Internat. J. Math. & Math. Sci.
Vol. 22, No. 4 (1999) 795–805
S 0161-1712(99)22795-0

APPLICATIONS OF RUSCHEWEYH DERIVATIVES AND HADAMARD PRODUCT TO ANALYTIC FUNCTIONS

M. L. MOGRA

(Received 31 August 1995 and in revised form 10 February 1998)

Abstract. For given analytic functions $\phi(z) = z + \sum_{m=2}^{\infty} \lambda_m z^m$, $\psi(z) = z + \sum_{m=2}^{\infty} \mu_m z^m$ in $U = \{z \mid |z| < 1\}$ with $\lambda_m \geq 0$, $\mu_m \geq 0$ and $\lambda_m \geq \mu_m$, let $E_n(\phi, \psi; A, B)$ be the class of analytic functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ in U such that $(f * \Psi)(z) \neq 0$ and

$$\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\Psi)(z)} \ll \frac{1+A_z}{1+B_z}, \quad -1 \leq A < B \leq 1,\ z \in U,$$

where $D^n h(z) = z(z^{n-1}h(z))^{(n)}/n!$, $n \in N_0 = \{0,1,2,\dots\}$ is the nth Ruscheweyh derivative; $\ll$ and $*$ denote subordination and the Hadamard product, respectively. Let T be the class of analytic functions in U of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$, and let $E_n[\phi, \psi; A, B] = E_n(\phi, \psi; A, B) \cap T$. Coefficient estimates, extreme points, distortion theorems and radius of starlikeness and convexity are determined for functions in the class $E_n[\phi, \psi; A, B]$. We also consider the quasi-Hadamard product of functions in $E_n[z/(1-z), z/(1-z); A, B]$ and $E_n[z/(1-z)^2, z/(1-z)^2; A, B]$.

Keywords and phrases. Ruscheweyh derivatives, Hadamard product, subordination, quasi-Hadamard product.

1991 Mathematics Subject Classification. 30C45.

1. Introduction. Let H denote the class of functions $f(z)$ analytic in the unit disc $U = \{z \mid |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. The Hadamard product of two functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$ in H is given by

$$(f*g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m. \tag{1.1}$$

Let $D^\alpha f(z) = z/(1-z)^{\alpha+1} * f(z), (\alpha \geq -1)$. Ruscheweyh [9] observed that $D^n f(z) = z(z^{n-1}f(z))^{(n)}/n!$ when $n \in N_0 = \{0,1,2,\dots\}$. This symbol $D^n f(z), n \in N_0$, was called the nth Ruscheweyh derivative of $f(z)$ by Al-Amiri [2]. Recently, several subclasses of H have been introduced and studied by using either the Hadamard product or Ruscheweyh derivatives (see [1, 4, 7, 8], etc.). To provide a unified approach to the study of various properties of these classes, we introduce the following most generalized subclass of H by using both the Hadamard product and Ruscheweyh derivatives.

Definition 1.1. Given the functions

$$\phi(z) = z + \sum_{m=2}^{\infty} \lambda_m z^m, \qquad \psi(z) = z + \sum_{m=2}^{\infty} \mu_m z^m$$

analytic in U such that $\lambda_m \ge 0$, $\mu_m \ge 0$ and $\lambda_m \ge \mu_m$ for $m = 2,3,\dots$, we say that $f \in H$ is in the class $E_n(\phi,\psi;A,B)$ if $(f*\psi)(z) \ne 0$ and

$$\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)} \ll \frac{1+Az}{1+Bz}, \quad z \in U, \tag{1.2}$$

where $\ll$ denote subordination, $-1 \le A < B \le 1$ and $n \in N_0$.

Let G be the class of functions w analytic in U and satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. By the definition of subordination, condition (1.2) is equivalent to

$$\frac{D^{n+1}(f*\phi)(z)}{D^n(f*\psi)(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad w \in G. \tag{1.3}$$

Let T denote the subclass of H consisting of functions of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \ge 0$, and let $E_n[\phi,\psi;A,B] = E_n(\phi,\psi;A,B) \cap T$. It is easy to check that various subclasses of T can be represented as $E_n[\phi,\psi;A,B]$ for suitable choices of $\phi(z),\psi(z),A,B$, and n. For example,

$$\begin{gathered}
E_n\left[\frac{z}{1-z},\frac{z}{1-z};A,B\right] = S_n[A,B],\\
E_n\left[\frac{z}{(1-z)^2},\frac{z}{(1-z)^2};A,B\right] = K_n[A,B],\\
E_0\left[\frac{z}{(1-z)^{2(1-\gamma)}},\frac{z}{(1-z)^{2(1-\gamma)}};(2\alpha-1)\beta,\beta\right] = R_\gamma[\alpha,\beta], \qquad (1.4)\\
E_0\left[\frac{z}{(1-z)^{2(1-\gamma)}},z;(2\alpha-1)\beta,\beta\right] = P_\gamma[\alpha,\beta], \quad 0 \le \alpha < 1, 0 < \beta \le 1, 0 \le \gamma < 1,\\
E_n\left[\frac{z}{(1-z)},z;A,B\right] = V_n[A,B],
\end{gathered}$$

etc. The classes $S_n[A,B]$ and $K_n[A,B]$ were introduced and studied by Padmanabhan and Manjini [8] whereas $R_\gamma[\alpha,\beta]$, $P_\gamma[\alpha,\beta]$, and $V_n[A,B]$ were, respectively, studied by Ahuja and Silverman [1], Owa and Ahuja [7], and Kumar [4]. Several other subclasses of T, introduced and studied by Silverman [10], Silverman and Silvia [11], Gupta and Jain [3], and others, can also be obtained from the class $E_n[\phi,\psi;A,B]$ by suitably choosing $\phi(z),\psi(z),A,B$, and n.

Now, we make a systematic study of the class $E_n[\phi,\psi;A,B]$. It is assumed throughout that $\phi(z)$ and $\psi(z)$ satisfy the conditions stated in Definition 1.1 and that $(f*\psi)(z) \ne 0$ for $z \in U$.

2. Coefficient inequalities. In this section, we find a necessary and sufficient condition for a function to be in $E_n[\phi,\psi;A,B]$ and, consequently, calculate coefficient estimates for functions in $E_n[\phi,\psi;A,B]$.

Theorem 2.1. *Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in H. If, for some $A,B\,(-1 \le A < B \le 1)$,*

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!\,\sigma_m}{(m-1)!\,(n+1)!}|a_m| \le B - A, \quad n \in N_0, \tag{2.1}$$

where $\sigma_m = (B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m$, then $f \in E_n(\phi,\psi;A,B)$.

Proof. Suppose that condition (2.1) holds for all admissible values of A, B, and n. In view of (1.3), it is sufficient to show that

$$\left| \frac{D^{n+1}(f*\phi)(z) - D^n(f*\psi)(z)}{BD^{n+1}(f*\phi)(z) - AD^n(f*\psi)(z)} \right| < 1, \quad z \in U. \tag{2.2}$$

For $|z| = r$, $0 \le r < 1$, we have

$$\begin{aligned}
&\left|D^{n+1}(f*\phi)(z) - D^n(f*\psi)(z)\right| - \left|BD^{n+1}(f*\phi)(z) - AD^n(f*\psi)(z)\right| \\
&\quad \le \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[(m+n)\lambda_m - (n+1)\mu_m\right]|a_m| r^m \\
&\qquad - \left\{(B-A)r - \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[B(m+n)\lambda_m - A(n+1)\mu_m\right]|a_m| r^m\right\} \\
&\quad < \left[\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[(B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m\right] \right. \\
&\qquad\qquad \left. \times |a_m| - (B-A)\right] \; |z| \le 0,
\end{aligned} \tag{2.3}$$

in view of (2.1). Thus, (2.2) is satisfied and, hence, $f \in E_n(\phi, \psi; A, B)$. □

Theorem 2.2. *Let $f \in T$. Then $f \in E_n[\phi, \psi; A, B]$ if and only if (2.1) is satisfied.*

Proof. In view of Theorem 2.1, it is sufficient to show the "only if" part. Thus, let $f \in E_n[\phi, \psi; A, B]$. Then, from (1.3), we get

$$|w(z)| = \left| \frac{\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[(m+n)\lambda_m - (n+1)\mu_m\right]|a_m| z^{m-1}}{(B-A) - \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[B(m+n)\lambda_m - A(n+1)\mu_m\right]|a_m| z^{m-1}} \right| < 1 \tag{2.4}$$

and, therefore,

$$\operatorname{Re}\left(\frac{\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[(m+n)\lambda_m - (n+1)\mu_m\right]|a_m| z^{m-1}}{(B-A) - \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[B(m+n)\lambda_m - A(n+1)\mu_m\right]|a_m| z^{m-1}} \right) < 1 \tag{2.5}$$

for all $z \in U$. We consider real values of z and take $z = r$ with $0 < r < 1$. Then, for $r = 0$, the denominator of (2.5) is positive and so is positive for all r, $0 \le r < 1$. Then (2.5) gives

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!}\left[(B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m\right]|a_m| r^{m-1} < B - A. \tag{2.6}$$

Letting $r \to 1^-$, we get (2.1). □

Corollary 2.1. *If $f \in E_n[\phi, \psi; A, B]$, then*

$$a_m \le \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} \quad \textit{for } m = 2, 3, \ldots \textit{ and } n \in N_0. \tag{2.7}$$

The equality holds, for each m, for functions of the form

$$f_m(z) = z - \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} z^m, \quad z \in U. \tag{2.8}$$

Remark 2.1. Taking different choices of $\phi(z)$, $\psi(z)$, A, B, and n as stated in Section 1, the above theorems lead to necessary and sufficient conditions and, consequently, coefficient inequalities for a function to be in $S_n[A,B]$, $K_n[A,B]$, $R_\gamma[\alpha,\beta]$, $P_\gamma[\alpha,\beta]$, $V_n[A,B]$, etc.

3. Closure theorems

Theorem 3.1. *The class $E_n[\phi,\psi;A,B]$ is closed under convex linear combinations.*

Proof. Let $f,g \in E_n[\phi,\psi;A,B]$ and let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $a_m \geq 0$, $b_m \geq 0$. For η such that $0 \leq \eta \leq 1$, it is sufficient to show that the function h, defined by $h(z) = (1-\eta)f(z) + \eta g(z)$, $z \in U$, belongs to $E_n[\phi,\psi;A,B]$.

Since $h(z) = z - \sum_{m=2}^{\infty}[(1-\eta)a_m + \eta b_m]z^m$, applying Theorem 2.2, we get

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{(m+n-1)!\sigma_m}{(m-1)!(n+1)!}\left[(1-\eta)a_m + \eta b_m\right] \\
&\quad \leq (1-\eta)\sum_{m=2}^{\infty} \frac{(m+n-1)!\sigma_m}{(m-1)!(n+1)!} a_m + \eta \sum_{m=2}^{\infty} \frac{(m+n-1)!\sigma_m}{(m-1)!(n+1)!} b_m \qquad (3.1)\\
&\quad \leq (1-\eta)(B-A) + \eta(B-A) = (B-A).
\end{aligned}$$

This implies that $h \in E_n[\phi,\psi;A,B]$. □

From Theorem 3.1 it follows that the closed convex hull of $E_n[\phi,\psi;A,B]$ is the same as $E_n[\phi,\psi;A,B]$. Now, we determine the extreme points of $E_n[\phi,\psi;A,B]$.

Theorem 3.2. *Let $f_1(z) = z$, $f_m(z) = z - ((m-1)!(n+1)!(B-A)/(m+n-1)!\sigma_m)z^m$, $m = 2,3,\ldots$, $z \in U$, and $n \in N_0$. Then $f \in E_n[\phi,\psi;A,B]$ if and only if it can be expressed as*

$$f(z) = \sum_{m=1}^{\infty} \rho_m f_m(z), \quad \text{where } \rho_m \geq 0 \text{ and } \sum_{m=1}^{\infty} \rho_m = 1. \qquad (3.2)$$

Proof. Suppose that

$$f(z) = \sum_{m=1}^{\infty} \rho_m f_m(z) = z - \sum_{m=2}^{\infty} \rho_m\left((m-1)!(n+1)!(B-A)(m+n-1)!\sigma_m\right)z^m. \qquad (3.3)$$

Since

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!\sigma_m}{(m-1)!(n+1)!(B-A)} \rho_m \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} = \sum_{m=2}^{\infty} \rho_m = 1 - \rho_1 \leq 1, \qquad (3.4)$$

it follows, from Theorem 2.2, that $f \in E_n[\phi,\psi;A,B]$.

Conversely, suppose that $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in E_n[\phi,\psi;A,B]$. Since

$$a_m \leq \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m}, \quad m = 2,3,\ldots, \qquad (3.5)$$

we may set

$$\rho_m = \frac{(m+n-1)!\sigma_m}{(m-1)!(n+1)!(B-A)} a_m, \quad m = 2,3,\ldots; n \in N_0 \text{ and } \rho_1 = 1 - \sum_{m=2}^{\infty} \rho_m. \qquad (3.6)$$

From Theorem 2.2, we have $\sum_{m=2}^{\infty}\rho_m \le 1$ and so $\rho_1 \ge 0$. It follows that $f(z) = \sum_{m=1}^{\infty}\rho_m f_m(z)$. □

Corollary 3.1. *The extreme points of $E_n[\phi,\psi;A,B]$ are the functions $f_m(z), m = 1,2,\dots$.*

4. Distortion theorems. With the aid of Theorem 3.2, we may now find bounds on the modulus of $f(z)$ and $f'(z)$ for $f \in E_n[\phi,\psi;A,B]$.

Theorem 4.1. *Let $f \in E_n[\phi,\psi;A,B]$ and $\sigma_m = (B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m$, $m = 2,3,\dots$. If n, m, σ_m, σ_{m+1} and $|z|$ satisfy the condition*

$$(m+n)\sigma_{m+1} - m\sigma_m|z| \ge 0, \tag{4.1}$$

then

$$\max\left\{0,|z| - \frac{B-A}{\sigma_2}|z|^2\right\} \le |f(z)| \le |z| + \frac{B-A}{\sigma_2}|z|^2. \tag{4.2}$$

The bounds are sharp.

Proof. By virtue of Theorem 3.2, we note that

$$\begin{aligned} |f(z)| &\ge \max\left\{0,|z| - \max_m \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m}|z|^m\right\}, \\ |f(z)| &\le |z| + \max_m \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m}|z|^m \end{aligned} \tag{4.3}$$

for $z \in U$. Thus, it suffices to show that

$$J(A,B,n,m,\sigma_m,|z|) = \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m}|z|^m \tag{4.4}$$

is a decreasing function of $m (m \ge 2)$. It is easily seen that, for $|z| \ne 0$,

$$J(A,B,n,m,\sigma_m,|z|) \ge J(A,B,n,m+1,\sigma_{m+1},|z|) \tag{4.5}$$

if and only if

$$(m+n)\sigma_{m+1} - m\sigma_m|z| \ge 0 \tag{4.6}$$

which is (4.1). Hence,

$$\max_m J(A,B,n,m,\sigma_m,|z|) \tag{4.7}$$

is attained at $m = 2$ and the proof is complete. □

Finally, since the functions $f_m(z), m \ge 2$, defined in Theorem 3.2, are extreme points of the class $E_n[\phi,\psi;A,B]$, we can see that the bounds of the theorem are attained for the function $f_2(z) = z - ((B-A)/\sigma_2)z^2$.

Corollary 4.1 [1]. *If $f \in R_\gamma[\alpha,\beta]$, $0 \le \alpha < 1$, $0 < \beta \le 1$, and either*

$$0 \le \gamma \le \frac{(2+3\beta-\alpha\beta)}{(2+4\beta-2\alpha\beta)} \quad \text{or} \quad |z| \le \frac{(1+2\beta-\alpha\beta)}{(1+3\beta-2\alpha\beta)}, \tag{4.8}$$

then

$$\max\left\{0, |z| - \frac{\beta(1-\alpha)}{(1-\gamma)[1+\beta(3-2\alpha)]}|z|^2\right\} \le |f(z)| \le |z| + \frac{\beta(1-\alpha)}{(1-\gamma)[1+\beta(3-2\alpha)]}|z|^2. \tag{4.9}$$

The bounds are sharp.

Proof. Choosing

$$\phi(z) = \psi(z) = \frac{z}{(1-z)^{2(1-\gamma)}} = z + \sum_{m=2}^{\infty} C(\gamma,m)z^m, \tag{4.10}$$

where

$$C(\gamma,m) = \frac{\left(\prod_{k=2}^{n}(k-2\gamma)\right)}{(m-1)!}, \tag{4.11}$$

so that $\lambda_m = \mu_m = C(\gamma,m)$ together with $A = (2\alpha-1)\beta$, $B = \beta$ and $n = 0$ in Theorem 4.1, the bounds (4.2) reduces to (4.9) provided

$$mC(\gamma,m+1)[m+\beta(m+2-2\alpha)] - mC(\gamma,m)[m-1+\beta(m+1-2\alpha)]|z| \ge 0. \tag{4.12}$$

Since

$$C(\gamma,m+1) = \frac{m+1-2\gamma}{m}C(\gamma,m), \tag{4.13}$$

the above inequality reduces to

$$(m+1-2\gamma)[m+\beta(m+2-2\alpha)] - m[m-1+\beta(m+1-2\alpha)]|z| \ge 0. \tag{4.14}$$

Now, proceeding exactly on the lines of Ahuja and Silverman [1], the result follows. □

Corollary 4.2 [7]. *If $f \in P_\gamma[\alpha,\beta]$, $0 \le \alpha < 1$, $0 < \beta \le 1$, and either $0 \le \gamma \le 5/6$ or $|z| \le 3/4$, then*

$$\max\left\{0, |z| - \frac{\beta(1-\alpha)}{2(1-\gamma)(1+\beta)}|z|^2\right\} \le |f(z)| \le |z| + \frac{\beta(1-\alpha)}{2(1-\gamma)(1+\beta)}|z|^2. \tag{4.15}$$

The bounds are sharp.

Proof. Taking

$$\phi(z) = \frac{z}{(1-z)^{2(1-\gamma)}} = z + \sum_{m=2}^{\infty} C(\gamma,m)z^m, \qquad \psi(z) = z, \tag{4.16}$$

so that $\lambda_m = C(\gamma, m)$ and $\mu_m = 0$ together with $A = (2\alpha - 1)\beta$, $B = \beta$ and $n = 0$ in Theorem 4.1, the bounds (4.2) reduces to (4.15) provided

$$m(m+1)(1+\beta)C(\gamma, m+1) - m^2(1+\beta)C(\gamma, m)|z| \geq 0. \tag{4.17}$$

Using

$$C(\gamma, m+1) = \frac{m+1-2\gamma}{m} C(\gamma, m), \tag{4.18}$$

the above inequality reduces to

$$(m+1)(m+1-2\gamma) - m^2|z| \geq 0. \tag{4.19}$$

Now, proceeding exactly on the lines of Owa and Ahuja [7], the result follows. □

Corollary 4.3 [8]. *Let $f \in S_n(A,B), -1 \leq A < B \leq 1$ and*

$$c_m = (B+1)(m+1) - (A+1)(n+1), \quad m = 2, 3, \ldots. \tag{4.20}$$

Then

$$\max\left\{0, |z| - \frac{B-A}{c_2}|z|^2\right\} \leq |f(z)| \leq |z| + \frac{B-A}{c_2}|z|^2. \tag{4.21}$$

The bounds are sharp.

Proof. Choosing $\phi(z) = \psi(z) = z/(1-z) = z + \sum_{m=2}^{\infty} z^m$ in Theorem 4.1 so that $\lambda_m = \mu_m = 1$ for $m \geq 2$, the bounds (4.2) reduces to (4.21) provided

$$\begin{aligned}(m+n)&[(B+1)(m+n+1) - (A+1)(n+1)] \\ &- m[(B+1)(m+n) - (A+1)(n+1)]|z| \geq 0.\end{aligned} \tag{4.22}$$

On simplification, the above inequality becomes

$$\begin{aligned}m(1-|z|)&[(m-1)(B+1) + (n+1)(B-A)] \\ &+ (n+1)[m(B+1) + (B-A)n] \geq 0\end{aligned} \tag{4.23}$$

which is true for all admissible values of m, n, A, B, and $|z|$. Hence, the result follows. □

Remark 4.1. The bounds for the functions in the classes $K_n[A,B]$ and $V_n[A,B]$ can be similarly deduced from Theorem 4.1 by choosing $\phi(z)$ and $\psi(z)$ suitably as indicated in Section 1.

Theorem 4.2. *Let $f \in E_n[\phi, \psi; A, B]$ and $\sigma_m = (B+1)(m+1)\lambda_m - (A+1)(n+1)\mu_m$, $m = 2, 3, \ldots$. If $n, m, \sigma_m, \sigma_{m+1}$, and $|z|$ satisfy the condition*

$$(m+n)\sigma_{m+1} - (m+1)\sigma_m|z| \geq 0, \tag{4.24}$$

then

$$\max\left\{0, 1 - \frac{2(B-A)}{\sigma_2}|z|\right\} \leq |f'(z)| \leq 1 + \frac{(B-A)}{\sigma_2}|z|. \tag{4.25}$$

The bounds are sharp for the function $f(z) = z - (2(B-A)/\sigma_2)z^2$.

Proof. By means of Theorem 3.2, we note that

$$|f'(z)| \geq 1 - \max_m \frac{m!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^{m-1},$$
$$|f'(z)| \leq 1 + \max_m \frac{m!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^{m-1} \tag{4.26}$$

for $z \in U$. Thus, it suffices to show that

$$J^*(A,B,n,m,\sigma_m,|z|) = \frac{m!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^{m-1} \tag{4.27}$$

is a decreasing function of $m(m \geq 2)$. But we can see that, for $|z| \neq 0$,

$$J^*(A,B,n,m,\sigma_m,|z|) \geq J^*(A,B,n,m+1,\sigma_{m+1},|z|) \tag{4.28}$$

if and only if

$$(m+n)\sigma_{m+1} - (m+1)\sigma_m|z| \geq 0 \tag{4.29}$$

which is (4.24). Hence,

$$\max_m J^*(A,B,n,m,\sigma_m,|z|) \tag{4.30}$$

is attained at $m = 2$ and the result follows. □

Remark 4.2. For suitable choices of $\phi(z), \psi(z)$, A, B, and n as stated in Section 1, the above theorem leads to the corresponding bounds for f', where f is in $S_n[A,B]$, $K_n[A,B]$, $P_\gamma[\alpha,\beta]$, $R_\gamma[\alpha,\beta]$, $V_n[A,B]$, etc. The different cases can be deduced from Theorem 4.2 as we did in the case of Theorem 4.1 and, hence, we omit the details.

Corollary 4.4. *Let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$ be in the class $E_n[\phi,\psi;A,B]$. Then, $f(z)$ is included in a disc with center at the origin and radius r_1 given by $r_1 = (\sigma_2 + B - A)/\sigma_2$ and $f'(z)$ is included in a disc with center at the origin and radius r_2 given by $r_2 = [\sigma_2 + 2(B-A)]/\sigma_2$.*

5. Radius of starlikeness and convexity.

Padmanabhan and Manjini [8] have shown that the functions in $E_n[\phi,\psi;A,B]$ are starlike in U if $\phi(z) = \psi(z) = z/(1-z)$ and convex in U if $\phi(z) = \psi(z) = z/(1-z)^2$. Now, we determine the largest disc in which functions in $E_n[\phi,\psi;A,B]$ are starlike and convex of order $\delta(0 \leq \delta < 1)$ in U for all admissible choices of $\phi(z), \psi(z), A, B$, and n.

Theorem 5.1. *If $f \in E_n[\phi,\psi;A,B]$, then f is starlike of order $\delta, 0 \leq \delta < 1$ for $|z| < r_1$, where*

$$r_1 = \inf_m \left\{ \frac{(m+n-1)!(1-\delta)\sigma_m}{(m-1)!(n+1)!(m-\delta)(B-A)} \right\}^{1\backslash m-1}, \tag{5.1}$$

$m = 2,3,\ldots$, and $n \in N_0$.

Proof. Let $f \in E_n[\phi,\psi;A,B]$. It is sufficient to show that $|zf'(z)/f(z)-1| \le 1-\delta$ for $|z| < r_1$, where r_1 is as specified in the statement of the theorem. We have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum_{m=2}^{\infty}(m-1)a_m|z|^{m-1}}{1-\sum_{m=2}^{\infty}a_m|z|^{m-1}}. \tag{5.2}$$

Thus, $|zf'(z)/(f(z)-1)| \le 1-\delta$ if $\sum_{m=2}^{\infty}((m-\delta)/(1-\delta))a_m \le 1$. By virtue of Theorem 2.2, we only need to find the values of $|z|$ for which the inequality

$$\frac{m-\delta}{1-\delta}|z|^{m-1} \le \frac{(m+n-1)!\sigma_m}{(m-1)!(n+1)!(B-A)} \tag{5.3}$$

is valid for all $m = 2,3,\ldots$, which is true when $|z| < r_1$. □

Theorem 5.2. *If $f \in E_n[\phi,\psi;A,B]$, then f is convex of order $\delta, 0 \le \delta < 1$ for $|z| < r_2$, where*

$$r_2 = \inf_m \left\{\frac{(m+n-1)!(1-\delta)\sigma_m}{m!(n+1)!(m-\delta)(B-A)}\right\}^{1\backslash m-1}, \quad m = 2,3,\ldots, \text{ and } n \in N_0. \tag{5.4}$$

Proof. Since $f(z)$ is convex of order δ if and only if $zf'(z)$ is starlike of order δ, the result follows by replacing m with ma_m in Theorem 5.1. □

6. Quasi-Hadamard product.

The quasi-Hadamard product of two or more functions has recently been defined and used by several researchers (see [5, 6] etc.). Accordingly the quasi-Hadamard product of $f(z) = z-\sum_{m=2}^{\infty}a_m z^m$, $a_m \ge 0$, and $g(z) = z-\sum_{m=2}^{\infty}b_m z^m$, $b_m \ge 0$, is given by $(f*g)_1(z) = z-\sum_{m=2}^{\infty}a_m b_m z^m$. Choosing $\phi(z) = \psi(z) = z/(1-z)$ and $\phi(z) = \psi(z) = z/(1-z)^2$, respectively, in Theorem 2.2, we get the following necessary and sufficient conditions for the functions in $S_n[A,B]$ and $K_n[A,B]$, obtained in [8].

Let $f \in T$. Then $f \in S_n[A,B]$ if and only if

$$\sum_{m=2}^{\infty}\frac{(m+n-1)!c_m}{(m-1)!(n+1)!}a_m \le B-A, \tag{6.1}$$

and $f \in K_n[A,B]$ if and only if

$$\sum_{m=2}^{\infty}\frac{(m+n-1)!mc_m}{(m-1)!(n+1)!}a_m \le B-A, \tag{6.2}$$

where $c_m = (B+1)(m+1)-(A+1)(n+1)$, $n \in N_0$, and $-1 \le A < B \le 1$. In this section, we introduce the following new class and establish a theorem concerning the quasi-Hadamard product for functions in $f \in S_n[A,B]$ and $f \in K_n[A,B]$. The theorem and its applications extend the corresponding results obtained by Kumar [5] when $a_{1,i} = 1$, $b_{1,j} = 1$, $i = 1,2,\ldots,p$, $j = 1,2,\ldots,q$.

Definition 6.1. A function $f(z) = z-\sum_{m=2}^{\infty}a_m z^m$, $a_m \ge 0$, which is analytic in U, belongs to the class $S_n^k[A,B]$ if and only if

$$\sum_{m=2}^{\infty}\frac{(m+n-1)!m^k c_m}{(m-1)!(n+1)!}a_m \le B-A, \tag{6.3}$$

where $c_m = (B+1)(m+n)-(A+1)(n+1)$, $-1 \le A < B \le 1$, $n \in N_0$ and k is any fixed nonnegative real number.

It is evident that $S_n^0[A,B] = S_n[A,B]$ and $S_n^1[A,B] = K_n[A,B]$. Further, $S_n^k[A,B] \subset S_n^h[A,B]$ if $k > h \ge 0$, the containment being proper. Whence, for any positive integer k, we have the following inclusion relation:

$$S_n^k[A,B] \subset S_n^{k-1}[A,B] \subset \cdots \subset S_n^2[A,B] \subset K_n[A,B] \subset S_n[A,B]. \tag{6.4}$$

We also note that, for every nonnegative real number k, the class $S_n^k[A,B]$ is nonempty as the functions of the form

$$f(z) = z - \sum_{m=2}^{\infty} \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!m^k c_m} \xi_m z^m, \tag{6.5}$$

where $\xi_m \ge 0$, $\sum_{m=2}^{\infty} \xi_m \le 1$, and $n \in N_0$, satisfy the required inequality.

Theorem 6.1. *Let the functions $f_i(z) = z - \sum_{m=2}^{\infty} a_{m,i} z^m$, $a_{m,i} \ge 0$, belong to the class $K_n[A,B]$ for every $i = 1,2,\dots,p$ and let the functions $g_j(z) = z - \sum_{m=2}^{\infty} b_{m,j} z^m$, $b_{m,j} \ge 0$, belong to the class $S_n[A,B]$ for every $j = 1,2,\dots,q$. Then the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_p * g_1 * g_2 * \cdots * g_q)_1(z)$ belongs to the class $S_n^{2p+q-1}[A,B]$.*

Proof. Since $f_i \in K_n[A,B]$, we have

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!mc_m}{(m-1)!(n+1)!} a_{m,i} \le B-A \tag{6.6}$$

or

$$a_{m,i} \le \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!mc_m} \tag{6.7}$$

for every $i = 1,2,\dots,p$. The right-hand expression of the last inequality is not greater than m^{-2} for all $A,B(-1 \le A < B \le 1)$, and $n \in N_0$. Hence,

$$a_{m,i} \le m^{-2} \quad \text{for every } i = 1,2,\dots,p. \tag{6.8}$$

Similarly, for $g_j \in S_n[A,B]$, we have

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!c_m}{(m-1)!(n+1)!} b_{m,j} \le B-A \tag{6.9}$$

and, hence,

$$b_{m,j} \le m^{-1} \quad \text{for every } j = 1,2,\dots,q. \tag{6.10}$$

Using (6.8) for $i = 1,2,\dots,p$; (6.10) for $j = 1,2,\dots,q-1$; and (6.9) for $j = q$, we get

$$\begin{aligned}
&\sum_{m=2}^{\infty} \left[\frac{(m+n-1)!m^{2p+q-1}c_m}{(m-1)!(n+1)!} \prod_{i=1}^{p} a_{m,i} \prod_{j=1}^{q} b_{m,j} \right] \\
&\quad \le \sum_{m=2}^{\infty} \left[\frac{(m+n-1)!m^{2p+q-1}c_m}{(m-1)!(n+1)!} (m^{-2p}m^{-(q-1)}) b_{m,q} \right] \\
&\quad = \sum_{m=2}^{\infty} \left[\frac{(m+n-1)!c_m}{(m-1)!(n+1)!} b_{m,q} \right] \le B-A.
\end{aligned} \tag{6.11}$$

Hence, $(f_1 * f_2 * \cdots * f_p * g_1 * g_2 * \cdots * g_q)_1(z) \in S_n^{2p+q-1}[A,B]$. □

We note that the required estimate can also be obtained by using (6.8) for $i = 1,2,\dots, p-1$; (6.10) for $j = 1,2,\dots,q$; and (6.6) for $i = p$.

Taking into account the quasi-Hadamard product of the functions $f_1(z), f_2(z), \dots, f_p(z)$ only in the proof of Theorem 6.1, and using (6.8) for $i = 1,2,\dots,p-1$; and (6.6) for $i = p$, we are led to the following corollary:

Corollary 6.1. *Let the functions* $f_i(z) = z - \sum_{m=2}^{\infty} a_{m,i} z^m, a_{m,i} \geq 0$*, belong to the class* $K_n[A,B]$ *for every* $i = 1,2,\dots,p$*. Then the quasi-Hadamard product* $(f_1 * f_2 * \cdots * f_p)_1(z)$ *belongs to the class* $S_n^{2p-1}[A,B]$*.*

Next, taking the quasi-Hadamard product of functions $g_1(z)$, $g_2(z), \dots, g_q(z)$ only in the proof of Theorem 6.1, and using (6.10) for $j = 1,2,\dots,q-1$; and (6.9) for $j = q$, we get the following corollary:

Corollary 6.2. *Let the functions* $g_j(z) = z - \sum_{m=2}^{\infty} b_{m,j} z^m, b_{m,j} \geq 0$*, belong to the class* $S_n[A,B]$ *for every* $j = 1,2,\dots,q$*. Then the quasi-Hadamard product* $(g_1 * g_2 * \cdots * g_q)_1(z)$ *belongs to the class* $S_n^{q-1}[A,B]$*.*

Acknowledgement. The author wishes to thank the referee for suggesting certain improvements in the paper.

References

[1] O. P. Ahuja and H. Silverman, *Convolutions of prestarlike functions*, Internat. J. Math. Math. Sci. **6** (1983), no. 1, 59–68. MR 84f:30013. Zbl 514.30011.

[2] H. S. Al-Amiri, *On Ruscheweyh derivatives*, Ann. Polon. Math. **38** (1980), no. 1, 88–94. MR 82c:30010. Zbl 452.30008.

[3] V. P. Gupta and P. K. Jain, *Certain classes of univalent functions with negative coefficients. II*, Bull. Austral. Math. Soc. **15** (1976), no. 3, 467–473. MR 55 8334. Zbl 335.30010.

[4] V. Kumar, *On a new criterion for univalent functions*, Demonstratio Math. **17** (1984), no. 4, 875–886. MR 86m:30018. Zbl 582.30012.

[5] ______, *Hadamard product of certain starlike functions. II*, J. Math. Anal. Appl. **113** (1986), no. 1, 230–234. MR 87j:30021. Zbl 606.30018.

[6] S. Owa, *On the Hadamard products of univalent functions*, Tamkang J. Math. **14** (1983), no. 1, 15–21. MR 84f:30021. Zbl 527.30001.

[7] S. Owa and O. P. Ahuja, *A class of functions defined by using Hadamard product*, Hokkaido Math. J. **15** (1986), no. 2, 217–232. MR 87j:30029. Zbl 612.30012.

[8] K. S. Padmanabhan and R. Manjini, *Certain classes of analytic functions with negative coefficients*, Indian J. Pure Appl. Math. **17** (1986), no. 10, 1210–1223. MR 87m:30025. Zbl 608.30014.

[9] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115. MR 51 3418. Zbl 303.30006.

[10] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116. MR 51 5910. Zbl 311.30007.

[11] H. Silverman and E. Silvia, *Subclasses of prestarlike functions*, Math. Japon. **29** (1984), no. 6, 929–935. MR 87a:30026. Zbl 558.30009.

Mogra: Department of Mathematics, University of Bahrain, P. O. Box 32038, Isa Town, Bahrain