem 4.3 from Theorem 8.1 by considering the characteristic functions of the sets A_i and B_i .

- (2) The conclusion of Theorem 8.1 can be stated as follows: if

$$\min_{x_i \in X_i} \sup_{x^i \in X^i} g_i(x^i, x_i) > t_i \quad \forall i \in I, \quad (8.4)$$

then (b) holds (see Fan [12, 13]).

From Theorem 4.3, we also obtain the following generalization of the Nash-Ma type equilibrium theorems.

THEOREM 8.2. *Let $\{X_i\}$ be a family of compact convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \rightarrow \mathbb{R}$ be functions such that*

- (1) $g_i(x) \leq f_i(x)$ for each $x \in X$;
- (2) for each $x^i \in X^i$, $x_i \mapsto f_i(x^i, x_i)$ is quasiconcave on X_i ;
- (3) for each $x^i \in X^i$, $x_i \mapsto g_i(x^i, x_i)$ is u.s.c. on X_i ; and
- (4) for each $x_i \in X_i$, $x^i \mapsto g_i(x^i, x_i)$ is l.s.c. on X^i .

Then there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i(\hat{x}^i, y_i) \quad \forall i \in I. \quad (8.5)$$

PROOF. For any $\varepsilon > 0$, we define

$$\begin{aligned} A_{\varepsilon,i} &= \{x \in X : f_i(x) > \max_{y_i \in X_i} g_i(x^i, y_i) - \varepsilon\}, \\ B_{\varepsilon,i} &= \{x \in X : g_i(x) > \max_{y_i \in X_i} g_i(x^i, y_i) - \varepsilon\} \end{aligned} \quad (8.6)$$

for each i . Then

- (1) for each $x^i \in X^i$, $B_{\varepsilon,i}(x^i) \subset A_{\varepsilon,i}(x^i)$;
- (2) for each $x^i \in X^i$, $A_{\varepsilon,i}(x^i)$ is convex;
- (3) for each $x^i \in X^i$, $B_{\varepsilon,i}(x_i) = \emptyset$ since $x_i \mapsto g_i(x^i, x_i)$ is u.s.c. on the compact space X_i ; and
- (4) for each $x_i \in X_i$, $B_{\varepsilon,i}(x_i)$ is open since $x^i \mapsto g_i(x^i, x_i)$ is l.s.c. on X^i .

Therefore, by applying Theorem 4.3, we have

$$\bigcap_{i \in I} A_{\varepsilon, i} \neq \emptyset \quad \forall \varepsilon > 0. \quad (8.7)$$

Since X is compact, there exists an $\hat{x} \in X$ such that $f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i(\hat{x}^i, y_i)$ for all $i \in I$. \square

EXAMPLES. (1) In case of $f_i = g_i$ and for a finite I , Theorem 8.2 reduces to Tan et al. [68, Theorem 2.1].

(2) From Theorem 8.2, we obtain the following generalization of the Nash equilibrium theorem due to Ma [37, Theorem 4].

THEOREM 8.3. *Let $\{X_i\}_{i \in I}$ be a family of compact convex spaces and, for each $i \in I$, let $f_i : X \rightarrow \mathbb{R}$ be a continuous function such that for each $x^i \in X^i$, $x_i \mapsto f_i(x^i, x_i)$ is quasiconcave on X_i . Then there exists a point $\hat{x} \in X$ such that*

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i(\hat{x}^i, y_i) \quad \forall i \in I. \quad (8.8)$$

EXAMPLES. (1) Nash [41, Theorem 1]: I is finite and X_i are subsets of Euclidean spaces.

(2) Nikaido and Isoda [42, Theorem 3.2]: I is finite.

(3) Fan [13, Theorem 4]: I is finite.

REMARK. In view of Theorem 8.2, the continuity of f_i in Theorem 8.3 can be weakened to component-wise u.s.c. and l.s.c., as Sion [65] generalized von Neumann's min-max theorem [70].

Particular forms of results in this section can also be seen in Browder [7].

9. The Tarafdar type equilibrium theorems. In this section, we apply Theorem 3.4 to the existence of equilibrium points and maximal elements of an abstract economy.

An *abstract economy* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ consists of an index set I of agents, a choice set X_i in a t.v.s. E_i , constraint correspondences $A_i, B_i : X = \prod_{i \in I} X_i \rightarrow X_i$, and a preference correspondence $P_i : X \rightarrow X_i$ for each $i \in I$. An *equilibrium point* $x = \{x_i\}_{i \in I} \in X$ is the one satisfying $x_i \in B_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for each $i \in I$. We say that $x \in X$ is a *maximal point* of the game $(X_i, P_i)_{i \in I}$ if $P_i(x) = \emptyset$ for each $i \in I$.

THEOREM 9.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that, for each $i \in I$,*

- (1) X_i is a convex subset of a t.v.s. E_i and K_i is a nonempty compact subset of X_i ;
- (2) for each $x \in X$, $\text{co} A_i(x) \subset B_i(x) \subset K_i$;
- (3) $D = \text{co} K \subset \bigcup_{y \in K_i} \text{Int}(A_i^-(y) \cap (P_i^-(y) \cup F_i))$, where $F_i = \{x \in X : A_i(x) \cap P_i(x) = \emptyset\}$; and
- (4) for each $x = \{x_i\}_{i \in I} \in X$, $x_i \notin \text{co} P_i(x)$.

If D is admissible, Γ has an equilibrium point in K .

PROOF. Let

$$G_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} \quad \forall i \in I. \quad (9.1)$$

For each $i \in I$, we define two maps $S_i, T_i : X \rightarrow K_i$ by

$$\begin{aligned} S_i(x) &= \begin{cases} A_i(x) \cap \text{co}P_i(x) & \text{if } x \in G_i, \\ A_i(x) & \text{if } x \in F_i, \end{cases} \\ T_i(x) &= \begin{cases} B_i(x) \cap \text{co}P_i(x) & \text{if } x \in G_i, \\ B_i(x) & \text{if } x \in F_i. \end{cases} \end{aligned} \quad (9.2)$$

Then for each $i \in I$ and $x \in X$, we have $\emptyset = \text{co}S_i(x) \subset T_i(x)$; and for each $y \in K_i$, we have

$$\begin{aligned} S_i^-(y) &= [(A_i^-(y) \cap (\text{co}P_i)^-(y)) \cap G_i] \cup [A_i^-(y) \cap F_i] \\ &\supset [(A_i^-(y) \cap P_i^-(y)) \cap G_i] \cup [A_i^-(y) \cap F_i] \\ &= [A_i^-(y) \cap P_i^-(y)] \cup [A_i^-(y) \cap F_i] \\ &= A_i^-(y) \cap (P_i^-(y) \cup F_i), \end{aligned} \quad (9.3)$$

which implies $D = \text{co}K \subset \bigcup_{y \in K_i} \text{Int}S_i^-(y)$ by (3). Therefore, all of the requirements of Theorem 3.4 are satisfied. Hence, there exists an $\bar{x} \in K$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$. By (4), $\bar{x}_i \notin \text{co}P_i(\bar{x})$. Therefore, $\bar{x}_i \in B_i(\bar{x})$ for each $i \in I$ by the definition of T_i and hence $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$. This shows that \bar{x} is an equilibrium point of Γ . \square

REMARK. We followed the proof of Tarafdar [69, Theorem 3.1]. If all of E_i 's are locally convex, then Theorem 9.1 reduces to Ding et al. [10, Theorem 5] and Husain and Tarafdar [20, Theorem 3.1].

THEOREM 9.2. Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that, for each $i \in I$,

- (1) X_i is a convex subset of a t.v.s. E_i , and K_i is a nonempty compact convex subset of X_i ;
- (2) $K \subset \bigcup_{y \in K_i} \text{Int}_X(P_i^-(y) \cup F_i)$, where $F_i = \{x \in X : P_i(x) = \emptyset\}$; and
- (3) for each $x = \{x_i\}_{i \in I} \in X$, $x_i \notin \text{co}P_i(x)$.

If $\text{co}K$ is admissible, then Γ has a maximal element in K .

PROOF. For each $i \in I$, define a map $A_i : X \rightarrow K_i$ by $A_i(x) = K_i$ for $x \in X$. Now we can apply Theorem 9.1 with $A_i = B_i$, and the conclusion follows. \square

REMARK. If each E_i is locally convex, then Theorem 9.2 reduces to Husain and Tarafdar [20, Theorem 3.2].

10. The Yannelis-Prabhakar type equilibrium theorems. In this section, we point out that some modifications or generalizations of the Yannelis-Prabhakar type equilibrium theorems [73] can also be obtained in the frame of our method in this paper for not-necessarily locally convex t.v.s.

We list some of them as follows.

THEOREM 10.1. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy such that, for each $i \in I$,

- (1) X_i is a convex subset of a t.v.s. E_i and K_i is a nonempty compact subset of X_i ;

- (2) for each $x \in X$, there exists a nonempty subset $B_i(x)$ such that $\text{co}B_i(x) \subset A_i(x) \subset K_i$;
- (3) $\bar{A}_i : X \rightarrow K_i$ defined by $\bar{A}_i(x) = \overline{A_i(x)}$ for $x \in X$ is a closed map;
- (4) $\bar{P}_i : X \rightarrow X_i$ is a closed correspondence, where $P_i(x)$ is (possibly empty) convex for each $x \in X$;
- (5) the set $W_i = \{x \in X : B_i(x) \cap P_i(x) \neq \emptyset\}$ is a (possibly empty) closed proper subset of X ; and
- (6) for each $x \in W_i$, $x_i \notin \bar{P}_i(x)$.

If X is admissible in E , then Γ has an equilibrium choice $\hat{x} \in K$; more precisely, for each $i \in I$,

$$\hat{x}_i \in \bar{A}_i(\hat{x}), \quad B_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset. \tag{10.1}$$

PROOF. Just follow the proof of Kim et al. [28] or Kim [29] using Theorem 2.1 instead of Himmelberg's. □

EXAMPLE. If each E_i is locally convex, then Theorem 10.1 reduces to Kim et al. [28] and Kim [29].

THEOREM 10.2. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy such that, for each $i \in I$, conditions (1)–(3) of Theorem 10.1 and the following hold.

- (4) for each $y \in K_i$, $B_i^-(y)$ is open in X ;
- (5) for each $y \in X_i$, $P_i^-(y)$ is open in X ;
- (6) for each $x \in X$, $x_i \notin \text{co}P_i(x)$; and
- (7) the set $\{x \in X : \text{co}B_i(x) \cap \text{co}P_i(x) = \emptyset\}$ is paracompact.

If X is admissible in E , then Γ has an equilibrium choice $\hat{x} \in K$ as in the conclusion of Theorem 10.1.

PROOF. Just follow the proof of Ding et al. [10, Theorem 4] using Theorem 2.1 instead of Himmelberg's. □

EXAMPLE. If each E_i is locally convex, then Theorem 10.2 reduces to Ding et al. [10, Theorem 4].

From Theorem 10.2, we immediately have the following.

THEOREM 10.3. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy, where I is countable, such that, for each $i \in I$, the following hold.

- (1) X_i is a nonempty compact convex subset of a t.v.s. E_i ;
- (2) $A_i(x)$ is convex for all $x \in X$;
- (3) the map $\bar{A}_i : X \rightarrow X_i$ defined by $\bar{A}_i(x) = \overline{A_i(x)}$ for all $x \in X$ is closed;
- (4) for each $y \in X_i$, $A_i^-(y)$ is open in X ;
- (5) for each $y \in X_i$, $P_i^-(y)$ is open in X (P may have empty values); and
- (6) $x_i \notin \text{co}P_i(x)$ for all $x \in X$.

If X_i is metrizable for each $i \in I$ and if X is admissible in E , then Γ has an equilibrium choice $\hat{x} \in X$; that is, $\hat{x}_i \in \bar{A}_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$.

EXAMPLE. If each E_i is locally convex, then Theorem 10.3 reduces to Yannelis and Prabhakar [73, Theorem 6.1].

REMARKS. (1) If X_i is not metrizable and X is not admissible, then we may not have an equilibrium in Theorem 10.3. But, even in this case, we have an equilibrium choice for certain “subeconomy.”

Since X is compact, from (2) and (3), there exists a continuous selection $f_i : X \rightarrow K_i$ of $A_i : X \rightarrow X_i$, by Lemma 3.1, where K_i is a polytope in X_i . Note that K_i is metrizable. Now define $A'_i, P'_i : K \rightarrow K_i$ by

$$A'_i(x) = A_i(x) \cap K_i, \quad P'_i(x) = P_i(x) \cap K_i, \quad \text{for } x \in K. \quad (10.2)$$

Note that $A'_i(x)$ is nonempty since $f_i(x) \in A_i(x) \cap K_i = A'_i(x)$. It is clear that the “subeconomy” $\Gamma' = (K_i, A'_i, P'_i)_{i \in I}$ satisfies all of the requirements (1)–(6) replacing (X, X_i, A_i, P_i) by (K, K_i, A'_i, P'_i) . Moreover, X is admissible as a subset of a locally convex t.v.s. as in the proof of Theorem 3.5. Therefore, there exists an $\hat{x} \in K \subset X$ such that

$$\hat{x}_i \in \bar{A}'_i(x) \subset \bar{A}_i(x), \quad P'_i(\hat{x}) \cap A'_i(\hat{x}) = \emptyset, \quad \forall i \in I. \quad (10.3)$$

(2) In [49], different versions of Theorems 8.1, 8.2, 8.3, 9.1, 9.2, 10.1, 10.2, and 10.3 were given by using the Idzik fixed point theorem [56].

(3) Finally, for further applications of Theorem 2.1, the reader may consult [52, 54, 44].

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