Internat. J. Math. & Math. Sci.
Vol. 24, No. 11 (2000) 785–791
S0161171200002325

ON SOME TOPOLOGICAL PROPERTIES OF GENERALIZED DIFFERENCE SEQUENCE SPACES

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(Received 22 October 1998 and in revised form 12 June 2000)

ABSTRACT. We obtain some topological results of the sequence spaces $\Delta^m(X)$, where $\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$, $(m \in \mathbb{N})$, and X is any sequence space. We compute the $p\alpha$-, $p\beta$-, and $p\gamma$-duals of l_∞, c, and c_0 and we investigate the N-(or null) dual of the sequence spaces $\Delta^m(l_\infty), \Delta^m(c)$, and $\Delta^m(c_0)$. Also we show that any matrix map from $\Delta^m(l_\infty)$ into a BK-space which does not contain any subspace isomorphic to $\Delta^m(l_\infty)$ is compact.

Keywords and phrases. Difference sequence spaces, statistical convergence, the N-dual, the $p\alpha$-, $p\beta$-, and $p\gamma$-dual.

2000 Mathematics Subject Classification. Primary 40A05, 40C05, 46A45.

1. Introduction. w denotes the space of all scalar sequences and any subspace of w is called a sequence space. The following sequence spaces will be used in what follows:

l_∞, the space of all bounded scalar sequences;
c, the space of all convergent scalar sequences;
c_0, the space of all null scalar sequences;
l_1, the space of all absolutely 1-summable scalar sequences;
s, the space of all real sequences;
s_0, the space of all statistically convergent sequences of real numbers;
$\Delta^m(l_\infty)$, the space of all Δ^m-bounded scalar sequences;
$\Delta^m(c)$, the space of all Δ^m-convergent scalar sequences;
$\Delta^m(c_0)$, the space of all Δ^m-null scalar sequences;
$\Delta^m(s_0)$, the space of all Δ^m-statistically convergent sequences of real numbers.

It is known that l_∞, c, and c_0 are B-spaces (Banach spaces) with their usual norm $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N} = \{1,2,\dots\}$. The sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, $c_0(\Delta^m)$ have been introduced by Et and Çolak [1]. These sequence spaces are BK-spaces (Banach coordinate spaces) with norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty, \tag{1.1}$$

where $m \in \mathbb{N}$, $\Delta^\circ x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$, and so

$$\Delta^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k+\nu}. \tag{1.2}$$

For convenience we denote these spaces $\Delta^m(l_\infty)$, $\Delta^m(c)$, and $\Delta^m(c_0)$ and call Δ^m-bounded, Δ^m-convergent, and Δ^m-null sequences, respectively. The operators

$$\Delta^{(m)}, \qquad \sum\nolimits^{(m)} : w \longrightarrow w \tag{1.3}$$

are defined by

$$\begin{gathered}\Delta^{(1)}x_k = x_k - x_{k-1}, \quad \sum\nolimits^{(1)} x_k = \sum_{j=0}^{k} x_j, \quad (k = 0,1,\dots), \\ \Delta^{(m)} = \Delta^{(1)}{}_0\Delta^{(m-1)}, \quad \sum\nolimits^{(m)} = \sum\nolimits^{(1)}{}_0\sum\nolimits^{(m-1)}, \quad (m \geq 2),\end{gathered} \tag{1.4}$$

and

$$\sum\nolimits^{(m)}{}_0\Delta^{(m)} = \Delta^{(m)}{}_0\sum\nolimits^{(m)} = \text{id}, \tag{1.5}$$

the identity on w (see [4]).

For any subset X of w let

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}. \tag{1.6}$$

Now we define

$$\Delta^{(m)}x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k-\nu}. \tag{1.7}$$

It is trivial that $(\Delta^m x_k) \in X$ if and only if $(\Delta^{(m)}x_k) \in X$, for $X = l_\infty$, c or c_0. In [4], Malkowsky and Parashar also showed that the sequence spaces $\Delta^m(l_\infty)$, $\Delta^m(c)$, and $\Delta^m(c_0)$ are also BK-spaces with norm

$$\|x\|_{\Delta 1} = \sup_k \left|\Delta^{(m)}x_k\right|. \tag{1.8}$$

It is trivial that the norms (1.1) and (1.8) are equivalent. Obviously

$$\begin{gathered}\Delta^{(m)} : \Delta^{(m)}(X) \longrightarrow X, \qquad \Delta^{(m)}x = y = (\Delta^{(m)}x_k), \\ \sum\nolimits^{(m)} : X \longrightarrow \Delta^{(m)}(X), \qquad \sum\nolimits^{(m)} x = y = \left(\sum\nolimits^{(m)} x_k\right)\end{gathered} \tag{1.9}$$

are isometric isomorphism, for $X = l_\infty$, c or c_0.

Hence $\Delta^m(l_\infty)$, $\Delta^m(c)$, and $\Delta^m(c_0)$ are isometrically isomorphic to l_∞, c, and c_0, respectively. Thus l_1 is continuous dual of $\Delta^m(c)$ and $\Delta^m(c_0)$.

Throughout the paper, we write $\sum_k$ for $\sum_{k=1}^{\infty}$ and $\lim_n$ for $\lim_{n\to\infty}$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers. Let E and F be BK-spaces. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. If $Ax = (A_n(x)) \in E$ for each $x = (x_k) \in F$, then we say that A defines a matrix map from F into E and we denote it by $A : F \to E$. By (F,E) we mean the class of matrices A such that $A : F \to E$. We denote the set $\{x \in w : Ax$ exists and $Ax \in E\}$ by E_A. Note that A is a matrix map from F into E if and only if $F \subseteq E_A$. From now on, E unless specified shall denote a BK-space.

In B-space E, the following statements are equivalent (see [5]).

(i) $\sum_n x_n$ is unconditionally convergent.

(ii) $\sum_n x_n$ is weakly subseries convergent; that is, weak $\lim_n \sum_{j=1}^{n} x_{k_j}$ exists for each increasing sequence (k_n) of positive integers.

(iii) $\sum_n x_n$ is subseries convergent; that is, norm $\lim_n \sum_{j=1}^{n} x_{k_j}$ exists with (k_n) above.

(iv) $\sum_n x_n$ is bounded multiplier convergent; that is, $\sum_n x_n t_n$ exists for each sequence $t = (t_n)$ of bounded scalars.

2. Some properties of $\Delta^m(X)$.

In this section, we will give some properties of $\Delta^m(X)$.

Theorem 2.1. *Let X be a vector space and let $A \subset X$. If A is a convex set, then $\Delta^m(A)$ is a convex set in $\Delta^m(X)$,*

Proof. Let $x, y \in \Delta^m(A)$, then $\Delta^m x, \Delta^m y \in A$. Since Δ^m is linear, we have

$$\lambda\Delta^m x + (1-\lambda)\Delta^m y = \Delta^m(\lambda x + (1-\lambda)y), \quad (0 \le \lambda \le 1). \tag{2.1}$$

Since A is convex $(\lambda\Delta^m x + (1-\lambda)\Delta^m y) \in A$ and so $(\lambda x + (1-\lambda)y) \in \Delta^m(A)$, $(0 \le \lambda \le 1)$.

Lemma 2.2. *Let m be a positive integer. Then*

(i) $\Delta^m(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \Delta^m(A_n)$,

(ii) $\Delta^m(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} \Delta^m(A_n)$.

The proof is clear. □

Lemma 2.3. *Let X be a Banach space and let $A \subset X$. Then*

(i) *If A is nowhere dense in X, then $\Delta^m(A)$ is nowhere dense in $\Delta^m(X)$.*

(ii) *If A is dense in X, then $\Delta^m(A)$ is dense in $\Delta^m(X)$.*

(iii) *$\Delta^m(w) = w$, where m is a positive integer.*

Proof. (i) Suppose that $\overset{\circ}{\bar{A}} = \emptyset$, but $\overset{\circ}{\overline{\Delta^m(A)}} \neq \emptyset$. Then $\bar{A}$ contains no neighborhood and $B(a) \subset \overline{\Delta^m(A)}$, where $B(a)$ is a neighborhood (or open ball) of center a and radius r. Hence $a \in B(a) \subset \overline{\Delta^m(A)} = \Delta^m(\bar{A})$. This implies that $\Delta^m(a) \in \bar{A}$. So $B(\Delta^m(a)) \cap A \neq \emptyset$. On the other hand, $B(\Delta^m(a)) \cap A \subset \bar{A}$. This contradicts to $\overset{\circ}{\bar{A}} = \emptyset$. Hence $\overset{\circ}{\overline{\Delta^m(A)}} = \emptyset$.

(ii) and (iii) are trivial. □

Theorem 2.4. (i) *The set $\Delta^m(s_0)$ is dense in the space s.*

(ii) *The set $\Delta^m(s_0)$ is a set of the first Baire category in the space s.*

(iii) *The set s-$\Delta^m(s_0)$ is a set of the second Baire category in the space s.*

Proof. The proof follows from [6, Theorem 3.1], Lemmas 2.2, and 2.3, we recall that the complement M^c of a meager (or of the first category) subset M of a complete metric space X is nonmeager (or of the second category). □

Theorem 2.5. $l_\infty \cap \Delta^m(c) = l_\infty \cap \Delta^m(c_0)$.

Proof. It is trivial that $l_\infty \cap \Delta^m(c_0) \subset l_\infty \cap \Delta^m(c)$. Now let $x \in l_\infty \cap \Delta^m(c)$, then $x \in l_\infty$ and $\Delta^{m-1}x_k$-$\Delta^{m-1}x_{k+1} \to l$, $(k \to \infty)$, $\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1} = l + \varepsilon_k$ $(\varepsilon_k \to 0,\ k \to \infty)$. This implies that

$$l = n^{-1}\Delta^{m-1}x_1 - n^{-1}\Delta^{m-1}x_{n+1} + n^{-1}\sum_{k=1}^{n}\varepsilon_k. \tag{2.2}$$

This yields $l = 0$ and $x \in l_\infty \cap \Delta^m(c_0)$. □

3. Dual spaces. In this section, we give the N-dual (null dual) of the sequence spaces $\Delta^m(l_\infty)$, $\Delta^m(c)$, and $\Delta^m(c_0)$ and the $p\alpha$-, $p\beta$-, and $p\gamma$-duals of the sequence spaces of l_∞, c, and c_0.

Definition 3.1. Let X be a sequence space and define

$$\begin{aligned}
X^\alpha &= \Big\{a = (a_k) : \sum_k |a_k x_k| < \infty,\ \forall x \in X\Big\},\\
X^\beta &= \Big\{a = (a_k) : \sum_k a_k x_k \text{ is convergent },\ \forall x \in X\Big\},\\
X^\gamma &= \Big\{a = (a_k) : \sup_n \Big|\sum_k a_k x_k\Big| < \infty,\ \forall x \in X\Big\},\\
X^N &= \Big\{a = (a_k) : \lim_k a_k x_k = 0,\ \forall x \in X\Big\},
\end{aligned} \tag{3.1}$$

then X^α, X^β, X^γ, and X^N are called the α-, β-, γ-, and N-(or nul) duals of X, respectively. It is known that $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha$-, β-, γ-, and N-, and ${c_0}^N = l_\infty$, ${l_\infty}^N = c^N = c_0$ [2, 3].

Lemma 3.2 (see [4]). *Let m be a positive integer. Then there exist positive constants M_1 and M_2 such that*

$$M_1 k^m \le \binom{m+k}{k} \le M_2 k^m \quad \forall k = 0, 1, \dots. \tag{3.2}$$

Lemma 3.3. *Let $x \in \Delta^m(c_0)$, then $\binom{m+k}{k}^{-1}|x_k| \to 0$, $(k \to \infty)$.*

Proof. The proof is trivial. □

Theorem 3.4. *Let m be a positive integer. Then $(\Delta^m(l_\infty))^N = (\Delta^m(c))^N = U_1$ and $(\Delta^m(c_0))^N = U_2$, where $U_1 = \{a = (a_n) : (n^m a_n) \in c_0\}$ and $U_2 = \{a = (a_n) : (\sum_{k=0}^{n}\binom{n+m-k-1}{m-1}a_n) \in l_\infty\}$.*

Proof. The proof of the part $(\Delta^m(l_\infty))^N = (\Delta^m(c))^N = U_1$ is easy. We show that $(\Delta^m(c_0))^N = U_2$. It is clear that $\sum_{k=0}^{n}\binom{n+m-k-1}{m-1} = \binom{n+m}{m} = \binom{n+m}{n}$. Let $a \in U_2$ and $x \in \Delta^m(c_0)$. Then

$$\lim_n a_n x_n = \lim_n \left(\sum_{k=0}^{n}\binom{n+m-k-1}{m-1}\right)a_n\left(\sum_{k=0}^{n}\binom{n+m-k-1}{m-1}\right)^{-1}x_n = 0. \tag{3.3}$$

Hence $a \in (\Delta^m(c_0))^N$.

Now let $a \in (\Delta^m(c_0))^N$. Then $\lim_n a_n x_n = 0$ for all $x \in \Delta^m(c_0)$. On the other hand, for each $x \in \Delta^m(c_0)$ there exists one and only one $y = (y_k) \in c_0$ such that

$$x_n = \sum_{k=1}^{n} \binom{n+m-k-1}{m-1} y_k = \sum_{k=0}^{n} \binom{n+m-k-1}{m-1} y_k, \quad y_0 = 0, \tag{3.4}$$

by (1.9). Hence

$$\lim_n a_n x_n = \lim_n \sum_{k=0}^{n} \binom{n+m-k-1}{m-1} a_n y_k = 0 \quad \forall y \in c_0. \tag{3.5}$$

If we take

$$a_{nk} = \begin{cases} \binom{n+m-k-1}{m-1} a_n, & 1 \le k \le n, \\ 0, & k > n, \end{cases} \tag{3.6}$$

then, we get

$$\lim_n \sum_{k=0}^{\infty} a_{nk} y_k = \lim_n \sum_{k=0}^{n} \binom{n+m-k-1}{m-1} a_n y_k = 0 \quad \forall y \in c_0. \tag{3.7}$$

Hence $A \in (c_0, c_0)$ and so $\sup_n \sum_{k=0}^{\infty} |a_{nk}| = \sup_n \sum_{k=0}^{n} \binom{n+m-k-1}{m-1} |a_n| < \infty$. This completes the proof. □

Now we give a new kind of duals of sequence sets.

Definition 3.5. Let X be a sequence spaces, $p > 0$ and define

$$\begin{aligned} X^{p\alpha} &= \Big\{ a = (a_k) : \sum_k |a_k x_k|^p < \infty, \ \forall x \in X \Big\}, \\ X^{p\beta} &= \Big\{ a = (a_k) : \sum_k (a_k x_k)^p \text{ is convergent }, \ \forall x \in X \Big\}, \\ X^{p\gamma} &= \Big\{ a = (a_k) : \sup_n \Big| \sum_{k=0}^{n} (a_k x_k)^p \Big| < \infty, \ \forall x \in X \Big\}, \end{aligned} \tag{3.8}$$

then $X^{p\alpha}$, $X^{p\beta}$, $X^{p\gamma}$ are called the $p\alpha$-, $p\beta$-, and $p\gamma$-duals of X, respectively. It can be shown that $X^{p\alpha} \subset X^{p\beta} \subset X^{p\gamma}$. If we take $p = 1$ in this definition, then we obtain the α-, β-, and γ-duals of X.

Theorem 3.6. *Let X stand for l_∞, c, and c_0 and $0 < p < \infty$. Then $X^{p\eta} = U$, for $\eta = \alpha, \beta$ or γ, where $U = \{a = (a_k) : \sum_k |a_k|^p < \infty\} = l_p$.*

Proof. We give the proof for the case $X = c_0$ and $\eta = \alpha$. If $a \in U$, then

$$\sum_k |a_k x_k|^p \le \sup_k |x_k|^p \sum_k |a_k|^p < \infty \tag{3.9}$$

for each $x \in c_0$. Hence $a \in (c_0)^{p\alpha}$.

Now suppose that $a \in (c_0)^{p\alpha}$ and $a \notin U$. Then there is a strictly increasing sequence (n_i) of positive integers n_i such that

$$\sum_{k=n_i+1}^{k=n_{i+1}} |a_k|^p > i^p. \tag{3.10}$$

Define $x \in c_0$ by $x_k = \operatorname{sgn} a_k / i$ for $n_i < k \le n_{i+1}$ and $x_k = 0$ for $1 \le k \le n_1$. Then we may write

$$\begin{aligned} \sum_k |a_k x_k|^p &= \sum_{k=n_1+1}^{k=n_2} |a_k x_k|^p + \cdots + \sum_{k=n_i+1}^{k=n_{i+1}} |a_k x_k|^p + \cdots \\ &= \sum_{k=n_1+1}^{k=n_2} |a_k|^p + \cdots + \frac{1}{i^p} \sum_{k=n_i+1}^{k=n_{i+1}} |a_k|^p + \cdots \\ &> 1 + 1 + \cdots = \sum_k 1 = \infty. \end{aligned} \tag{3.11}$$

This contradicts to $a \in (c_0)^{p\alpha}$. Hence $a \in U$. The proof for the cases $X = l_\infty$ or c and $\eta = \beta$ or γ is similar. □

The proofs of Lemmas 3.7 and 3.8 and Theorem 3.10 are easily obtained by using the same techniques of Mishra [5, Lemmas 1 and 2 and Theorem 1], therefore we give them without proofs.

Lemma 3.7. *Let $A : \Delta^m(l_\infty) \to E$ defines a matrix map. If A is weakly compact, then $\sum_k a_k$ is unconditionally convergent in E.*

Lemma 3.8. *If $\sum_k a_k$ is unconditionally convergent in E, then $A : \Delta^m(l_\infty) \to E$ defines a matrix map, and $A(\alpha) = \sum_k a_k \alpha_k$ for every $\alpha = (\alpha_k) \in \Delta^m(l_\infty)$.*

Corollary 3.9. *If $\sum_k a_k$ is unconditionally convergent in E, then $\Delta^m(l_\infty) \subseteq E_A$.*

Theorem 3.10. *If $A : \Delta^m(l_\infty) \to E$ is a weakly compact matrix map, then A is compact map.*

Corollary 3.11. *Let E be a BK-space such that it contains no subspace isomorphic to $\Delta^m(l_\infty)$. If $A : \Delta^m(l_\infty) \to E$ defines a matrix map, then A is compact map.*

Acknowledgement. The author wishes to thank Prof. Dr Rifat Çolak for his valuable comments on the manuscript.

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