Internat. J. Math. & Math. Sci.
Vol. 23, No. 1 (2000) 45–54
S0161171200001769

A NOTE ON CONNECTEDNESS IN INTUITIONISTIC FUZZY SPECIAL TOPOLOGICAL SPACES

SELMA ÖZÇAĞ and DOĞAN ÇOKER

(Received 31 July 1998)

ABSTRACT. We prove some properties of several types of connectedness defined in intuitionistic fuzzy special topological spaces.

Keywords and phrases. Intuitionistic fuzzy special set, intuitionistic fuzzy special topological space, connectedness.

2000 Mathematics Subject Classification. 54A99, 03E99.

1. Introduction. After the introduction of the concept of fuzzy sets by Zadeh [12], several researches were conducted on the generalizations of the notion of fuzzy set. The idea of "intuitionistic fuzzy set" was first given by Atanassov [2, 3]. Later this concept is generalized to intuitionistic sets in Çoker [6] and intuitionistic topological spaces in [5, 9, 10]. An introduction to connectedness in these spaces is given in [10].

2. Preliminaries. First we present the fundamental definitions (see [6]).

Definition 2.1 (cf. [5, 9])**.** Let X be a nonempty fixed set. An intuitionistic fuzzy special set (IFSS for short) A is an object having the form $A = \langle x, A_1, A_2\rangle$, where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. The set A_1 is called the set of members of A, while A_2 is called the set of nonmembers of A.

The reader may consult [6, 9] to see several types of relations and operations on IFSS's, and intuitionistic fuzzy special points (IFSP's for short) and vanishing intuitionistic fuzzy special points (VIFSP's for short).

Definition 2.2 (cf. [5, 7, 8, 9, 10, 11])**.** An intuitionistic fuzzy special topology (IFST for short) on a nonempty set X is a family τ of IFSS's in X containing $\underset{\sim}{\emptyset}$, $\underset{\sim}{X}$ and closed under finite infima and arbitrary suprema. In this case the pair (X,τ) is called an intuitionistic fuzzy special topological space (IFSTS for short) and any IFSS in τ is known as an intuitionistic fuzzy special open set (IFSOS for short) in X. The complement $\bar{A}$ of an IFSOS A in an IFSTS (X,τ) is called an intuitionistic fuzzy special closed set (IFSCS for short) in X.

Using a similar construction as in [7], one can easily define the interior and closure operators in IFSTS's.

3. Types of connectedness in intuitionistic fuzzy special topological spaces. Throughout this section (X,τ) and (Y,Φ) will always denote IFSTS's. Here we define several types of connectedness in IFSTS's.

Notice that two IFSS's A and B in (X,τ) are said to be weakly separated, if $\mathrm{cl}(A) \subseteq \bar{B}$ and $\mathrm{cl}(B) \subseteq \bar{A}$; and q-separated, if $\mathrm{cl}(A) \cap B = \underset{\sim}{\emptyset} = A \cap \mathrm{cl}(B)$.

Lemma 3.1.

$$A \cap B = \underset{\sim}{\emptyset} \Longrightarrow A \subseteq \bar{B}; \tag{3.1}$$

$$A \nsubseteq \bar{B} \Longrightarrow A \cap B \neq \underset{\sim}{\emptyset}. \tag{3.2}$$

Definition 3.1 (cf. [1, 10, 11]). Let (X,τ) be an IFSTS in X.

(a) X is called C_S-disconnected, if there exist weakly separated nonzero IFSS's A and B in (X,τ) such that $\underset{\sim}{X} = A \cup B$. (X,τ) is called C_S-connected, if (X,τ) is not C_S-disconnected.

(b) X is called C_M-disconnected, if there exist q-separated nonzero IFSS's A and B in (X,τ) such that $\underset{\sim}{X} = A \cup B$. X is called C_M-connected, if X is not C_M-disconnected.

The idea of C_i-connectedness in fuzzy topological spaces and in intuitionistic fuzzy topological spaces (see [1, 11]) can be generalized to the intuitionistic case.

Definition 3.2 (cf. [10]). Let N be an IFSS in (X,τ).

(a) If there exist IFSOS's M and W in X satisfying the following properties, then N is called C_i-disconnected $(i = 1,2,3,4)$.
  - C_1: $N \subseteq M \cup W$, $M \cap W \subseteq \bar{N}$, $N \cap M \neq \underset{\sim}{\emptyset}$, $N \cap W \neq \underset{\sim}{\emptyset}$,
  - C_2: $N \subseteq M \cup W$, $M \cap W \cap N = \underset{\sim}{\emptyset}$, $N \cap M \neq \underset{\sim}{\emptyset}$, $N \cap W \neq \underset{\sim}{\emptyset}$,
  - C_3: $N \subseteq M \cup W$, $M \cap W \subseteq \bar{N}$, $M \nsubseteq \bar{N}$, $W \nsubseteq \bar{N}$,
  - C_4: $N \subseteq M \cup W$, $M \cap W \cap N = \underset{\sim}{\emptyset}$, $M \nsubseteq \bar{N}$, $W \nsubseteq \bar{N}$.

(b) N is said to be C_i-connected $(i = 1,2,3,4)$ if N is not C_i-disconnected $(i = 1,2,3,4)$.

Corollary 3.1. *P, Q are weakly separated if and only if $\exists M, W \in \tau$ such that $P \subseteq M$, $Q \subseteq W$, $P \subseteq \bar{W}$, and $Q \subseteq \bar{M}$.*

Proof. $(\Leftarrow)$ Suppose there exist M, $W \in \tau$ such that $P \subseteq M$, $Q \subseteq W$, $P \subseteq \bar{W}$, and $Q \subseteq \bar{M}$. Then $\mathrm{cl}(P) \subseteq \mathrm{cl}(\bar{W}) = \bar{W}$ (since $\bar{W}$ is an IFSCS) and $\mathrm{cl}(Q) \subseteq \mathrm{cl}(\bar{M}) = \bar{M} \Rightarrow \mathrm{cl}(P) \subseteq \bar{W} \subseteq \bar{Q} \Rightarrow \mathrm{cl}(P) \subseteq \bar{Q}$ and $\mathrm{cl}(Q) \subseteq \bar{M} \subseteq \bar{P} \Rightarrow \mathrm{cl}(Q) \subseteq \bar{P} \Rightarrow P, Q$ are weakly separated.

$(\Rightarrow)$ Let $\mathrm{cl}(P) \subseteq \bar{Q}$, $\mathrm{cl}(Q) \subseteq \bar{P}$. Now take $W = \overline{\mathrm{cl}(P)}$ and $M = \overline{\mathrm{cl}(Q)}$ which are IFSOS's in (X,τ). Hence $\bar{W} \subseteq \bar{Q}$ and $\bar{M} \subseteq \bar{P} \Rightarrow P \subseteq M$, $Q \subseteq W$. We also have $W = \overline{\mathrm{cl}(P)} \subseteq \bar{P} \Rightarrow P \subseteq \bar{W}$ and $M = \overline{\mathrm{cl}(Q)} \subseteq \bar{Q} \Rightarrow Q \subseteq \bar{M}$. □

Here we define C_S-connectedness and C_M-connectedness of an IFSS in (X,τ).

Definition 3.3 (cf. Ajmal-Kohli [1]). An IFSS N in (X,τ) is said to be C_S-disconnected (C_M-disconnected) if and only if there are two nonempty weakly separated (q-separated) IFSS's A and B in (X,τ) such that $N = A \cup B$. N is called C_S-connected (C_M-connected) if and only if N is not C_S-disconnected (C_M-disconnected).

Theorem 3.1. *If N is C_3-connected, then N is C_M-connected.*

Proof. Let N be C_M-disconnected. Then there exist IFSS's A, B such that $N = A \cup B$, $A, B \neq \underset{\sim}{\emptyset}$ and A, B are q-separated. Let $P = \overline{\mathrm{cl}(A)}$ and $Q = \overline{\mathrm{cl}(B)}$. Then P, Q are IFSOS's.

Now

$$\begin{aligned} \operatorname{cl}(A)\cap\operatorname{cl}(B) &\subseteq \bar{A}\cap\bar{B} = \overline{A\cup B} = \bar{N} \Longrightarrow N \\ &\subseteq \overline{\operatorname{cl}(A)\cap\operatorname{cl}(B)} = \overline{\operatorname{cl}(A)}\cup\overline{\operatorname{cl}(B)} \\ &= P\cup Q \Longrightarrow N\subseteq P\cup Q, \end{aligned} \tag{3.3}$$

$$P\cap Q = \overline{\operatorname{cl}(A)}\cap\overline{\operatorname{cl}(B)} = \overline{\operatorname{cl}(A)\cup\operatorname{cl}(B)} = \overline{\operatorname{cl}(A\cup B)} \subseteq \overline{A\cup B} = \bar{N} \Longrightarrow P\cap Q\subset \bar{N}. \tag{3.4}$$

If $P\subseteq\bar{N}$, then $N\subseteq\operatorname{cl}(A)\Rightarrow N\cap B=\underset{\sim}{\emptyset}$ (since $\operatorname{cl}(A)\cap B=\underset{\sim}{\emptyset}$) and $N\cap B=(A\cup B)\cap B = B=\underset{\sim}{\emptyset}$. This is a contradiction. Hence $P\nsubseteq\bar{N}$ follows. $Q\nsubseteq\bar{N}$ can be proved similarly. □

Theorem 3.2. *If N is C_1-connected, then N is C_S-connected.*

Proof. Let N be C_S-disconnected. Then there exist IFSS's A, B such that $N=A\cup B$, A, $B\neq\underset{\sim}{\emptyset}$ and A, B are weakly separated. Let $P=\overline{\operatorname{cl}(A)}$ and $Q=\overline{\operatorname{cl}(B)}$. Then P, Q are IFSOS's. We have seen that $N\subseteq P\cup Q$ and $P\cap Q\subseteq\bar{N}$. If $P\cap N=\underset{\sim}{\emptyset}$, then $P\subseteq\bar{N}\Rightarrow N\subseteq\bar{P}\Rightarrow N\subseteq\operatorname{cl}(A)\subseteq\bar{B}\Rightarrow N\subseteq\bar{B}$. Since $N=A\cup B$ and $A\cup B\subseteq\bar{B}$, we obtain a contradiction. Hence $P\cap N\neq\underset{\sim}{\emptyset}$ follows. Similarly, it can be proved that $Q\cap N\neq\underset{\sim}{\emptyset}$. □

Theorem 3.3. *If N is C_S-connected, then N is C_2-connected.*

Proof. Suppose, on the contrary, that N is C_2-disconnected. Hence there exist IFSOS's M, W such that $N\subseteq M\cup W$, $N\cap M\cap W=\underset{\sim}{\emptyset}$, $N\cap M\neq\underset{\sim}{\emptyset}$, $N\cap W\neq\underset{\sim}{\emptyset}$. Now, take $P=N\cap M$ and $Q=N\cap W$. Since $N\subseteq M\cup W$, we get $N=N\cap(M\cup W)=(N\cap M)\cup(N\cap W)=P\cup Q$. We show that P and Q are weakly separated. Let $P\subseteq M$, $Q\subseteq W$. Suppose that $P\nsubseteq\bar{W}$. Then $P\cap W\neq\underset{\sim}{\emptyset}\Rightarrow(N\cap M)\cap W\neq\underset{\sim}{\emptyset}$, a contradiction, in other words $P\subseteq\bar{W}$ follows. Similarly one can also show that $Q\subseteq\bar{M}$. Thus P, Q are weakly separated, which is a contradiction. Therefore N is C_2-connected. □

Theorem 3.4. *If N is C_S-connected, then N is C_3-connected.*

Proof. Similar to the previous one. □

C_S-connectedness does not imply C_1-connectedness in general:

Counterexample 3.1. Let $X=\{a,b,c\}$ and $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A_1,A_2,A_3\}$ where

$$A_1=\langle x,\{c\},\{a,b\}\rangle,\qquad A_2=\langle x,\{a\},\{b,c\}\rangle,\qquad A_3=\langle x,\{a,c\},\{b\}\rangle. \tag{3.5}$$

If $N=\langle x,\{a\},\{b\}\rangle$, then N is C_S-connected, since there exist no two nonempty weakly separated IFSS's A, $B\neq\underset{\sim}{\emptyset}$ such that $N=A\cup B$. But N is C_1-disconnected.

If N is C_2-connected (C_3-connected), then N may not be C_S-connected.

Counterexample 3.2. Let $X=\{a,b,c,d\}$ and $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A_1,A_2,A_3,A_4\}$, where

$$\begin{aligned} A_1&=\langle x,\{c\},\{a,b\}\rangle, & A_2&=\langle x,\{a,c\},\{b\}\rangle, \\ A_3&=\langle x,\{a\},\{b\}\rangle, & A_4&=\langle x,\emptyset,\{a,b\}\rangle. \end{aligned} \tag{3.6}$$

Now take $N = \langle x, \{a\}, \{b,c\}\rangle$. N is C_2-connected (C_3-connected) but not C_S-connected, since there exist two nonempty weakly separated IFSS's A, $B \neq \underset{\sim}{\emptyset}$ such that $N = A \cup B$; namely

$$A = \langle x, \emptyset, \{a,b,c\}\rangle, \qquad B = \langle x, \{a\}, \{b,c\}\rangle. \tag{3.7}$$

C_2-connectedness does not imply C_M-connectedness in general as shown below.

Counterexample 3.3. Let $X = \{a,b,c\}$ and $\tau = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}, A_1, A_2, A_3, A_4\}$, where

$$\begin{aligned} A_1 &= \langle x, \{b\}, \{c\}\rangle, & A_2 &= \langle x, \{c\}, \{a\}\rangle, \\ A_3 &= \langle x, \{b,c\}, \emptyset\rangle, & A_4 &= \langle x, \emptyset, \{a,c\}\rangle. \end{aligned} \tag{3.8}$$

$N = \langle x, \{c\}, \{a\}\rangle$ is C_2-connected, but not C_M-connected, since N can be expressed as the join of two nonempty q-separated IFSS's

$$A = \langle x, \{c\}, \{a,b\}\rangle, \qquad B = \langle x, \emptyset, \{a,c\}\rangle. \tag{3.9}$$

Similarly, C_M-connectedness does not imply C_3- (C_4-)connectedness in general:

Counterexample 3.4. Let $X = \{a,b,c\}$ and $\tau = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}, A_1, A_2, A_3\}$, where

$$A_1 = \langle x, \{c\}, \{a,b\}\rangle, \qquad A_2 = \langle x, \{a\}, \{b,c\}\rangle, \qquad A_3 = \langle x, \{a,c\}, \{b\}\rangle. \tag{3.10}$$

Let $N = \langle x, \{a\}, \{b\}\rangle$. N is C_M-connected, since there exist no two nonempty q-separated IFSS's A, $B \neq \underset{\sim}{\emptyset}$ such that $N = A \cup B$. But N is C_3-disconnected (C_4-disconnected). If N is C_4-connected, then N may not be C_M-connected.

Counterexample 3.5. Let $X = \{a,b,c,d\}$ and $\tau = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}, A_1, A_2, A_3, A_4\}$, where

$$\begin{aligned} A_1 &= \langle x, \{c\}, \{a,b\}\rangle, & A_2 &= \langle x, \{a,c\}, \{b\}\rangle, \\ A_3 &= \langle x, \{a\}, \{b\}\rangle, & A_4 &= \langle x, \emptyset, \{a,b\}\rangle. \end{aligned} \tag{3.11}$$

If $N = \langle x, \{a\}, \{b,d\}\rangle$, then N is C_4-connected, but not C_M-connected. This is because, N can be expressed as the join of two nonempty q-separated IFSS's A and B, where

$$A = \langle x, \emptyset, \{a,b,d\}\rangle, \qquad B = \langle x, \{a\}, \{b,c,d\}\rangle. \tag{3.12}$$

Now, we summarize the relations between several types of connectedness.

$$\begin{array}{ccccc} C_1\text{-connectedness} & \longrightarrow & C_S\text{-connectedness} & \longrightarrow & C_2\text{-connectedness} \\ & & \downarrow & & \downarrow \\ C_M\text{-connectedness} & \longleftarrow & C_3\text{-connectedness} & \longrightarrow & C_4\text{-connectedness}. \end{array} \tag{3.13}$$

None of these implications are reversible, as given here and in [10]. The following example shows that the closure of C_1- (C_2-)connected IFSS need not be C_1-connected (C_2-connected).

Counterexample 3.6. Let $X = \{a,b,c,d\}$ and $\tau = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}, A_1, A_2, A_3\}$, where

$$A_1 = \langle x, \{a,b\}, \{c,d\}\rangle, \qquad A_2 = \langle x, \{d\}, \{a,b\}\rangle, \qquad A_3 = \langle x, \{a,b,d\}, \emptyset\rangle. \tag{3.14}$$

If $N = \langle x, \{b\}, \{c,d\}\rangle$, then N is C_1-connected (C_2-connected), but $\mathrm{cl}(N)$ is C_1-disconnected (C_2-disconnected).

Theorem 3.5. *The closure of C_3-connected (C_4-connected) IFSS is C_3-connected (C_4-connected)*

Proof. Let N be C_3-connected, but $\mathrm{cl}(N)$ be C_3-disconnected. Hence there exist IFSOS's $M, W \neq \underset{\sim}{\emptyset}$ such that $\mathrm{cl}(N) \subseteq M \cup W$, $M \cap W \subseteq \overline{\mathrm{cl}(N)}$, $M \not\subseteq \overline{\mathrm{cl}(N)}$, $W \not\subseteq \overline{\mathrm{cl}(N)}$. We easily deduce $N \subseteq \mathrm{cl}(N) \subseteq M \cup W$ and $M \cap W \subseteq \overline{\mathrm{cl}(N)} \subseteq \bar{N}$. Since N is C_3-connected, $M \subseteq \bar{N}$ or $W \subseteq \bar{N}$ follows. If $M \subseteq \bar{N}$, then $N \subseteq \bar{M} \Rightarrow \mathrm{cl}(N) \subseteq \mathrm{cl}(\bar{M}) = \overline{\mathrm{int}(M)} = \bar{M}$, i.e., $\mathrm{cl}(N) \subseteq \bar{M}$ or $M \subseteq \overline{\mathrm{cl}(N)}$. But this is a contradiction to the fact $M \not\subseteq \overline{\mathrm{cl}(N)}$. Similarly, we obtain a contradiction in case $W \subseteq \bar{N}$. Therefore $\mathrm{cl}(N)$ is also C_3-connected. The other case can be proved similarly. □

Theorem 3.6. *If N is C_3-connected (C_4-connected) IFSS in (X,τ) and $N \subseteq P \subseteq \mathrm{cl}(N)$, then P is C_3-connected (C_4-connected) IFSS in (X,τ), too.*

Proof. Assume the contrary and let M, W be IFSOS's in X such that $N \subseteq P \subseteq M \cup W$, $M \cap W \subseteq \bar{P} \subseteq \bar{N}$. Since N is C_3-connected, $M \subseteq \bar{N}$ or $W \subseteq \bar{N}$ follows. If $M \subseteq \bar{N}$, then $N \subseteq \bar{M} \Rightarrow \mathrm{cl}(N) \subseteq \mathrm{cl}(\bar{M}) = \overline{\mathrm{int}(M)} = \bar{M} \Rightarrow \mathrm{cl}(N) \subseteq \bar{M}$. On the other hand, if $N \subseteq \bar{W}$, then $\mathrm{cl}(N) \subseteq \mathrm{cl}(\bar{W}) = \overline{\mathrm{int}(W)} = \bar{W} \Rightarrow \mathrm{cl}(N) \subseteq \bar{W}$. $P \subseteq \mathrm{cl}(N) \subseteq \bar{M}$ and $P \subseteq \mathrm{cl}(N) \subseteq \bar{W}$. Therefore P is C_3-connected. □

This theorem fails in the cases of C_1- (C_2-)connectedness as shown by the following example.

Counterexample 3.7. Let $X = \{a,b,c,d\}$ and $\tau = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}, A_1, A_2, A_3\}$, where

$$A_1 = \langle x, \{a,b\}, \{c,d\}\rangle, \qquad A_2 = \langle x, \{d\}, \{a,b\}\rangle, \qquad A_3 = \langle x, \{a,b,d\}, \emptyset\rangle. \tag{3.15}$$

If $N = \langle x, \{a\}, \{c,d\}\rangle$, then N is C_2-connected. If we take the IFSS $P = \langle x, \{a\}, \{d\}\rangle$, then P satisfies the inclusions $N \subseteq P \subseteq \mathrm{cl}(N)$, and P is not C_2-connected. On the other hand, if we consider the C_1-connected IFSS $N = \langle x, \{b\}, \{c,d\}\rangle$ in (X,τ), then $P = \langle x, \{b\}, \{d\}\rangle$ satisfies the inclusions $N \subseteq P \subseteq \mathrm{cl}(N)$, but it is not C_1-connected.

Theorem 3.7. *If N_1 and N_2 are intersecting C_1-connected IFSS's, then $N_1 \cup N_2$ is also C_1-connected.*

Proof. Assume that $N_1 \cup N_2$ is C_1-disconnected. Thus there exist IFSOS's M and W such that $N_1 \cup N_2 \subseteq M \cup W$ and $M \cap W \subseteq \overline{N_1 \cup N_2}$, $(N_1 \cup N_2) \cap M \neq \underset{\sim}{\emptyset}$ and $(N_1 \cup N_2) \cap W \neq \underset{\sim}{\emptyset}$. Since N_1 and N_2 are C_1-connected, then ($N_1 \cap M = \underset{\sim}{\emptyset}$ or $N_1 \cap W = \underset{\sim}{\emptyset}$) and ($N_2 \cap M = \underset{\sim}{\emptyset}$ or $N_2 \cap W = \underset{\sim}{\emptyset}$) follow. Since $N_1 \cap N_2 \neq \underset{\sim}{\emptyset}$, $\exists \underset{\approx}{p} \in (N_1 \cap N_2)$, there exist four cases:

Case 1. Let $N_1 \cap M = \underset{\sim}{\emptyset}$ and $N_2 \cap M = \underset{\sim}{\emptyset}$. In this case we get $(N_1 \cap M) \cup (N_2 \cap M) = (N_1 \cup N_2) \cap M = \underset{\sim}{\emptyset}$, a contradiction.

Case 2. Let $N_1 \cap M = \underset{\sim}{\emptyset}$ and $N_2 \cap W = \underset{\sim}{\emptyset}$. Then $\underset{\approx}{p} \notin M$, $\underset{\approx}{p} \notin W$. But this is impossible, since $\underset{\approx}{p} \in N_1 \cup N_2 \subseteq M \cup W$.

Case 3 and Case 4. $N_1 \cap W = \underset{\sim}{\emptyset}$ and $N_2 \cap M = \underset{\sim}{\emptyset}$, or $N_1 \cap W = \underset{\sim}{\emptyset}$ and $N_2 \cap W = \underset{\sim}{\emptyset}$. These cases may be treated similarly.

Hence it is seen that $N_1 \cup N_2$ is C_1-connected. □

Theorem 3.8. *If N_1 and N_2 are intersecting C_2-connected IFSS's, then $N_1 \cup N_2$ is also C_2-connected.*

Proof. Assume that $N_1 \cup N_2$ is C_2-disconnected. Then there exist IFSOS's M and W such that $N_1 \cup N_2 \subseteq M \cup W (N_1 \cup N_2) \cap M \cap W = \underset{\sim}{\emptyset}$, $(N_1 \cup N_2) \cap M \neq \underset{\sim}{\emptyset}$ and $(N_1 \cup N_2) \cap W \neq \underset{\sim}{\emptyset}$. Since $N_1 \cap N_2 \neq \underset{\sim}{\emptyset}$, $\exists \underset{\approx}{p} \in N_1 \cap N_2$, and since N_1 and N_2 are C_2-connected, then $(N_1 \cap M = \underset{\sim}{\emptyset}$ or $N_1 \cap W = \underset{\sim}{\emptyset})$ and $(N_2 \cap M = \underset{\sim}{\emptyset}$ or $N_2 \cap W = \underset{\sim}{\emptyset})$.

Case 1. Let $N_1 \cap M = \underset{\sim}{\emptyset}$ and $N_2 \cap M = \underset{\sim}{\emptyset}$. Then $(N_1 \cup N_2) \cap M = (N_1 \cap M) \cup (N_2 \cap M) = \underset{\sim}{\emptyset}$, a contradiction.

Case 2. Let $N_1 \cap M = \underset{\sim}{\emptyset}$ and $N_2 \cap W = \underset{\sim}{\emptyset}$. Then we obtain $\underset{\approx}{p} \notin M$, $\underset{\approx}{p} \notin W$ a contradiction to $\underset{\approx}{p} \in N_1 \cup N_2 \subseteq M \cup W$.

Case 3 and Case 4. They are similar to the ones given above.

Hence $N_1 \cup N_2$ is C_2-connected. □

Definition 3.4. Two IFSS's A and B are said to be overlapping, if $N_1 \not\subseteq \overline{N_2}$. Conversely, N_1 and N_2 are said to be nonoverlapping, if $N_1 \subseteq \overline{N_2}$.

Notice that

$$\begin{aligned} N_1 \not\subseteq \overline{N_2} &\iff N_1^{(1)} \not\subseteq N_2^{(2)} \text{ or } N_1^{(2)} \not\supseteq N_2^{(1)} \\ &\iff \exists x \left(x \in N_1^{(1)}, x \notin N_2^{(2)} \right) \text{ or } \exists y \left(y \in N_2^{(1)}, y \notin N_1^{(2)} \right) \\ &\iff \exists x \left(\underset{\sim}{x} \in N_1, \underset{\approx}{x} \in N_2 \right) \text{ or } \exists y \left(\underset{\sim}{y} \in N_2, \underset{\approx}{y} \in N_1 \right). \end{aligned} \tag{3.16}$$

Theorem 3.9. *If N_1 and N_2 are overlapping C_3-connected IFSS's, then so is $N_1 \cup N_2$.*

Proof. Let $N_1 \cup N_2$ be C_3-disconnected. Then there exist IFSOS's M and W such that $N_1 \cup N_2 \subseteq M \cup W$, $M \cap W \subseteq \overline{N_1 \cup N_2}$, $M \not\subseteq \overline{N_1 \cup N_2}$, $W \not\subseteq \overline{N_1 \cup N_2}$. Since N_1 and N_2 are overlapping, $\exists x (\underset{\sim}{x} \in N_1, \underset{\approx}{x} \in N_2)$ or $\exists y (\underset{\sim}{y} \in N_2, \underset{\approx}{y} \in N_1)$. Since N_1 and N_2 are C_3-connected, then we obtain: $(M \subseteq \overline{N_1}$ or $W \subseteq \overline{N_1})$ and $(M \subseteq \overline{N_2}$ or $W \subseteq \overline{N_2})$.

Case 1. Let $M \subseteq \overline{N_1}$ and $M \subseteq \overline{N_2}$. Then $M \subseteq \overline{N_1} \cap \overline{N_2} = \overline{N_1 \cup N_2}$, a contradiction to $M \not\subseteq \overline{N_1 \cup N_2} \Rightarrow$

Case 2. Let $M \subseteq \overline{N_1}$ and $W \subseteq \overline{N_2}$. Now suppose that $\exists x (\underset{\sim}{x} \in N_1, \underset{\approx}{x} \in N_2)$. From $M \subseteq \overline{N_1}$ and $W \subseteq \overline{N_2}$, we obtain $N_1 \cup N_2 \subseteq M \cup W \subseteq \overline{N_1} \cup \overline{N_2} = \overline{N_1 \cap N_2} \Rightarrow N_1 \cap N_2 \subseteq \overline{N_1 \cup N_2} = \overline{N_1} \cap \overline{N_2}$. But $\underset{\sim}{x} \in N_1, \underset{\approx}{x} \in N_2 \Rightarrow \underset{\approx}{x} \in N_1 \Rightarrow \underset{\approx}{x} \in N_2 \Rightarrow \underset{\approx}{x} \in N_1 \cap N_2 \subseteq \overline{N_1} \cap \overline{N_2} \Rightarrow \underset{\approx}{x} \in \overline{N_1}, \underset{\approx}{x} \in \overline{N_2}$ means a contradiction. Similarly, if $\exists y (\underset{\sim}{y} \in N_2, \underset{\approx}{y} \in N_1)$, we arrive at a contradiction again.

Case 3 and Case 4. They are similar to the previous ones.

Hence it follows that $N_1 \cup N_2$ is also C_3-connected. □

Theorem 3.10. *If N_1 and N_2 are overlapping C_4-connected IFSS's, then so is $N_1 \cup N_2$.*

Proof. Similar to the previous one. □

Using the last two theorems we get the following lemmas immediately:

Lemma 3.2. *If N_1 and N_2 are C_3-connected IFSS's such that $[\,](N_1 \cap N_2) \neq \underset{\sim}{\emptyset}$, then $N_1 \cup N_2$ is C_3-connected, too.*

Proof. For IFSS A, the set $[\,]A$ was defined as $[\,]A = \langle x, A_1, A_2\rangle$ if $A = \langle x, A_1, A_2\rangle$. If $[\,](N_1 \cap N_2) \neq \underset{\sim}{\emptyset}$, then we see that $N_1^{(1)} \cap N_2^{(1)} \neq \phi$, i.e., $\exists x \in N_1^{(1)} \cap N_2^{(1)} \Rightarrow x \in N_1^{(1)}$ and $x \in N_2^{(1)} \Rightarrow \underset{\sim}{x} \in N_1$ and $x \notin N_2^{(2)} \Rightarrow \underset{\sim}{x} \in N_1$ and $\underset{\approx}{x} \in N_2$, i.e., N_1 and N_2 are overlapping. Hence, the required result follows from a previous theorem. □

Lemma 3.3. *If N_1 and N_2 are C_4-connected IFSS's such that $[\,](N_1 \cap N_2) \neq \underset{\sim}{\emptyset}$, then $N_1 \cup N_2$ is C_4-connected, too.*

Now, we give generalized versions of these theorems. Here, a family $(N_i)_{i\in J}$ of IFSS's is said to be nonoverlapping if and only if for each $i \in J$, N_i and $\cap_{j\neq i} N_j$ are nonoverlapping, i.e., $N_i \subseteq \overline{\cap_{j\neq i} N_j}$.

Theorem 3.11. *Let $(N_i)_{i\in J}$ be a family of C_1-connected IFSS's such that $\cap N_j \neq \underset{\sim}{\emptyset}$. Then $\cup N_i$ is C_1-connected, too.*

Proof. Let $N = \cup N_i$ be C_1-disconnected. Then there exist IFSOS's M and W such that $N \subseteq M \cup W$, $M \cap W \subseteq \bar{N}$, $N \cap M \neq \underset{\sim}{\emptyset}$, $N \cap W \neq \underset{\sim}{\emptyset}$.

Now consider any index $i_0 \in J$. Since N_{i_0} is C_1-connected, we have $N_{i_0} \cap M = \underset{\sim}{\emptyset}$ or $N_{i_0} \cap W = \underset{\sim}{\emptyset}$. Hence there exist three cases:

Case 1. Let $N_i \cap M = \underset{\sim}{\emptyset}$ for each $i \in J$. Then, we may write down $N \cap M = (\cup N_i) \cap M = \cup(N_i \cap M) = \cup \underset{\sim}{\emptyset} = \underset{\sim}{\emptyset}$, which is a contradiction.

Case 2. Let $N_i \cap W = \underset{\sim}{\emptyset}$ for each $i \in J$. Then we obtain a similar contradiction.

Case 3. Let $N_i \cap M = \underset{\sim}{\emptyset}$ for each $i \in J_1$ and $N_i \cap W = \underset{\sim}{\emptyset}$ for each $i \in J_2$, where $J = J_1 \cup J_2$ and $J_1 \neq \emptyset$, $J_2 \neq \emptyset$. Since $\cap N_j \neq \underset{\sim}{\emptyset}$, $\exists \underset{\approx}{p} \in \cap N_j$. in this case we get $\underset{\approx}{p} \notin M$ and $\underset{\approx}{p} \notin W$, which is a contradiction with $\underset{\approx}{p} \in N \subseteq M \cup W$. Therefore, N is also C_1-connected. □

Theorem 3.12. *Let $(N_i)_{i\in J}$ be a family of C_2-connected IFSS's such that $\cap N_j \neq \underset{\sim}{\emptyset}$. Then $\cup N_i$ is C_2-connected, too.*

Proof. Similar to the previous one. □

Theorem 3.13. *Let $(N_i)_{i\in J}$ be an overlapping family of C_3-connected IFSS's. Then $\cup N_i$ is C_3-connected, too.*

Proof. Let $N = \cup N_i$ be C_3-disconnected. Then there exist IFSOS's M and W such that $N \subseteq M \cup W$, $M \cap W \subseteq \bar{N}$, $M \nsubseteq \bar{N}$, $W \nsubseteq \bar{N}$. Now consider any index $i \in J$. Since N_i is C_3-connected, we have $M \subseteq \overline{N_i}$ or $W \subseteq \overline{N_i}$. Since (N_i) is an overlapping family, suppose further that $\exists i_0 \in J$ such that

$$\exists x\Big(\underset{\sim}{x} \in N_{i_0}, \underset{\approx}{x} \in \bigcap_{j\neq i_0} N_j\Big) \qquad \text{or} \qquad \exists y\Big(\underset{\sim}{y} \in \bigcap_{j\neq i_0} N_j, \underset{\approx}{y} \in N_{i_0}\Big). \tag{3.17}$$

Hence there exist three cases:

Case 1. Let $M \subseteq \overline{N_i}$ for each $i \in J$. Then we may write down $M \subseteq \cap\overline{N_i} = \overline{\cup N_i} = \overline{N}$, which is an obvious contradiction.

Case 2. Let $W \subseteq \overline{N_i}$ for each $i \in J$. Then we obtain a similar contradiction.

Case 3. Let $M \subseteq \overline{N_i}$ for each $i \in J_1$ and $W \subseteq \overline{N_i}$ for each $i \in J_2$, where $J = J_1 \cup J_2$ and $J_1 \neq \emptyset$, $J_2 \neq \emptyset$. Hence

$$\begin{aligned} N \subseteq M \cup W &\subseteq \Big(\bigcap_{i\in J_1} \overline{N_i}\Big) \cup \Big(\bigcap_{i\in J_2} \overline{N_i}\Big) = \Big(\overline{\bigcup_{i\in J_1} N_i}\Big) \cup \Big(\overline{\bigcup_{i\in J_2} N_i}\Big) \\ &\Longrightarrow \Big(\bigcup_{i\in J_1} N_i\Big) \cap \Big(\bigcup_{i\in J_2} N_i\Big) \subseteq \overline{N} = \bigcap_{i\in J} \overline{N_i} \end{aligned} \tag{3.18}$$

follows.

Now, let $\exists x(\underset{\sim}{x} \in N_{i_0}, \underset{\approx}{x} \in \bigcap_{j\neq i_0} N_j)$. Since $\underset{\approx}{x} \in N_{i_0}$ and hence $\underset{\approx}{x} \in \cap N_i$. We see that $\underset{\approx}{x} \in \overline{N} \Rightarrow \underset{\approx}{x} \in \overline{N_{i_0}}$, a contradiction to $\underset{\sim}{x} \in N_{i_0}$. Secondly, let $\exists y(\underset{\sim}{y} \in \bigcap_{j\neq i_0} N_j, \underset{\approx}{y} \in N_{i_0})$. From these data we get $\underset{\approx}{y} \in \cap N_j$ and hence $\underset{\approx}{y} \in \overline{N}$. Without loss of generality, we may assume that the index set $J\setminus\{i_0\}$ has cardinality greater than 1; in other words, $\exists i_1 \in J$ such that $i_1 \neq i_0$. Thus $\underset{\sim}{y} \in N_{i_1}$ and $\underset{\approx}{y} \in \overline{N_{i_1}}$, an obvious contradiction. Therefore, N is also C_3-connected. □

Theorem 3.14. *Let $(N_i)_{i\in J}$ be an overlapping family of C_4-connected IFSS's. Then $\cup N_i$ is C_4-connected, too.*

Proof. Similar to the above proof. □

Now, we show that intuitionistic points are always C_i connected, unless X is one-point space $(i = 1,2,3,4)$.

Lemma 3.4. *Let (X,τ) be an IFSTS and $p \in X$. Then*

(a) *$\underset{\sim}{p}$ is C_1-connected.*

(b) *$\underset{\sim}{p}$ is C_2-connected.*

(c) *$\underset{\sim}{p}$ is C_3-connected.*

(d) *$\underset{\sim}{p}$ is C_4-connected.*

Proof. (a) Assume the contrary, and let $\underset{\sim}{p}$ be C_1-disconnected. Hence there exist IFSOS's M and W such that $\underset{\sim}{p} \subseteq M \cup W$, $M \tilde{\cap} W \subseteq \overline{\underset{\sim}{p}} = \langle x, \{p\}^c, \{p\}\rangle$, $\underset{\sim}{p} \cap M \neq \emptyset$, $\underset{\sim}{p} \cap W \neq \emptyset$. Since $\underset{\sim}{p} \cap M \neq \emptyset$, and $\underset{\sim}{p} \cap W \neq \emptyset$, we get $\underset{\sim}{p} \in M$ and $\underset{\sim}{p} \in W$; but from $M \tilde{\cap} W \subseteq \overline{\underset{\sim}{p}}$, we see that $M_1 \cap W_1 \subseteq \{p\}^c$ and $M_2 \cup W_2 \supseteq \{p\}$, which is impossible. Hence $\underset{\sim}{p}$ is C_1-connected.

(c) Assume the contrary, and let $\underset{\sim}{p}$ be C_3-disconnected. Hence there exist IFSOS's M and W such that $\underset{\sim}{p} \subseteq M \cup W$, $M \tilde{\cap} W \subseteq \overline{\underset{\sim}{p}} = \langle x, \{p\}^c, \{p\}\rangle$, $M \nsubseteq \overline{\underset{\sim}{p}}$ and $W \nsubseteq \overline{\underset{\sim}{p}}$. Since $M \nsubseteq \overline{\underset{\sim}{p}}$ and $W \nsubseteq \overline{\underset{\sim}{p}}$, we get $\underset{\sim}{p} \in M$ and $\underset{\sim}{p} \in W$; and the same reasoning may be applied in this case, too. Hence $\underset{\sim}{p}$ is C_3-connected.

(b) and (d) are similar to the first part. □

Lemma 3.5. (a) $\underset{\approx}{p}$ *is* C_2*-connected.*

(b) $\underset{\approx}{p}$ *is* C_3*-connected.*

(c) $\underset{\approx}{p}$ *is* C_4*-connected.*

Proof. (a) Suppose the contrary, i.e., let there exist IFSOS's M and W such that $\underset{\approx}{p} \subseteq M \cup W$, $M \cap W \cap \underset{\approx}{p} = \underset{\sim}{\emptyset}$, $\underset{\approx}{p} \cap M \neq \underset{\sim}{\emptyset}$, and $\underset{\approx}{p} \cap W \neq \underset{\sim}{\emptyset}$. Hence, $\{p\} \cap M_2^c \cap W_2^c = \underset{\sim}{\emptyset}$, $p \in M_2^c$, $p \in W_2^c$ follow, which is a contradiction.

(b) Suppose not, i.e., let there exist IFSOS's M and W such that $\underset{\approx}{p} \subseteq M \cup W$, $M \cap W \subseteq \underset{\approx}{\overline{p}}$, $M \not\subseteq \underset{\approx}{\overline{p}}$, and $W \not\subseteq \underset{\approx}{\overline{p}}$. Hence $M_1 \cap W_1 \subseteq \{p\}^c$, $p \in M_1$, $p \in W_1$, a contradiction, i.e., $\underset{\approx}{\overline{p}}$ is C_3-connected.

(c) Similar to (a) and (b). □

Notice that IFSS $N = \langle x, N_1, N_2 \rangle$ is called proper if and only if $N_1 \cup N_2 \neq X$.

Corollary 3.2. *In discrete intuitionistic fuzzy special topological space* $(X, I(X))$ *any nonempty proper IFSS,* N *is* C_1*-disconnected.*

Proof. Take $M := \overline{N}$, $W := N \in I(X)$. Then $N \subseteq N \cup \overline{N}$, $N \cap \overline{N} \subseteq \overline{N}$, $N \cap N = N \neq \underset{\sim}{\emptyset}$ and $N \cap \overline{N} \neq \underset{\sim}{\emptyset}$ hold, since, for example

$$N \cap \overline{N} = \langle x, N_1 \cap N_2, N_1 \cup N_2 \rangle = \langle x, \emptyset, N_1 \cup N_2 \rangle \neq \langle x, \emptyset, X \rangle = \underset{\sim}{\emptyset}. \tag{3.19}$$

□

Corollary 3.3. *In discrete intuitionistic fuzzy special topological space* $(X, I(X))$ *any proper IFSS* $N = \langle x, N_1, N_2 \rangle$*, where* $N_1 \neq \emptyset$*, is* C_2*-disconnected.*

Proof. Take a point $p \in X$ such that $p \in N_1^c$ and $p \in N_2^c$ and let $M := p$, $W := \overline{p}$ in this IFST. Then we get $N \subseteq M \cup W$, $M \cap W \cap N = \underset{\sim}{\emptyset}$, $N \cap M \neq \underset{\sim}{\emptyset}$ and $N \cap \tilde{W} \neq \underset{\sim}{\emptyset}$, as required. □

References

[1] N. Ajmal and J. K. Kohli, *Connectedness in fuzzy topological spaces*, Fuzzy Sets and Systems **31** (1989), no. 3, 369–388. MR 90i:54012. Zbl 684.54004.

[2] K. T. Atanassov, VII ITKR's Session, June 1983 (Sofia) (V. Sgurev, ed.), Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984.

[3] ______, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), no. 1, 87–96. MR 87f:03151. Zbl 631.03040.

[4] A. K. Chaudhuri and P. Das, *Fuzzy connected sets in fuzzy topological spaces*, Fuzzy Sets and Systems **49** (1992), no. 2, 223–229. MR 93f:54005. Zbl 762.54005.

[5] D. Çoker, *An introduction to intuitionistic topological spaces*, to appear in Turk. J. Math.

[6] ______, *A note on intuitionistic sets and intuitionistic points*, Turk. J. Math. **20** (1996), no. 3, 343–351. MR 99c:03100. Zbl 862.04007.

[7] ______, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Systems **88** (1997), no. 1, 81–89. MR 97m:54009. Zbl 990.27983.

[8] D. Çoker and A. H. Eş, *On fuzzy compactness in intuitionistic fuzzy topological spaces*, J. Fuzzy Math. **3** (1995), no. 4, 899–909. MR 96j:54010. Zbl 846.54003.

[9] E. Coşkun and D. Çoker, *On neighborhood structures in intuitionistic topological spaces*, Math. Balkanica **12** (1998), no. 3-4, 283–293. CMP 1 688 660.

[10] S. Özçağ and D. Çoker, *On connectedness in intuitionistic fuzzy special topological spaces*, Internat. J. Math. Math. Sci. **21** (1998), no. 1, 33–40. CMP 1 486 955. Zbl 892.54005.

[11] N. Turanli and D. Çoker, *On fuzzy connectedness in intuitionistic fuzzy topological spaces*, to appear in Fuzzy Sets and Systems.

[12] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353. MR 36#2509. Zbl 139.24606.

Özçağ: Department of Mathematics, Hacettepe University, Beytepe, 06532 Ankara, Turkey

E-mail address: sozcag@eti.cc.hun.edu.tr

Çoker: Department of Mathematics, Akdeniz University, 07200 Antalya, Turkey

E-mail address: coker@eti.cc.hun.edu.tr, coker@pascal.sci.akdeniz.edu.tr