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BANACH-MACKEY, LOCALLY COMPLETE SPACES, AND $\ell_{p,q}$-SUMMABILITY

CARLOS BOSCH and ARMANDO GARCÍA

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Abstract. We defined the $\ell_{p,q}$-summability property and study the relations between the $\ell_{p,q}$-summability property, the Banach-Mackey spaces and the locally complete spaces.

We prove that, for c_0-quasibarrelled spaces, Banach-Mackey and locally complete are equivalent. Last section is devoted to the study of CS-closed sets introduced by Jameson and Kakol.

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1. Introduction. Let (E,τ) be a locally convex space. If A is absolutely convex its linear span E_A may be endowed with the seminorm topology given by the Minkowski functional of A, we denote it by (E_A,ρ_A). If A is bounded then (E_A,ρ_A) is a normed space. If every bounded set B is contained in an absolutely convex, closed, bounded set, called a disk A such that (E_A,ρ_A) is complete (barrelled) then E is said to be locally complete (barrelled).

A locally convex space is a Banach-Mackey space if $\sigma(E,E')$-bounded sets are $\beta(E,E')$-bounded sets.

Finally, let us define the $\ell_{p,q}$-summability property. For $1 \le p \le \infty$ let q be such that $(1/p)+(1/q)=1$. A sequence $(x_n)_n \subset E$ is p-absolutely summable if for every ρ continuous seminorm in (E,τ) the sequence $(\rho(x_n))_n$ is in ℓ_p. A p-absolutely summable sequence is $\ell_{p,q}$-summable if for every $(\lambda_n)_n \in \ell_q$, the series $\Sigma_{n=1}^{\infty}\lambda_n x_n$ converges to x for some $x \in E$. A locally convex space E has the $\ell_{p,q}$-summability property if each p-absolutely summable sequence is $\ell_{p,q}$-summable.

2. $\ell_{p,q}$-summability. Let $(E,\tau)=(c_0,\sigma(c_0,\ell_1))$. (E,τ) is a locally complete space. Take $\alpha=(\alpha_n)_n \in \ell_1$ and $(e_n)_n$ the canonical unit vectors in c_0. Then $\rho_\alpha(e_n)=|\alpha_n|$ so $\Sigma_{n=1}^{\infty}\rho_\alpha(e_n)=\Sigma_{n=1}^{\infty}|\alpha_n|<\infty$ which means that $(e_n)_n$ is absolutely summable for every continuous seminorm in $\sigma(c_0,\ell_1)$. Now, since $\Sigma_{n=1}^{\infty}(e_n)\notin c_0$ we have here an example of a space that has the $\ell_{\infty,1}$-summability property and does not have the $\ell_{1,\infty}$-summability property.

Now let us establish some properties of the spaces with the $\ell_{p,q}$-summability property.

Theorem 2.1. *Let (E,τ) be a locally convex space. If E satisfies the $\ell_{p,q}$-summability property for $1 \le p,\, q \le \infty$ with $(1/p)+(1/q)=1$, then E is locally complete.*

Proof. Let A be a bounded set and $B = \overline{\operatorname{abconv} A}$; B is a disk. Take $(x_n)_n \subset E_B$ a sequence such that $(\rho_B(x_n))_n \in \ell_p$. Since $i : (E_B,\rho_B) \hookrightarrow (E,\tau)$ is continuous, for every continuous seminorm ρ in E, we have $(\rho(x_n))_n \in \ell_p$. So for every $(a_n)_n \in \ell_q$, we have $\Sigma_{n=1}^{\infty} a_n x_n \to x$ with respect to τ since E has the $\ell_{p,q}$-summability property.

Now the sequence of partial sums $\Sigma_{n=1}^{k} a_n x_n$ is ρ_B-bounded since it is a ρ_B-Cauchy sequence as we can see

$$\rho_B\left(\sum_{n=1}^{k+r} a_n x_n - \sum_{n=1}^{k} a_n x_n\right) = \rho_B\left(\sum_{k+1}^{k+r} a_n x_n\right) \le \left|(a'_n)_n\right|_q \cdot \left|(\rho_B(x'_n))_n\right|_p, \tag{2.1}$$

which is small for k big enough, $(a'_n)_n = (0,\dots,0,a_{k+1},\dots,a_{k+r},0,\dots)$ and $(x'_n)_n = (0,\dots,0,x_{k+1},\dots,x_{k+r},0,\dots)$.

So $\{\Sigma_{n=1}^{K} a_n x_n : K \in \mathbb{N}\}$ is a ρ_B bounded set in (E_B,ρ_B).

By [5, Theorem 3.2.4] we have that $(\Sigma_{n=1}^{K} a_n x_n)_K$ converges to x in (E_B,ρ_B). So (E_B,ρ_B) has also the $\ell_{p,q}$-summability property.

Now, we will prove the space (E_B,ρ_B) is complete. Let $(x_n)_n \subset E_B$ be an absolutely summable sequence with $x_n \ne 0$ for every $n \in \mathbb{N}$, so $(\rho_B(x_n))_n \in \ell_1$ then

$$(\alpha_n)_n = \left(\rho_B^{1/p}(x_n)\right)_n \in \ell_p, \qquad (\beta_n)_n = \left(\rho_B^{1/q}(x_n)\right)_n \in \ell_q. \tag{2.2}$$

Let $y_n = x_n/\rho_B(x_n)$ then $(y_n)_n$ is ρ_B-bounded. So $(\alpha_n y_n)_n \subset E_B$, $(\rho_B(\alpha_n y_n)_n) \in \ell_p$ and $\Sigma_{n=1}^{\infty} x_n = \Sigma_{n=1}^{\infty} \alpha_n \beta_n y_n$ converges in (E_B,ρ_B) since (E_B,ρ_B) has the $\ell_{p,q}$-summability property so (E_B,ρ_B) is a Banach disk. □

Corollary 2.2. *Let (E,τ) be a locally convex space. (E,τ) is locally complete if and only if (E,τ) has the $\ell_{\infty,1}$-summability property.*

Proof. Let (E,τ) be a locally complete space and $(x_n)_n \subset (E,\tau)$ be a bounded sequence, so there exists a Banach disk $B \subset E$ such that $\{x_n\}_n \subset B$ and $\{x_n\}_n$ is bounded in (E_B,ρ_B).

Let $(\alpha_n)_n \in \ell_1$, then $(\alpha_n x_n)_n$ is ρ_B-absolutely summable, that is $\Sigma_{n=1}^{\infty} \rho_B(\alpha_n x_n) < \infty$. Hence $\Sigma_{n=1}^{\infty} \alpha_n x_n$ converges in (E_B,ρ_B) so it also converges in (E,τ) since $i : (E_B,\rho_B) \hookrightarrow (E,\tau)$ is continuous. So E has the $\ell_{\infty,1}$-summability property. □

Corollary 2.3. *E is a Banach space if and only if E is normed and has the $\ell_{p,q}$-summability property.*

Proof. We can reproduce the last part of the proof of Theorem 2.1 to show that E normed and with the $\ell_{p,q}$-summability property is a locally complete normed space and so a Banach space.

Now suppose E is a Banach space and denote the norm by $\|\ \|$. Let $(x_n)_n \subset E$ be a sequence such that $(\|x_n\|)_n \in \ell_p$ and let $(\beta_n)_n \in \ell_q$ then the sequence $(\beta_n x_n)_n$ is absolutely summable that is

$$\sum_{n=1}^{\infty}\|\beta_n x_n\| \le \left(\sum_{n=1}^{\infty}\|x_n\|^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\|\beta_n\|^q\right)^{1/q} < \infty \tag{2.3}$$

hence summable, since E is a Banach space so E has the $\ell_{p,q}$-summability property. □

3. Banach-Mackey space

Definition 3.1. E is a c_0-barrelled (c_0-quasibarrelled) space if each null sequence in $(E', \sigma(E',E))$ $((E', \beta(E',E)))$ is E-equicontinuous.

Note that a c_0-barrelled space is a c_0-quasibarrelled space.

Lemma 3.2. *If $(E, \mu(E,E'))$ is a Banach-Mackey space, where $\mu(E,E')$ denotes the Mackey topology, and c_0-quasibarrelled space then it is a c_0-barrelled space.*

Proof. Let $A \subset (E', \sigma(E',E))$ be a bounded set, since E is a Banach-Mackey space, E' is also a Banach-Mackey space (cf. [9, Theorem 5, page 158]), and then A is $\beta(E',E)$-bounded so it is contained in a bounded Banach disk by [2, Observation 8.2.23], since the space is c_0-quasibarrelled. Then by the same observation we have that $(E', \sigma(E',E))$ is locally complete. □

Corollary 3.3. *$(E, \mu(E,E'))$ is c_0-quasibarrelled and Banach-Mackey if and only if $(E', \sigma(E',E))$ is locally complete.*

Proof. Necessity follows from previous lemma and [2, Observation 8.2.23]. The other implication follows from the same observation, the note following Definition 3.1 and the fact that by [7, Corollary 3, Theorem 1] we have that $(E', \sigma(E',E))$ locally complete implies $(E, \mu(E,E'))$ is a Banach-Mackey space. □

Following Saxon and Sánchez [8], a space E is dual locally complete if $(E, \sigma(E',E))$ is locally complete; then we can extend the result shown in [8, Theorem 2.6].

Corollary 3.4. *$(E, \mu(E,E'))$ is dual locally complete if and only if it is Banach-Mackey and c_0-quasibarrelled.*

A locally convex space E is quasibarrelled if each barrel that absorbs bounded sets is a neighborhood of zero in E. It is clear that a barrelled space is quasibarrelled, in certain cases they are equivalent.

Note that using [7, Theorem 1] we can easily prove that: a locally convex space E is quasibarrelled and Banach-Mackey if and only if it is a barrelled space. Next proposition summarizes what we know about Banach-Mackey spaces in the case of quasibarrelled spaces.

Proposition 3.5. *Let (E, τ) be a locally convex quasibarrelled space, then the following properties are equivalent:*

(a) *E' is a Banach-Mackey space.*
(b) *E is a Banach-Mackey space.*
(c) *E is barrelled.*
(d) *E' is semireflexive.*
(e) *In E', $\operatorname{abconv} K$ is compact for each $K \subset E'$ compact.*

(f) *For every* $x_n \to 0$ *in* E' *and every* $(\alpha_n)_n \in \ell_1$, $\Sigma_{n=1}^{\infty} \alpha_n x_n \to x$ *for some* $x \in E'$.
(g) E' *is locally complete.*
(h) E' *is locally barrelled.*

Proof. (a)⟹(b) using [9, Theorem 5, page 158]. (b)⟹(c) from the previous note. (c)⟹(d) by [9, Theorem 4, page 153]. (d)⟹(e) is obtained using the same theorem and the fact that a convex hull of a compact set is totally bounded together with [9, Exercise 5, page 122]. (e)⟹(f) by [7, Theorems 2 and 3]. (f)⟹(g) using [3, Proposition III.1.4] and [2, Theorem 5.1.11]. (g)⟹(h) is trivial. (h)⟹(a) using [1, Theorem 1]. □

Note that (f) and (g) are equivalent in general, [3, Proposition III.1.4] and [2, Theorem 5.1.11] prove (f)⟹(g) and do not assume E is quasibarrelled, and the other implication can be obtained using an argument similar to the one in Corollary 2.2.

4. CS-closed sets.

In this section, we give a more precise definition of the convex series and their properties, first studied by Jameson [4] and Käkol [6].

Definition 4.1. Let (E,τ) be a locally convex space.

(a) Let $A \subset E$, $(a_n)_n \subset A$ and $(c_n) \subset [0,1]$ such that $\Sigma_{n=1}^{\infty} c_n = 1$ if $\Sigma_{n=1}^{\infty} c_n a_n$ is convergent we say that it is a convex convergent series of elements of A.

(b) $A \subset E$ is CS-closed if each convex convergent series of elements of A belongs to A.

(c) $A \subset E$ is CS-compact if each convex series of elements of A converges to an element of A.

(d) $A \subset E$ is ultrabounded if each convex series of elements of A is convergent in E.

(e) The CS-closure of A is the intersection of all CS-closed sets that contain A.

Observation. (i) An ultrabounded set is bounded.

(ii) The intersection of CS-closed sets is a CS-closed set.

For convenience let us introduce another definition.

Definition 4.2. (a) $B \subset E$ is called a CS-barrel if it is absolutely convex, absorbent and CS-closed.

(b) E is a locally CS-barrelled (barrelled) space if for each bounded set $A \subset E$ there exists a disk B such that $A \subset B$ and E_B is a CS-barrelled (barrelled) space, that is that each CS-barrel (barrel) is a neighborhood of zero.

Now several properties of barrels also hold for CS-barrels although the last sets are somehow "smaller" than the first sets.

It is clear that if E is a CS-barrelled space then it is a barrelled space.

Now if (E,τ) is locally barrelled, then for each bounded set $A \subset E$ there exists a closed bounded disk B such that $A \subset B \subset E$ and (E_B,ρ_B) is barrelled, so for each CS-barrel U in E_B, $\overline{U}$ is a barrel so it is a zero neighborhood with respect to ρ_B, since (E_B,ρ_B) is metrizable by [4, Theorem 1], U is also a zero neighborhood with respect to ρ_B. So we have proved the following.

Proposition 4.3. (E,τ) *is a locally barrelled space if and only if it is locally CS-barrelled space.*

The CS-compact hull of a set A is the set of convex convergent series of its elements. A is CS-compact if each convex series of elements of A converges to an element of A, so we have that the CS-compact hull of a set is not necessarily a CS-compact set. This is the moment to bring in the ultrabounded sets, since the CS-compact hull of an ultrabounded set is a CS-compact set.

Proposition 4.4. *In a locally convex space (E,τ), CS-barrels absorb ultrabounded sets.*

Proof. Let W be a CS-barrel and A an ultrabounded set in E. Let D be the balanced CS-compact hull of A, by [6, Corollaries 2–4] D is a Banach disk so E_D is barrelled, and the identity map $i : E_D \to E$ is continuous so $\overline{W}^{\tau} \cap E_D$ is a barrel in (E_D, ρ_D), furthermore it is a neighborhood of zero in E_D, so $A \subset D \subset \lambda \overline{W}^{\tau} \cap E_D$ for some $\lambda > 0$. Now for $(x_n)_n \subset W \cap E_D$ and $(a_n)_n \in [0,1]$, with $\Sigma_n a_n = 1$ such that $\Sigma_n a_n x_n \to x$ in (E_D, ρ_D), since W is a CS-barrel in (E,τ), we have $\Sigma_n a_n x_n \to x$ in (E,τ) and $x \in W$, then $x \in W \cap E_D$ and it is a CS-barrel in (E_D, ρ_D). By [4, Theorem 1], $W \cap E_D$ and $\overline{W}^{\tau} \cap E_D$ have the same interior with respect to ρ_B, so $A \subset D \subset \lambda(W \cap E_D) \subset \lambda W$. □

Remark 4.5. Since every Banach disk is ultrabounded (cf. [6, Proposition 2.2]) then each CS-barrel absorbs Banach disks.

To close this section let us mention that if E is locally barrelled then each CS-barrel is a bornivorous (see [7, proof of Theorem 2(1)]).

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Bosch: Departmento de Matemáticas, ITAM, Rio Hondo #1, 01000 México D. F., Mexico
E-mail address: bosch@itam.mx

García: Instituto de Matemáticas, Zona de la Investigacion Científica, Circuito Exterior, Ciudad Universitaria, 04510 México D. F., Mexico