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THE ℓ-TRANSLATIVITY OF ABEL-TYPE MATRIX

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ABSTRACT. Lemma introduced the Abel-type matrix $A_{\alpha,t}$ defined by $a_{nk} = \binom{k+\alpha}{k} t_n^{k+1}(1-t_n)^{\alpha+1}$, where $\alpha > -1$, $0 < t_n < 1$, for all n, and $\lim t_n = 1$; and studied it as mappings into ℓ. In this paper, we extend our study of this matrix and investigate its translativity in the ℓ-ℓ setting.

Keywords and phrases. ℓ-ℓ method, ℓ-translative, Abel-type power series method, Abel-type matrix.

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1. Background. In [1], Borwein proved that the Abel-type power series method of summability denoted by $A_\alpha(\alpha > -1)$, is translative in the ordinary summability (c-c) setting. So, it natural to ask if there is a theory in the ℓ-ℓ setting that parallel the theory of A_α in the c-c setting. The answer is affirmative, and have provided the present study.

2. Basic notation and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence to a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \tag{2.1}$$

where $(Ax)_n$ denotes the nth term of the image sequence Ax. Let y be a complex number sequence. Throughout this paper, we use the following basic notations and definitions:

(i) $c = \{$the set of all convergent complex number seqences$\}$,
(ii) $\ell = \left\{y : \sum_{k=0}^{\infty} |y_k| \text{ converges}\right\}$,
(iii) $\ell(A) = \{y : Ay \in \ell\}$,
(iv) $c(A) = \{y : y \text{ is summable by A}\}$.

Definition 1. If X and Y are sets of complex number sequences, then the matrix A is called an X-Y matrix if the image Au of u under the transformation A is in Y whenever u is in X.

Definition 2. The summability matrix A is said to be ℓ-translative for the sequence u in $\ell(A)$ provided that each of the sequences T_u and S_u is in $\ell(A)$, where $T_u = \{u_1, u_2, u_3, \ldots\}$ and $S_u = \{0, u_0, u_1, \ldots\}$.

3. The main results

Proposition 1. *Every ℓ-ℓ $A_{\alpha,t}$ matrix is ℓ-translative for each sequence $x \in \ell$.*

Theorem 1. *Every ℓ-ℓ $A_{\alpha,t}$ matrix is ℓ-translative for those sequences $x \in \ell(A_{\alpha,t})$ for which $\{x_k/k\} \in \ell, k = 1,2,3,\dots$.*

Proof. Suppose that x is a sequence in $\ell(A_{\alpha,t})$ for which $\{x_k/k\} \in \ell$. We show that

(1) $T_x \in \ell(A_{\alpha,t})$, and

(2) $S_x \in \ell(A_{\alpha,l})$, where T_x and S_x are as defined in Definition 2. Let us first show that (1) holds.

Note that

$$\begin{aligned} |(A_{\alpha,t}T_x)_n| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right| \\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k+1}\right| \\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^k\right| \\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\frac{k}{k+\alpha}\right| \\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\left(1-\frac{\alpha}{k+\alpha}\right)\right| \\ &\le A_n + B_n, \end{aligned} \tag{3.1}$$

where

$$A_n = \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right| \tag{3.2}$$

and

$$B_n = \frac{|\alpha|(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\alpha}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|. \tag{3.3}$$

The use of the triangle inequality in equation (3.1) is legitimate as the radii of convergence of the two power series are at least 1. Now if we show that both A and B are in ℓ, then (1) holds. But the conditions that $A \in \ell$ and $B \in \ell$ follow easily from the hypotheses that $x \in \ell(A_{\alpha,t})$ and $\{x_k/k\} \in \ell$, respectively. Next, we show that (2) holds as follows. We have

$$\begin{aligned} |(A_{\alpha,t}S_x)_n| &= (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_{k-1}t_n^k\right| \\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k+1}x_k t_n^{k+1}\right| \end{aligned}$$

$$
\begin{aligned}
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(\frac{k+\alpha+1}{k+1}\right)\right| \\
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(1+\frac{\alpha}{k+1}\right)\right| \\
&\le E_n + F_n,
\end{aligned}
\tag{3.4}
$$

where

$$
E_n = (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k}\right|
\tag{3.5}
$$

and

$$
F_n = (1-t_n)^{\alpha+1}|\alpha|\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+1}t_n^{k+1}\right|.
\tag{3.6}
$$

The use of the triangle inequality in (3.4) is justified as above. If we show that E and F are in ℓ, then (2) holds. But the hypothesis that $x \in \ell(A_{\alpha,t})$ and $\{x_k/k\} \in \ell$ implies that both E and F are in ℓ, respectively, and hence the theorem follows. □

Here, we remark that a sequence x defined by $x_k = (-1)^k/k$ is one of the sequences which satisfies the condition of Theorem 1.

Theorem 2. *Suppose that $-1 < \alpha \le 0$, then every ℓ-$\ell A_{\alpha,t}$ matrix is ℓ-traslative for each A_α-summable sequence x in $\ell(A_{\alpha,t})$.*

Proof. Let $x \in c(A_\alpha) \cap \ell(A_{\alpha,t})$. We show that

(1) $T_x \in \ell(A_{\alpha,t})$,

(2) $S_x \in \ell(A_{\alpha,t})$.

First, let us show that (1) holds. Since the case $\alpha = 0$ was already proved by J. Fridy in [2], here we only consider the case $-1 < \alpha < 0$. Note that

$$
\begin{aligned}
\left|(A_{\alpha,t}T_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k+1}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^{k}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k}\frac{k}{k+\alpha}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k}\left(1-\frac{\alpha}{k+\alpha}\right)\right| \\
&\le A_n + B_n,
\end{aligned}
\tag{3.7}
$$

where

$$
A_n = (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k}\right|
\tag{3.8}
$$

and

$$B_n = -\alpha(1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}t_n^k\frac{x_k}{k+\alpha}\right|. \tag{3.9}$$

Now if we show that both A and B are in ℓ, then (1) holds. But the condition that $A \in \ell$ follows from the hypothesis that $x \in \ell(A_{\alpha,t})$ and $B \in \ell$ are shown as follows. Observe that

$$B_n < (1-t_n)^{\alpha+1}|x_1| + (1-t_n)^{\alpha+1}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}t_n^k\frac{x_k}{k+\alpha}\right| = C_n + D_n, \tag{3.10}$$

where

$$C_n = |x_1|(1-t_n)^{\alpha+1} \tag{3.11}$$

and

$$D_n = (1-t_n)^{\alpha+1}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}t_n^k\frac{x_k}{k+\alpha}\right|. \tag{3.12}$$

By [3, Thm. 1], the hypothesis that $A_{\alpha,t}$ ℓ-ℓ implies that $C \in \ell$, hence there remains only to show $D \in \ell$ to show that $B \in \ell$. Observe that

$$\begin{aligned} D_n &= \frac{(1-t_n)^{\alpha+1}}{t_n^{\alpha}}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}x_k\left(\int_0^{t_n} t^{k+\alpha-1}\,dt\right)\right| \\ &= \frac{(1-t_n)^{\alpha+1}}{t_n^{\alpha}}\left|\int_0^{t_n} dt\left(\sum_{k=2}^{\infty}\binom{k+\alpha}{k}x_k t^{k+\alpha-1}\right)\right|. \end{aligned} \tag{3.13}$$

The interchanging of the integral and the summation is legitimate as the radius of convergence of the power series

$$\sum_{k=2}^{\infty}\binom{k+\alpha}{k}x_k t^{k+\alpha-1} \tag{3.14}$$

is at least 1 by [3, Prop. 1], and hence the power series converges absolutely and uniformly for $0 \le t \le t_n$. Now we let

$$F(t) = \sum_{k=2}^{\infty}\binom{k+\alpha}{k}x_k t^{k+\alpha-1}. \tag{3.15}$$

Then, we have

$$F(t)(1-t)^{\alpha+1} = (1-t)^{\alpha+1}\sum_{k=2}^{\infty}\binom{k+\alpha}{k}x_k t^{k+\alpha-1} \tag{3.16}$$

and the hypothesis that $x \in c(A_\alpha)$ implies that

$$\lim_{t\to 1^-} F(t)(1-t)^{\alpha+1} = A \text{ (finite)} \quad \text{for } 0 < t < 1. \tag{3.17}$$

We also have

$$\lim_{t\to 0} F(t)(1-t)^{\alpha+1} = 0. \tag{3.18}$$

Now equations (3.17) and (3.18) yield that

$$\left|F(t)(1-t)^{\alpha+1}\right| \le M_1 \quad \text{for some } M_1 > 0, \tag{3.19}$$

and hence

$$|F(t)| \le M_1(1-t)^{-(\alpha+1)}. \tag{3.20}$$

So, we have

$$\begin{aligned} D_n &= \frac{(1-t_n)^{\alpha+1}}{t_n^{\alpha}}\left|\int_0^{t_n} F(t)\,dt\right| \\ &\le M_2(1-t_n)^{\alpha+1}\int_0^{t_n}|F(t)|\,dt \quad \text{for some } M_2>0, \\ &\le M_1M_2(1-t_n)^{\alpha+1}\int_0^{t_n}(1-t)^{-(\alpha+1)}\,dt \\ &= \frac{M_1M_2}{\alpha}(1-t_n) - \frac{M_1M_2}{\alpha}(1-t_n)^{\alpha+1} \\ &\le -\frac{2M_1M_2}{\alpha}(1-t_n)^{\alpha+1}. \end{aligned} \tag{3.21}$$

By [3, Thm. 1], the hypothesis that $A_{\alpha,t}$ ℓ-ℓ implies that $(1-t)^{\alpha+1} \in \ell$, and hence $D \in \ell$. Next we show that (2) holds. We have

$$\begin{aligned} \left|(A_{\alpha,t}S_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_{k-1}t_n^k\right| \\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k+1}x_k t_n^{k+1}\right| \\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(\frac{k+\alpha+1}{k+1}\right)\right| \\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(1+\frac{\alpha}{k+1}\right)\right| \\ &\le E_n + F_n, \end{aligned} \tag{3.22}$$

where

$$E_n = (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right| \tag{3.23}$$

and

$$F_n = -(1-t_n)^{\alpha+1}\alpha\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+1}t_n^{k+1}\right|. \tag{3.24}$$

Now if we show that both E and F are in ℓ, then (2) follows. But the hypothesis that $x \in \ell(A_{\alpha,t})$ implies that $E \in \ell$, and $F \in \ell$ follows using the same technique used in showing $D \in \ell$ in (1) in the proof of Theorem 2. Hence the theorem is proved. □

Corollary 1. *Suppose that* $-1 \le \alpha \le 0$*; then every* ℓ*-*$\ell A_{\alpha,t}$ *matrix is* ℓ*-translative for the class of all sequence* x *whose partial sum is bounded.*

Proof. By [3, Thm. 8], x is in $\ell(A_{\alpha,t})$ and it is easy to see that x is also in $c(A_\alpha)$. Hence the assertion follows by Theorem 2. □

Corollary 2. *Suppose that* $-1 \le \alpha \le 0$*; then every* ℓ*-*$\ell A_{\alpha,t}$ *matrix is* ℓ*-translative for the unbounded sequence* x *defined by*

$$x_k = (-1)^k \frac{k+\alpha+1}{\alpha+1}. \tag{3.25}$$

Proof. Since $x \in c(A_\alpha) \cap \ell(A_{\alpha,t})$, the corollary easily follows by Theorem 2. □

Theorem 3. *Suppose that* $\alpha > 0$ *and* $(1-t) \in \ell$*; then every* ℓ*-*$\ell A_{\alpha,t}$ *matrix is* ℓ*-translative for each* A_α*-summable sequence in* $\ell(A_{\alpha,t})$*.*

Proof. Suppose that $x \in c(A_\alpha) \cap \ell(A_{\alpha,t})$. To prove the theorem, we need to show that both $A_{\alpha,t}T_x$ and $A_{\alpha,t}S_x$ are in ℓ. We have

$$\left|(A_{\alpha,t}T_x)_n\right| = (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right|, \tag{3.26}$$

and by referring to the proof of Theorem 1, we can express $|(A_{\alpha,t}T_x)_n|$ as

$$\left|(A_{\alpha,t}T_x)_n\right| \le A_n + B_n, \tag{3.27}$$

where

$$A_n = \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right| \tag{3.28}$$

and

$$B_n = \frac{\alpha(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|. \tag{3.29}$$

Now if we show that both A and B are in ℓ, then $A_{\alpha,t}T_x$ is in ℓ. The condition that $A \in \ell$ follows easily since $A_{\alpha,t}$ is an ℓ-ℓ matrix and $x \in \ell(A_{\alpha,t})$. The condition that $B \in \ell$ is shown as follows. Using the triangle inequality, we have

$$B_n \le C_n + D_n, \tag{3.30}$$

where

$$C_n = \alpha|x_1|(1-t_n)^{\alpha+1} \tag{3.31}$$

and

$$D_n = \alpha(1-t_n)^{\alpha+1}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}t_n^k\frac{x_k}{k+\alpha}\right|. \tag{3.32}$$

We have $B \in \ell$, if we show that both C and D are in ℓ. The condition that $C \in \ell$ follows easily as $(1-t)^{\alpha+1} \in \ell$ by [3, Thm. 1], and $D \in \ell$ can be shown as follows. Note that following exactly the same steps as in the proof of (1) of Theorem 2, we can easily show that

$$D_n \le \frac{M_1 M_2}{\alpha}(1-t_n) - \frac{M_1 M_2}{\alpha}(1-t_n)^{\alpha+1}. \tag{3.33}$$

Now since $(1-t) \in \ell$ is given and $A_{\alpha,t}$ is an ℓ-ℓ matrix implies that $(1-t)^{\alpha+1}$ is in ℓ by [3, Thm. 1], it follows that $D \in \ell$. Also using the same techniques as in the proof of (2) of Theorem 2, we can easily show that $A_{\alpha,t} S_x \in \ell$ and hence the assertion follows. □

Corollary 3. *Suppose that $\alpha > 0$, $q > 1$, and $t_n = 1-(n+2)^{-q}$, then every $A_{\alpha,t}$ matrix is ℓ-translative for each A_α-summable sequence x in $\ell(A_{\alpha,t})$.*

Proof. Since by [3, Thm. 5], $A_{\alpha,t}$ is an ℓ-ℓ matrix, the corollary easily follows by Theorem 3. □

Example 1. Suppose that $x_k = (-1)^k$ and $(1-t) \in \ell$; then every ℓ-$\ell A_{\alpha,t}$ matrix is ℓ-translative for the sequence x. Note that $x \in c(A_\alpha) \cap \ell(A_{\alpha,t})$. If $-1 < \alpha \le 0$, then every ℓ-$\ell A_{\alpha,t}$ matrix is ℓ-translative for the sequence x by Theorem 2. If $\alpha > 0$, then every ℓ-$\ell A_{\alpha,t}$ matrix is ℓ-translative for the sequence x by Theorem 3.

Example 2. Suppose that $\alpha > 0$, $q > 1$, and $t_n = 1-(n+2)^{-q}$. Then every $A_{\alpha,t}$ matrix is ℓ-translative for the unbounded sequence x defined by

$$x_k = (-1)^k \frac{k+\alpha+1}{\alpha+1}. \tag{3.34}$$

Since $x \in c(A_\alpha) \cap \ell(A_{\alpha,t})$, the assertion follows by Corollary 3.

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