Internat. J. Math. & Math. Sci.
Vol. 23, No. 3 (2000) 211–216
S0161171200001563

FREE MINIMAL RESOLUTIONS AND THE BETTI NUMBERS OF THE SUSPENSION OF AN n-GON

TILAK DE ALWIS

(Received 31 July 1998)

ABSTRACT. Consider the general n-gon with vertices at the points $1,2,\dots,n$. Then its suspension involves two more vertices, say at $n+1$ and $n+2$. Let R be the polynomial ring $k[x_1,x_2,\dots,x_n]$, where k is any field. Then we can associate an ideal I to our suspension in the Stanley-Reisner sense. In this paper, we find a free minimal resolution and the Betti numbers of the R-module R/I.

Keywords and phrases. Suspension, finite abstract simplicial complex, Stanley-Reisner ideal, free-minimal resolution, Betti numbers, double complex.

2000 Mathematics Subject Classification. Primary 13H10.

1. Introduction. Consider the suspension of the n-gon whose vertices are at the points $1,2,\dots,n$ (see [6]). This introduces two new vertices, say $n+1$ and $n+2$. The finite abstract simplicial complex Ω corresponding to this suspension is given by

$$\begin{aligned}\Omega = \{&\varnothing,\{1\},\{2\},\dots,\{n\},\{n+1\},\{n+2\},\{1,2\},\{2,3\},\dots,\{n,1\},\\ &\{1,n+1\},\{2,n+1\},\dots,\{n-1,n+1\},\{n,n+1\},\{1,n+2\},\\ &\{2,n+2\},\dots,\{n-1,n+2\},\{n,n+2\}\}.\end{aligned} \tag{1.1}$$

Let k be any field and $R = k[x_1,\dots,x_{n+2}]$. By definition, the Stanley-Reisner ideal associated to Ω is given by $I =$ The ideal in R generated by all the monomials of the form $x_{i_1}x_{i_2}\cdots x_{i_r}$, where $1 \le i_1 < i_2 < \cdots < i_r \le n+2$ and $\{i_1,\dots,i_r\} \notin \Omega$ (see [3, 7]). Then, it follows that $I = (x_1x_3,x_1x_4,\dots,x_1x_{n-1},x_2x_4,\dots,x_2x_n,\dots,x_{n-2}x_n,x_{n+1}x_{n+2})$ for $n > 3$, and $I = (x_1x_2x_3,x_4x_5)$ for $n = 3$. In the literature, the ring R/I is also known as the face ring or the Stanley-Reisner ring of the finite abstract simplicial complex Ω (see [3, 7]).

By definition, a free-minimal resolution of the R-module R/I is an exact sequence of the form

$$\cdots M_i \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \frac{R}{I} \longrightarrow 0, \tag{1.2}$$

where each M_i is a free R-module with the smallest possible rank. For material on free-minimal resolutions, the reader can refer to [5] or [7]. The Betti numbers $B_i(n)$ of the R-module R/I are just the ranks of those free modules M_i, i.e., $B_i(n) = \operatorname{rank}_R(M_i)$ for $i = 0,1,\dots$.

In this paper, we find a free-minimal resolution and the Betti numbers of the R-module R/I. Sometimes we simply refer to them as a free-minimal resolution and the Betti numbers of the suspension of the n-gon.

2. Some useful results. In this section, we recall some results on free-minimal resolutions and the Betti numbers of the n-gon. These results are needed to obtain the theorems on the suspension of the n-gon. The proofs of most of these theorems can be found in [1] or [2].

(1) Let Δ be the finite abstract simplicial complex corresponding to the n-gon with vertices at the points $1,2,\dots,n$. Let $S = k[x_1,\dots,x_n]$ and J_1 be the Stanley-Reisner ideal associated to Δ. Then, it easily follows that $J_1 = (x_1x_3, x_1x_4, \dots, x_2x_4, \dots, x_2x_n, \dots, x_{n-2}x_n)$ for $n > 3$, and $J_1 = (x_1x_2x_3)$ for $n = 3$.

(2) Let $\beta_i(n)$ denote the ith Betti number of the S-module S/J_1. In other words, it is the ith Betti number of the n-gon. Then, for $n \geq 3$,

$$\beta_i(n) = \begin{cases} 1, & i = 0, \\ \binom{n}{i+1}\dfrac{i(n-i-2)}{n-1}, & i = 1,2,\dots,n-3, \\ 1, & i = n-2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

(3) We can show that,

$$0 \longrightarrow S^{\beta_{n-2}} \xrightarrow{f_{n-2}} S^{\beta_{n-3}} \longrightarrow \cdots \longrightarrow S^{\beta_1} \xrightarrow{f_1} S^{\beta_0} \xrightarrow{f_0} \frac{S}{J_1} \longrightarrow 0 \tag{2.2}$$

is a free-minimal resolution of the S-module S/J_1. Even though we do not need the specific definitions of the maps f_j for what follows, the inquisitive reader can find them in [1].

3. Main results. Let J_1 be the ideal in the polynomial ring $S = k[x_1,\dots,x_n]$ as in Section 2. Let J be the ideal in the polynomial ring $R = k[x_1,\dots,x_n,x_{n+1},x_{n+2}]$ generated by the same generators as that of J_1.

Tensor the exact sequence (2.2) with the k-module $k[x_{n+1},x_{n+2}]$, which is a free module. Hence we obtain the following exact sequence of R-modules.

$$0 \longrightarrow R^{\beta_{n-2}} \xrightarrow{d_{n-2}} R^{\beta_{n-3}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{d_1} R^{\beta_0} \xrightarrow{d_0} \frac{R}{J} \longrightarrow 0, \tag{3.1}$$

where d_i are the same as the maps $f_i \otimes id$. This means that the following complex is exact at all places except at degree 0:

$$\begin{array}{ccccccccccc} & & n-2 & & n-3 & & & & 1 & & 0 \\ 0 & \longrightarrow & R^{\beta_{n-2}} & \xrightarrow{d_{n-2}} & R^{\beta_{n-3}} & \longrightarrow & \cdots & \longrightarrow & R^{\beta_1} & \xrightarrow{d_1} & R^{\beta_0} & \longrightarrow & 0. \end{array} \tag{3.2}$$

Consider the following diagram where the two rows are the same as the complex (3.2) and the vertical maps are multiplication by the element $y = x_{n+1}x_{n+2}$ of R.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R^{\beta_{n-2}} & \xrightarrow{d_{n-2}} & R^{\beta_{n-3}} & \longrightarrow \cdots \longrightarrow & R^{\beta_1} & \xrightarrow{d_1} & R^{\beta_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R^{\beta_{n-2}} & \xrightarrow{d_{n-2}} & R^{\beta_{n-3}} & \longrightarrow \cdots \longrightarrow & R^{\beta_1} & \xrightarrow{d_1} & R^{\beta_0} & \longrightarrow & 0. \end{array} \tag{3.3}$$

The squares in (3.3) commute, because $x_{n+1}x_{n+2}$ is an element of the ring R and our maps are R-module homomorphisms. Hence (3.3) is a double complex, and its total complex is given by

$$0 \longrightarrow R^{\beta_{n-2}} \xrightarrow{\partial_{n-1}} R^{\beta_{n-2}} \oplus R^{\beta_{n-3}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \oplus R^{\beta_0} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow 0, \tag{3.4}$$

where the differential maps $\partial_i : R^{\beta_i} \oplus R^{\beta_{i-1}} \to R^{\beta_{i-1}} \oplus R^{\beta_{i-2}}$, $i = 1,2,\dots,n-1$ are given by $\partial_i(p,q) = (d_i(p) + (-1)^i yq, d_{i-1}(q))$ for $i = 2,3,\dots,n-2$. Obvious definitions would apply for $i = 1$ and $i = n-1$. It is a routine exercise to verify that $\partial_{i-1} \circ \partial_i = 0$.

Theorem 3.1. *The complex (3.4) is exact at all places except degree* 0 *at which it has homology equal to* R/I. *In other words, the following is a free resolution of* R/I:

$$0 \longrightarrow R^{\beta_{n-2}} \xrightarrow{\partial_{n-1}} R^{\beta_{n-2}} \oplus R^{\beta_{n-3}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \oplus R^{\beta_0} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow \frac{R}{I} \longrightarrow 0. \tag{3.5}$$

Proof. Denote R^{β_i} by D_i. Then for $i > 1$, consider the sequence $D_{i+1} \oplus D_i \xrightarrow{\partial_{i+1}} D_i \oplus D_{i-1} \xrightarrow{\partial_i} D_{i-1} \oplus D_{i-2}$. Suppose $(p,q) \in \operatorname{Ker}\partial_i$. Then $\partial_i(p,q) = (d_i(p) + (-1)^i yq, d_{i-1}(q)) = 0$. Hence $d_i(p) + (-1)^i yq = 0$ and $d_{i-1}(q) = 0$. Therefore, $q \in \operatorname{Ker} d_{i-1} = \operatorname{im} d_i$, so $q = d_i(q_1)$ for some $q_1 \in D_i$. The equation $d_i(p) + (-1)^i yq = 0$ yields $d_i(p + (-1)^i yq_1) = 0$, which means that $p + (-1)^i yq_1 \in \operatorname{Ker} d_i = \operatorname{im} d_{i+1}$. Therefore, $p + (-1)^i yq_1 = d_{i+1}(p_1)$ for some $p_1 \in D_{i+1}$, which implies that $p = d_{i+1}(p_1) + (-1)^{i+1} yq_1$. Hence, $\partial_{i+1}(p_1,q_1) = (p,q)$, i.e., $(p,q) \in \operatorname{im}\partial_{i+1}$. This shows that $\operatorname{Ker}\partial_i = \operatorname{im}\partial_{i+1}$ for $i > 1$.

For $i = 1$, we have $D_2 \oplus D_1 \xrightarrow{\partial_2} D_1 \oplus D_0 \xrightarrow{\partial_1} D_0$. Let $(p,q) \in \operatorname{Ker}\partial_1$. Therefore, $\partial_1(p,q) = d_1(p) + (-1)yq = 0$. This yields $d_1(p) = yq \in \operatorname{im} d_1 = \operatorname{Ker} d_0 = J$. But $y \notin J$. Hence, even though J is not a prime ideal of R, by considering the primary decomposition of J, one can easily obtain that $q \in J$. Therefore, the exact sequence (3.1) gives us $q = d_1(q_1')$ for some $q_1' \in D_1$. Hence $d_1(p) = yq = yd_1(q_1') = d_1(yq_1')$, which implies that $p - yq_1' \in \operatorname{Ker} d_1 = \operatorname{im} d_2$. Therefore, $p - yq_1' = d_2(p_1')$ for some $p_1' \in D_2$. Hence, $p = d_2(p_1') + yq_1'$. Now we have two equations $d_2(p_1') + yq_1' = p$, and $d_1(q_1') = q$ where $(p_1', q_1') \in D_2 \oplus D_1$. This yields $\partial_2(p_1', q_1') = (p,q)$ and hence $\operatorname{Ker}\partial_1 = \operatorname{im}\partial_2$.

Finally, for $i = 0$, we have $D_1 \oplus D_0 \xrightarrow{\partial_1} D_0 \to 0$. We know that $\partial_1(p,q) = d_1(p) - yq$. However, the exact sequence (3.1) implies that $d_1(D_1) = J$ and hence $\operatorname{im}\partial_1 = \{j - yq \mid j \in J,\ q \in R\} = J + (y) = I$. Therefore the homology of the complex (3.4) at the zeroth spot is equal to R/I. □

Theorem 3.2 says more about the free resolution (3.5).

Theorem 3.2. *The sequence (3.5) is a free-minimal resolution of the R-module R/I.*

Proof. To show the minimality, it is enough to show that the maps $\partial_i \otimes id$: $(R^{\beta_i} \oplus R^{\beta_{i-1}}) \otimes_R k \to (R^{\beta_{i-1}} \oplus R^{\beta_{i-2}}) \otimes_R k$ are zero for $i = 1, 2, \dots, n-1$ (see [4, p. 136]). However, this is an easy consequence of commutativity of the following diagram and the minimality of (2.2):

$$\begin{array}{ccc} (R^{\beta_i} \oplus R^{\beta_{i-1}}) \otimes_R k & \longrightarrow & (R^{\beta_{i-1}} \oplus R^{\beta_{i-2}}) \otimes_R k \\ \downarrow & & \downarrow \\ (R^{\beta_i} \otimes_R k) \oplus (R^{\beta_{i-1}} \otimes_R k) & \longrightarrow & (R^{\beta_{i-1}} \otimes_R k) \oplus (R^{\beta_{i-2}} \otimes_R k). \end{array} \tag{3.6}$$

□

Theorem 3.3 enables us to calculate the Betti numbers $B_i(n)$ of the suspension of the n-gon.

Theorem 3.3. *Let $n \geq 3$ be a positive integer. Then the ith Betti number $B_i(n)$ of the suspension of the n-gon is given by*

$$B_i(n) = \begin{cases} 1, & i = 0, \\ \binom{n-1}{2}, & i = 1, \\ \binom{n}{i} \dfrac{[ni - (i^2 + i + 1)]}{i+1}, & i = 2, 3, \dots, n-3, \\ \binom{n-1}{2}, & i = n-2, \\ 1, & i = n-1, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.7}$$

Proof. Let $n \geq 3$ be a positive integer. Since (3.5) is a free-minimal resolution, the Betti numbers of R/I are just the respective ranks of the free modules appearing in (3.5). Hence, we obtain, for $n \geq 3$,

$$B_i(n) = \begin{cases} \beta_0, & i = 0, \\ \beta_i + \beta_{i-1}, & i = 1, 2, \dots, n-2, \\ \beta_{n-2}, & i = n-1. \end{cases} \tag{3.8}$$

Let us denote $B_i(n)$ by B_i. The theorem is clear for $n = 3$. Therefore assume that $n > 3$. So

$$B_0 = \beta_0 = \beta_{n-2} = B_{n-1} = 1 \tag{3.9}$$

and

$$B_1 = \beta_1 + \beta_0 = \binom{n}{2} \frac{n-3}{n-1} + 1 = \frac{1}{2} n(n-3) + 1 = \frac{1}{2}(n-1)(n-2) = \binom{n-1}{2} \tag{3.10}$$

by using formula (2.1). A similar calculation shows that $B_{n-2} = \binom{n-1}{2}$. Now, let $1 < i < n-2$. The formula (2.1) again gives,

$$\begin{aligned}
B_i = \beta_i + \beta_{i-1} &= \binom{n}{i+1}\frac{i(n-i-2)}{n-1} + \binom{n}{i}\frac{(i-1)(n-i+1-2)}{n-1} \\
&= \frac{n!}{(i+1)!(n-i-1)!}\cdot\frac{i(n-i-2)}{n-1} + \frac{n!}{i!(n-i)!}\cdot\frac{(i-1)(n-i-1)}{n-1} \\
&= \frac{n!}{(n-1)(i+1)(n-i)i!(n-i-1)!} \\
&\quad\times\left[i(n-i)(n-i-2)+(i-1)(i+1)(n-i-1)\right] \qquad (3.11)\\
&= \frac{n!}{(n-1)(i+1)(n-i)i!(n-i-1)!}\left[n^2 i - n(i+1)^2 + i^2 + i + 1\right] \\
&= \frac{n!}{(n-1)(i+1)(n-i)i!(n-i-1)!}(n-1)\left[ni-(i^2+i+1)\right] \\
&= \frac{n!}{i!(n-i)!}\frac{\left[ni-(i^2+i+1)\right]}{i+1} = \binom{n}{i}\frac{\left[ni-(i^2+i+1)\right]}{i+1}
\end{aligned}$$

which proves the formula (3.7) for the case $n > 3$. This completes the proof. □

We can illustrate our theory with an example. For $n = 4$ we get the suspension of the square, which is nothing but the familiar octahedron. Hence,

$$\begin{aligned}
\Omega &= \{\emptyset,\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,2\},\{2,3\},\{3,4\},\{4,1\}, \\
&\qquad \{1,5\},\{2,5\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{3,6\},\{4,6\}\}, \\
R &= k[x_1,x_2,x_3,x_4,x_5,x_6], \\
I &= (x_1x_3, x_2x_4, x_5x_6).
\end{aligned} \qquad (3.12)$$

The formula (3.7) gives us $B_0(4) = 1$, $B_1(4) = 3$, $B_2(4) = 3$, and $B_3(4) = 1$ which are the Betti numbers of the octahedron.

Acknowledgement. The author wishes to thank his adviser John Eagon for all the valuable help.

References

[1] T. de Alwis, *Free minimal resolutions*, Comm. Algebra **21** (1993), no. 12, 4575–4585. MR 95f:13017. Zbl 806.18009.

[2] ______, *Pascal type properties of Betti numbers*, Internat. J. Math. Math. Sci. **17** (1994), no. 3, 545–552. MR 95c:13026. Zbl 809.13010.

[3] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Ring theory, II (New York), Lecture Notes in Pure and Appl. Math., vol. 26, Proc. Second Conf., Univ. Oklahoma, Dekker, 1977, pp. 171–223. MR 56 376. Zbl 351.13009.

[4] H. Matsumura, *Commutative algebra*, 2d ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, M.S., 1980. MR 82i:13003. Zbl 441.13001.

[5] ______, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, New York, 1986. MR 88h:13001. Zbl 603.13001.

[6] J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, Calif., 1984. MR 85m:55001. Zbl 673.55001.

[7] R. P. Stanley, *Combinatorics and commutative algebra*, Progress in Mathematics, vol. 41, Birkhauser Boston, Inc., Boston, M.A., 1983. MR 85b:05002. Zbl 537.13009.

de Alwis: Department of Mathematics, Southeastern Louisiana University, Hammond, Louisiana 70402, USA