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UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. We study some integral operators and determine conditions for the univalence of these integral operators.

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1. Introduction. Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the functions $f \in A$ which are univalent in U.

Many authors studied the problem of integral operators which preserve the class S. In this sense an important result is due to Pfaltzgraff [4].

THEOREM 1.1 [4]. *If f is univalent in U, α a complex number and $|\alpha| \le 1/4$, then the function*

$$G_\alpha(z) = \int_0^z [f'(\xi)]^\alpha \, d\xi \tag{1.1}$$

is univalent in U.

THEOREM 1.2 [3]. *If the function $g \in S$ and α is a complex number, $|\alpha| \le 1/(4n)$, then the function defined by*

$$G_{\alpha,n}(z) = \int_0^z [g'(u^n)]^\alpha \, du \tag{1.2}$$

is univalent in U for all positive integer n.

2. Preliminary results. We need the following theorems.

THEOREM 2.1 [2]. *Let α be a complex number,* $\operatorname{Re}\alpha > 0$ *and $f \in A$. If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \tag{2.1}$$

for all $z \in U$, then for any complex number β, $\operatorname{Re}\beta \ge \operatorname{Re}\alpha$ *the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) \, du \right]^{1/\beta} \tag{2.2}$$

is in the class S.

Theorem 2.2 [1]. *If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold:*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right| \le \left|\frac{\xi - z}{1-\bar{z}\xi}\right|, \tag{2.3}$$

$$|g'(z)| \le \frac{1-|g(z)|^2}{1-|z|^2}, \tag{2.4}$$

the equalities hold only in the case $g(z) = \epsilon(z+u)/(1+\overline{u}z)$, where $|\epsilon| = 1$ and $|u| < 1$.

Remark 2.3 [1]. For $z = 0$, from inequality (2.3)

$$\left|\frac{g(\xi)-g(0)}{1-\overline{g(0)}g(\xi)}\right| \le |\xi| \tag{2.5}$$

and, hence

$$|g(\xi)| \le \frac{|\xi|+|g(0)|}{1+|g(0)||\xi|}. \tag{2.6}$$

Considering $g(0) = a$ and $\xi = z$,

$$|g(z)| \le \frac{|z|+|a|}{1+|a||z|}. \tag{2.7}$$

for all $z \in U$.

Schwarz lemma [1]. *If the function g is regular in U, $g(0) = 0$ and $|g(z)| \le 1$ for all $z \in U$, then the following inequalities hold:*

$$|g(z)| \le |z| \tag{2.8}$$

for all $z \in U$, and $|g'(0)| \le 1$, the equalities (in inequality (2.8) for $z \ne 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. Main results

Theorem 3.1. *Let α, γ be complex numbers,* $\operatorname{Re}\alpha = a > 0$ *and $g \in A$.*
If

$$\left|\frac{g''(z)}{g'(z)}\right| \le \frac{1}{n} \tag{3.1}$$

for all $z \in U$ and

$$|\gamma| \le \frac{n+2a}{2}\left(\frac{n+2a}{n}\right)^{n/2a}, \tag{3.2}$$

then for any complex number β, $\operatorname{Re}\beta \ge a$, *the function*

$$G_{\beta,\gamma,n}(z) = \left\{\beta\int_0^z u^{\beta-1}[g'(u^n)]^\gamma\,du\right\}^{1/\beta} \tag{3.3}$$

is in the class S for all $n \in N^* - \{1\}$.

Proof. Let us consider the function

$$f(z) = \int_0^z [g'(u^n)]^\gamma \, du. \tag{3.4}$$

The function

$$p(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)}, \tag{3.5}$$

where the constant $|\gamma|$ satisfies the inequality (3.2), is regular in U.

From (3.4) and (3.5), we obtain

$$p(z) = \frac{\gamma}{|\gamma|} \left[\frac{nz^{n-1} g''(z^n)}{g'(z^n)} \right]. \tag{3.6}$$

Using (3.1) and (3.6) we obtain

$$|p(z)| < 1 \tag{3.7}$$

for all $z \in U$. For $z = 0$ we have $p(0) = 0$.

From (3.6) and Schwarz lemma it results that

$$\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \le |z|^{n-1} \le |z| \tag{3.8}$$

for all $z \in U$, and hence

$$\left(\frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \le |\gamma| \left(\frac{1 - |z|^{2a}}{a} \right) |z|^n. \tag{3.9}$$

Let us consider $Q : [0,1] \to R$, $Q(x) = ((1 - x^{2a})/a)x^n$, $x = |z|$. We have

$$Q(x) \le \frac{2}{n + 2a} \left(\frac{n}{n + 2a} \right)^{n/2a} \tag{3.10}$$

for all $x \in [0,1]$. From (3.2), (3.9), and (3.10) we obtain

$$\left(\frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \tag{3.11}$$

for all $z \in U$. Then, from (3.11) and Theorem 2.1 it follows that the function $G_{\beta,\gamma,n}$ is in the class S. □

Theorem 3.2. *Let* α, γ *be complex numbers,* $\operatorname{Re}\alpha = b > 0$ *and the function* $g \in A$, $g(z) = z + a_2 z^2 + \cdots$. *If*

$$\left| \frac{g''(z)}{g'(z)} \right| < 1 \tag{3.12}$$

for all $z \in U$ *and the constant* $|\gamma|$ *satisfies the condition*

$$|\gamma| \le \frac{1}{\max_{|z| \le 1} \left[((1 - |z|^{2b})/b) |z| ((|z| + 2|a_2|)/(1 + 2|a_2||z|)) \right]} \tag{3.13}$$

then for any complex number β, $\operatorname{Re}\beta \ge b$ the function

$$G_{\beta,\gamma}(z) = \left\{\beta \int_0^z u^{\beta-1}[g'(u)]^\gamma \, du\right\}^{1/\beta} \tag{3.14}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z [g'(u)]^\gamma \, du. \tag{3.15}$$

The function

$$h(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)}, \tag{3.16}$$

where the constant $|\gamma|$ satisfies the inequality (3.13), is regular in U.

From (3.15) and (3.16) we have

$$h(z) = \frac{\gamma}{|\gamma|} \frac{g''(z)}{g'(z)}. \tag{3.17}$$

Using (3.12) and (3.17) we obtain

$$|h(z)| < 1, \tag{3.18}$$

for all $z \in U$ and $|h(0)| = 2|a_2|$.

Remark 2.3 applied to the function h gives

$$\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \le \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \tag{3.19}$$

for all $z \in U$.

From (3.19) we obtain

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \le |\gamma| \frac{1 - |z|^{2b}}{b} |z| \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \tag{3.20}$$

for all $z \in U$. Hence, we have

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \le |\gamma| \max_{|z| \le 1} \left[\frac{1 - |z|^{2b}}{b} |z| \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \right]. \tag{3.21}$$

From (3.13) and (3.21) we obtain

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \tag{3.22}$$

for all $z \in U$. From Theorem 2.1, it follows that the function $G_{\beta,\gamma}$ defined by (3.14) is in the class S. □

References

[1] Z. Nehari, *Conformal Mapping*, McGraw-Hill Book Co., Inc., New York, Toronto, London, 1952. MR 13,640h. Zbl 048.31503.

[2] N. N. Pascu, *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session: Simion Stoïlow (Braşov, 1987) (Braşov), Univ. Braşov, 1987, pp. 43–48. MR 90j:30033. Zbl 666.30012.

[3] N. N. Pascu and V. Pescar, *On the integral operators of Kim-Merkes and Pfaltzgraff*, Mathematica (Cluj) **32(55)** (1990), no. 2, 185–192. MR 93d:30034. Zbl 761.30011.

[4] J. A. Pfaltzgraff, *Univalence of the integral of* $f'(z)^\lambda$, Bull. London Math. Soc. **7** (1975), no. 3, 254–256. MR 52#8410. Zbl 316.30012.

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