Internat. J. Math. & Math. Sci.
Vol. 22, No. 4 (1999) 869–883
S 0161-1712(99)22869-4

LARGE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR GRADIENT TERMS

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(Received 19 June 1998)

ABSTRACT. We show that large positive solutions exist for the equation $(P\pm): \Delta u \pm |\nabla u|^q = p(x)u^\gamma$ in $\Omega \subseteq \mathbf{R}^N$ $(N \geq 3)$ for appropriate choices of $\gamma > 1, q > 0$ in which the domain Ω is either bounded or equal to $\mathbf{R}^N$. The nonnegative function p is continuous and may vanish on large parts of Ω. If $\Omega = \mathbf{R}^N$, then p must satisfy a decay condition as $|x| \to \infty$. For $(P+)$, the decay condition is simply $\int_0^\infty t\phi(t)\,dt < \infty$, where $\phi(t) = \max_{|x|=t} p(x)$. For $(P-)$, we require that $t^{2+\beta}\phi(t)$ be bounded above for some positive β. Furthermore, we show that the given conditions on γ and p are nearly optimal for equation $(P+)$ in that no large solutions exist if either $\gamma \leq 1$ or the function p has compact support in Ω.

Keywords and phrases. Large solution, elliptic equation, existence of solution, semilinear elliptic equation.

1991 Mathematics Subject Classification. Primary 35J25, 35J60.

1. Introduction. We consider existence and nonexistence of large solutions of the equation

$$\Delta u \pm |\nabla u|^q = p(x)u^\gamma \tag{$P\pm$}$$

in which q and γ are positive constants, the function p is nonnegative, and the domain Ω is either bounded with smooth boundary or equal to $\mathbf{R}^N$. A solution $u(x)$ of $(P\pm)$ is called a *large* solution if $u \to \infty$ as $x \to \partial\Omega$. In the case $\Omega = \mathbf{R}^N$, $x \to \partial\Omega$ means $|x| \to \infty$ and such solutions are called *entire* large solutions.

Large solutions of semilinear elliptic equations have been studied for decades. Almost all such studies have dealt with equations of the form

$$\Delta u = g(x,u) \tag{1.1}$$

in which the function g takes various forms (see [1, 2, 6, 8, 10, 12] and their references). Except for [2] and [10], all of these have restricted their attention to bounded domains and functions g which are strictly positive when $u > 0$.

In [2], Bandle and Marcus proved the existence of large solutions when $g(x,u) = p(x)u^\gamma$ in which p is allowed to be zero at finitely many places in Ω. Lair and Shaker [10] proved the existence of large solutions in bounded domains and entire large solutions in $\mathbf{R}^N$ for $g(x,u) = p(x)f(u)$, allowing p to be zero on large parts of Ω.

A few authors considered large solutions of semilinear equations containing nonlinear gradient terms (see [1, 7, 11]). Motivation for the present study stems from the work of Bandle and Giarrusso [1] who developed existence and asymptotic behavior results for large solutions of

$$\Delta u \pm |\nabla u|^q = f(u) \tag{1.2}$$

on a bounded domain. Our goal here is to develop comprehensive (and nearly optimal) existence results for $(P+)$ when Ω is either bounded or equal to $\mathbf{R}^N$, and at the same time develop somewhat comparable results for problem $(P-)$. In particular, for Ω bounded, we show that problem $(P+)$ has a positive large solution if γ and q satisfy

$$\gamma > \max\{1, q\}, \tag{1.3}$$

and p satisfies the following condition.

Condition (P). $p(x) \geq 0$, $\forall x \in \Omega$; $p(x) \in C(\bar{\Omega})$; if $z \in \Omega$ and $p(z) = 0$, then there exists a domain D_z containing z for which $\overline{D_z} \subseteq \Omega$ and $p(x) > 0$ for all $x \in \partial D_z$.

Thus, p is allowed to vanish on significant portion of Ω. Indeed, the function p could be zero on all of Ω except for a small (in measure) open set containing $\partial\Omega$. For problem $(P-)$ with Ω bounded, we do not need inequality (1.3) when we establish the existence of a nonnegative solution (see Theorem 4).

For $\Omega = \mathbf{R}^N$, we prove the existence of a positive entire large solution if p also satisfies

$$\int_0^\infty r\phi(r)\,dr < \infty, \tag{1.4}$$

where $\phi(r) = \max_{|x|=r} p(x)$ (see Theorem 3). For problem $(P-)$, we require the stronger decay condition that $r^{2+\beta}\phi(r)$ be bounded above for some $\beta > 0$. In addition, restrictions are placed on the relationship between q and γ (see Theorem 5).

Furthermore, for both the bounded and unbounded cases, we show that our conditions on γ, q, and p are nearly optimal for $(P+)$ in that if $\gamma \leq 1$ or p has compact support in Ω, then $(P+)$ has no positive large solutions.

We also note that, among the many open problems related to $(P\pm)$, the existence of positive large solutions remains unproved for $(P+)$ if $1 < \gamma \leq q$ and for $(P-)$ if the function p is required to satisfy the weaker decay condition (1.4).

2. Existence results

2.1. Equation $(P+)$**.** In this section, we develop a critical boundedness result (Lemma 1), which will prove very useful in developing our existence theorem for bounded domains (Theorem 2). This result, in turn, provides the critical element for the existence proof when $\Omega = \mathbf{R}^N$. Interestingly, Bandle and Giarrusso [1] had no boundedness result comparable to that of Lemma 1. Indeed, their proof of their main existence result simply assumes that such an upper bound exists.

2.1.1. Ω is a bounded domain

Lemma 1. *Let u_n be a solution of the problem*

$$\begin{aligned} \Delta u_n + |\nabla u_n|^q &= p(x) u_n^\gamma \quad x \in \Omega, \\ u_n(x) &= n, \quad x \in \partial\Omega. \end{aligned} \tag{2.1}$$

Then $0 < u_n \leq n$ on $\bar{\Omega}$. Furthermore, let B_R be a ball of radius R such that $\overline{B_R} \subseteq \Omega$. Then *there exists a constant $M = M(R,q,\gamma)$ such that $u_n(x) \le M$ on $\overline{B_R}\ \forall n$, provided that $0 < m_o \le p(x) \le M_o$ in Ω and (1.3) holds.*

Proof. To prove that $u_n > 0$ in Ω, without loss of generality, we let $n = 1$. We observe from the maximum principle that $0 \le u_1 \le 1$. Furthermore, it is clear that, for any $0 < \epsilon_o < 1$, any solution, say z, to the problem (which exists by [9, Thm. 8.3, p. 301])

$$\begin{aligned} \Delta z + |\nabla z|^q &= p(x) z^\gamma, \quad x \in \Omega, \\ z &= \epsilon_o, \quad x \in \partial\Omega \end{aligned} \tag{2.2}$$

satisfies $z \le u_1$ and $0 \le z \le \epsilon_o$. Hence, if we show that $z > 0$ in Ω for some choice of $\epsilon_o \in (0,1)$, we will be done. To do this, let $x_o \in \mathbf{R}^N \backslash \bar{\Omega}$. We assume, without loss of generality, that $x_o = 0$. Let $r = |x|$ and choose $R_o > 0$ large so that $\bar{\Omega} \subseteq B(0,R_o)$. Choose $M_o > 0$ so that $p(x) < M_o$ on $\bar{\Omega}$. Now, choose $0 < \epsilon_o < 1$ so that

$$\frac{M_o \epsilon_o^\gamma R_o^2}{2N} \le \epsilon_o. \tag{2.3}$$

Let $v(x) = (M_o \epsilon_o^\gamma / 2N) r^2$ for $|x| \le R_o$. Define w on the ball $B(0,R_o)$ as $w(x) = z(x)$ for $x \in \bar{\Omega}$ and $w(x) = \epsilon_o$ on $\overline{B(0,R_o)} \backslash \bar{\Omega}$. We show that $v \le w$ in $\overline{B(0,R_o)}$.

Thus, we suppose that $\max(v - w)$ in $\overline{B(0,R_o)}$ is positive. In this case, the point where the maximum occurs must lie in Ω since

$$v(x) = \frac{M_o \epsilon_o^\gamma}{2N} r^2 \le \frac{M_o \epsilon_o^\gamma}{2N} R_o^2 \le \epsilon_o = w(x), \quad x \in \overline{B(0,R_o)} \backslash \Omega. \tag{2.4}$$

Therefore, at the point where $\max(v - w)$ occurs, we have

$$0 \ge \Delta(v - w) = \Delta(v - z) = M_o \epsilon_o^\gamma - p z^\gamma + |\nabla z|^q > p(\epsilon_o^\gamma - z^\gamma) \ge 0, \tag{2.5}$$

a contradiction. Therefore, $v \le w$ in $B(0,R_o)$ which yields $v \le w$ in Ω or $(M_o \epsilon_o^\gamma / 2N) r^2 \le z(x)$ in Ω. Since $r > 0$ in Ω, we get $z > 0$ in Ω. Hence, $u_1 > 0$ in Ω.

Now, let ϵ be a sufficiently small positive number so that $B_{R+\epsilon} \subseteq \Omega$ and let v_n be a solution of

$$\begin{aligned} \Delta v_n &= m_o v_n^\gamma, \quad x \in B_{R+\epsilon}, \\ v_n &= n, \quad x \in \partial B_{R+\epsilon}. \end{aligned} \tag{2.6}$$

A similar argument as above implies that $v_n > 0$ in $B_{R+\epsilon}$. By the maximum principle, it is clear that $v_n \le v_{n+1}, n = 1,2,\ldots$. By [4, Thm. A], it is easy to show that v_n is a radial solution. Thus, v_n satisfies

$$\begin{aligned} v_n'' + \frac{N-1}{r} v_n' &= m_o v_n^\gamma, \quad x \in B_{R+\epsilon}, \\ v_n &= n, \quad x \in \partial B_{R+\epsilon}. \end{aligned} \tag{2.7}$$

It is clear that $v_n'(0) = 0$ and $v_n'(r) \ge 0\ \forall n$ and $\forall r$. Equation (2.7) may be rewritten as

$$\left(r^{N-1} v_n'(r)\right)' = m_o r^{N-1} v_n^\gamma, \tag{2.8}$$

which may be integrated to get

$$\int_0^r \left(s^{N-1} v_n'(s)\right)' ds = m_o \int_o^r s^{N-1} v_n^\gamma ds. \tag{2.9}$$

Thus, we have

$$r^{N-1}v_n'(r) = m_o\int_0^r s^{N-1}v_n^\gamma\,ds \le m_o r^{N-1}v_n^\gamma(r)\int_0^r ds = m_o r^N v_n^\gamma(r) \tag{2.10}$$

which implies that

$$v_n'(r) \le m_o r v_n^\gamma. \tag{2.11}$$

Let v be a solution of

$$\begin{aligned} \Delta v &= m_o v^\gamma, \quad x \in B_{R+\epsilon}, \\ v &\to \infty, \quad x \in \partial B_{R+\epsilon}. \end{aligned} \tag{2.12}$$

The existence of v is justified by [10, Thm. 1]. By the maximum principle, $v_n \le v$ in $B_{R+\epsilon}$ for all n. Thus, v_n is bounded above on B_R by a constant which is independent of n. By (2.11), $v_n'(r)$ is also bounded above by a constant independent of n. Let K be an upper bound for both v_n and v_n' on $\overline{B_R}$.

If we can find a function w_n which satisfies

$$\begin{aligned} \Delta w_n + |\nabla w_n|^q &\le m_o w_n^\gamma, \quad x \in B_{R+\epsilon} \subseteq \Omega, \\ w_n &= n, \quad x \in \partial B_{R+\epsilon}, \\ w_n &\le K_o, \quad x \in \bar{B}_R, \end{aligned} \tag{2.13}$$

where K_o is a constant independent of n, then by the maximum principle, we have $u_n \le w_n \le K_o$, and we will be done.

Let $w_n = cv_n^\lambda$, where v_n is a solution of (2.7), the constants c and λ, both independent of n, are determined later. Since

$$\begin{aligned} &\Delta w_n + |\nabla w_n|^q - m_o w_n^\gamma \\ &\quad = \lambda c v_n^{\lambda-1}\Delta v_n + \lambda(\lambda-1)c v_n^{\lambda-2}|\nabla v_n|^2 + c^q\lambda^q v^{(\lambda-1)q}|\nabla v_n|^q - m_o c^\gamma v_n^{\lambda\gamma} \\ &\quad = \lambda c v_n^{\lambda-1}\Big[\Delta v_n + (\lambda-1)v_n^{-1}|\nabla v_n|^2 + c^{q-1}\lambda^{q-1}v_n^{(\lambda-1)(q-1)}|\nabla v_n|^q \\ &\qquad\qquad - \frac{m_o c^{\gamma-1}}{\lambda}v_n^{\lambda\gamma-\lambda+1}\Big], \end{aligned} \tag{2.14}$$

to complete the proof, it suffices to find c and λ such that

$$\Delta v_n + (\lambda-1)v_n^{-1}|\nabla v_n|^2 + c^{q-1}\lambda^{q-1}v_n^{(\lambda-1)(q-1)}|\nabla v_n|^q - \frac{m_o c^{\gamma-1}}{\lambda}v_n^{\lambda\gamma-\lambda+1} \le 0 \tag{2.15}$$

on $B_{R+\epsilon}$ for all n, for then we have $u_n \le w_n \le cK^\lambda \equiv K_o$. Since v_n satisfies (2.7) and (2.11), the left side of the above equals

$$\begin{aligned} &m_o v_n^\gamma + (\lambda-1)v_n^{-1}|\nabla v_n|^2 + c^{q-1}\lambda^{q-1}v_n^{(\lambda-1)(q-1)}|\nabla v_n|^q - \frac{m_o c^{\gamma-1}}{\lambda}v_n^{\lambda\gamma-\lambda+1} \\ &\quad = m_o v_n^\gamma + (\lambda-1)v_n^{-1}\left|v_n'\right|^2 + c^{q-1}\lambda^{q-1}v_n^{(\lambda-1)(q-1)}|v_n'|^q - \frac{m_o c^{\gamma-1}}{\lambda}v_n^{\lambda\gamma-\lambda+1} \\ &\quad \le m_o v_n^\gamma + (\lambda-1)m_o^2(R+\epsilon)^2 v_n^{2\gamma-1} + c^{q-1}\lambda^{q-1}m_o^q(R+\epsilon)^q v_n^{(\lambda+\gamma)q-q-\lambda+1} \\ &\qquad - \frac{m_o}{\lambda}c^{\gamma-1}v_n^{\lambda\gamma-\lambda+1}. \end{aligned} \tag{2.16}$$

Now, since $\gamma > q$, we can choose λ large so that

$$\begin{aligned}&\lambda\gamma-\lambda+1>\gamma \Longleftrightarrow \lambda\geq 1,\\&\lambda\gamma-\lambda+1>2\gamma-1 \Longleftrightarrow \lambda\geq 2,\\&\lambda\gamma-\lambda+1>(\lambda+\gamma)q-q-\lambda+1 \Longleftrightarrow \lambda\geq \frac{q(\gamma-1)}{\gamma-q},\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}&m_o s^{\gamma}+(\lambda-1)m_o^2(R+\epsilon)^2 s^{2\gamma-1}\\&\quad+c^{q-1}\lambda^{q-1}m_o^q(R+\epsilon)^q s^{(\lambda+\gamma)q-q-\lambda+1}-\frac{m_o}{\lambda}c^{\gamma-1}s^{\lambda\gamma-\lambda+1}<0\end{aligned}\tag{2.18}$$

for $s\geq 2$ and $c\geq 1$. Since $0<v_1\leq\cdots\leq v_n\leq v_{n+1}\leq\cdots$ in $B_{R+\epsilon}$, we may find $\beta>0$ such that $v_n(r)\geq\beta$ $\forall n$ and $\forall r$. For the above choice of λ, choose the constant $c\geq 1$ so that the following holds.

$$\begin{aligned}&m_o 2^{\gamma}+(\lambda-1)m_o^2(R+\epsilon)^2 2^{2\gamma-1}\\&\quad+c^{q-1}\lambda^{q-1}m_o^q(R+\epsilon)^q 2^{(\lambda+\gamma)q-q-\lambda+1}-\frac{m_o}{\lambda}c^{\gamma-1}\beta^{\lambda\gamma-\lambda+1}<0.\end{aligned}\tag{2.19}$$

Thus, whether $v_n(r)\leq 2$ or $v_n(r)\geq 2$, we get

$$\begin{aligned}&m_o v_n^{\gamma}+(\lambda-1)m_o^2(R+\epsilon)^2 v_n^{2\gamma-1}+c^{q-1}\lambda^{q-1}m_o^q(R+\epsilon)^q v_n^{(\lambda+\gamma)q-q-\lambda+1}\\&\quad<\frac{m_o}{\lambda}c^{\gamma-1}v_n^{\lambda\gamma-\lambda+1}.\end{aligned}\tag{2.20}$$

Hence, $\Delta w_n+|\nabla w_n|^q\leq m_o w_n^{\gamma}$ on $B_{R+\epsilon}$. □

Theorem 2. *Assume that (1.3) and condition (P) hold. Suppose that Ω is a bounded domain in $\mathbf{R}^N$, $N\geq 3$, with smooth boundary. Then, equation $(P+)$ has a large positive solution in Ω.*

Proof. By [9, Thm. 8.3, p. 301], it is easy to prove that, for each $k\in N$, the boundary value problem

$$\begin{aligned}\Delta v_k+|\nabla v_k|^q&=p(x)v_k^{\gamma},\quad x\in\Omega,\\v_k(x)&=k,\quad x\in\partial\Omega\end{aligned}\tag{2.21}$$

has a unique positive classical solution. By the maximum principle it can be shown that $v_k\leq v_{k+1}$, $k\geq 1$, in Ω. Indeed, suppose that there is a point where $v\equiv v_{k+1}-v_k<0$. Let $x_o\in\mathbf{R}^N\backslash\bar{\Omega}$. We assume, without loss of generality, that $x_o=0$. Let $r=|x|$. Then, for some small $\epsilon>0$, $v+\varepsilon/(1+r)$ has a negative minimum in Ω. At that minimum, we have

$$\begin{aligned}0&\leq\Delta\Big(v+\frac{\epsilon}{1+r}\Big)\\&=p\left[v_{k+1}^{\gamma}-v_k^{\gamma}\right]-|\nabla v_{k+1}|^q+|\nabla v_k|^q+\epsilon\left[\frac{2}{(1+r)^3}-\frac{N-1}{r(1+r)^2}\right]\\&\leq 0-\epsilon\frac{N-1}{r(1+r)^3}<0,\end{aligned}\tag{2.22}$$

a contradiction. Hence, $v_k\leq v_{k+1}$, for $k=1,2,\ldots$. Furthermore, by Lemma 1, $v_1>0$ in Ω.

In what follows, it is understood that the maximum principle is applied as above, where the factor $\varepsilon/(1+r)$ is used whenever the function p is not strictly positive.

To complete the proof, it suffices to show the following:

(C1) $\forall x_o \in \Omega$, there exists M (depending on x_o but independent of k) such that $v_k(x) \le M$ $\forall x$ near x_o;
(C2) $\lim_{x\to\partial\Omega} v(x) = \infty$, where $v(x) = \lim_{k\to\infty} v_k(x)$ for $x \in \Omega$;
(C3) v is classical solution of $(P+)$.

To prove (C1), we consider two cases.

Case (a). $p(x_o) > 0$. Since p is continuous, there exists a ball $B(x_o,r)$ such that $p(x) \ge m_o$ in $\overline{B(x_o,r)}$ for some $m_o > 0$. (C1) then follows easily from Lemma 1.

Case (b). $p(x_o) = 0$. By the hypothesis, there exists a domain $\Omega_o \subseteq \Omega$ such that $x_o \in \Omega_o$ and $p(x) > 0$ $\forall x \in \partial\Omega_o$. From Case (a) above, we know that $\forall x \in \partial\Omega_o$ there exists a ball $B(x,r_x)$ and a positive constant M_x such that $v_k \le M_x$ on $B(x,r_x/2)$. Since Ω is bounded (and hence Ω_o is bounded), $\partial\Omega_o$ is compact. Thus, there exists a finite number of such balls that cover $\partial\Omega_o$. Let $M = \max\{M_{x_1},\dots,M_{x_k}\}$, where the balls $B(x_i,r_{x_i}/2)$, $i = 1,\dots,k$, cover $\partial\Omega_o$. Clearly, $v_k \le M$ on $\partial\Omega_o$. Yet another maximum principle argument yields $v_k \le M$ on Ω_o.

The proof of (C2) is straightforward. For any $L > 0$ and any sequence $x_k \to x \in \partial\Omega$, since $v_{L+1} = L+1$ on $\partial\Omega$ and is continuous, there is some $K > 0$ such that $v_{L+1}(x_k) \ge L$ for $k \ge K$. Note that, since $v \ge v_{L+1}$ in Ω, we have $v(x_k) \ge L$, $k \ge K$. Hence, $v(x_k) \to \infty$ as $k \to \infty$. Thus, we have $v \to \infty$ as $x \to \partial\Omega$.

To prove (C3), we let $x_o \in \Omega$ and let $B(x_o,r)$ be the ball of radius r centered at x_o such that it is contained in Ω. Let ψ be a C^∞ function which is equal to 1 on $\overline{B(x_o,r/2)}$ and zero off $B(x_o,r)$.

Let $f(s) = 1/(1+s)$. Multiplying both sides of equation (2.21) by $\psi^2 f(v_k)$ and integrating over $B(x_o,r)$ yields

$$\int_{B(x_o,r)} \psi^2 f(v_k)\Delta v_k\,dx + \int_{B(x_o,r)} \psi^2 f(v_k)|\nabla v_k|^q\,dx = \int_{B(x_o,r)} \psi^2 f(v_k)pv_k^\gamma\,dx. \tag{2.23}$$

Integration by parts produces

$$\begin{aligned}-\int_{B(x_o,r)} \psi^2 f'(v_k)|\nabla v_k|^2\,dx - \int_{B(x_o,r)} 2\psi\nabla\psi f(v_k)\cdot\nabla v_k\,dx&\\ + \int_{B(x_o,r)} \psi^2 f(v_k)|\nabla v_k|^q\,dx = \int_{B(x_o,r)} \psi^2 f(v_k)pv_k^\gamma\,dx&.\end{aligned} \tag{2.24}$$

Thus, we have

$$\begin{aligned}\left(\frac{1}{1+M_r}\right)^2 &\int_{B(x_o,r)} \psi^2|\nabla v_k|^2\,dx\\ &\le \int_{B(x_o,r)} \frac{\psi^2}{(1+v_k)^2}|\nabla v_k|^2\,dx + \int_{B(x_o,r)} \psi^2 f(v_k)|\nabla v_k|^q\,dx\\ &= \int_{B(x_o,r)} 2\psi\nabla\psi\cdot\nabla v_k\left(\frac{1}{1+v_k}\right)dx + \int_{B(x_o,r)} \psi^2 f(v_k)pv_k^\gamma\,dx\\ &\le \int_{B(x_o,r)} (\psi|\nabla v_k|)\frac{2|\nabla\psi|}{1+v_1} + \int_{B(x_o,r)} \psi^2 f(v_k)pv_k^\gamma\,dx\\ &\le \epsilon\int_{B(x_o,r)} \psi^2|\nabla v_k|^2 dx + \frac{1}{4\epsilon}\int_{B(x_o,r)} \left(\frac{2|\nabla\psi|}{1+v_1}\right)^2 dx + M_1\\ &\le \epsilon\int_{B(x_o,r)} \psi^2|\nabla v_k|^2 dx + M_2,\end{aligned} \tag{2.25}$$

where M_r is an upper bound for v_k on $B(x_o,r)$, $k=1,2,\ldots,\epsilon$ is any positive number, and the constants M_1 and M_2 are independent of k. Hence, we get

$$\int_{B(x_o,r)} |\psi\nabla v_k|^2 dx \le M. \tag{2.26}$$

That is, the $L^2(B(x_o,r))$-norm of $|\psi\nabla v_k|$ is bounded independently of k. Thus, the $L^2(B(x_o,r/2))$-norm of $|\nabla v_k|$ is bounded independently of k.

By the standard regularity argument (see [10]), we may find a number $r_1>0$ such that there is a subsequence of $\{v_k\}_1^\infty$, which we still call $\{v_k\}_1^\infty$, that converges in $C^{1+\alpha}(\overline{B(x_o,r_1)})$ for some positive number $\alpha<1$.

Let ψ be as before but with r replaced by r_1.

Now, we consider two cases regarding the regularity of the function $p(x)$.

Case 1. $p(x)\in C^\alpha(\Omega)$. Note that, $\Delta v_k = pv_k^\gamma - |\nabla v_k|^q$ and $\Delta(\psi v_k) = 2\nabla\psi\cdot\nabla v_k + v_k\Delta\psi + \psi\Delta v_k$, $k\ge 1$, converges in $C^\alpha\overline{(B(x_o,r_1))}$ as $k\to\infty$. By Schauder theory, $\{\psi v_k\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_o,r_1))}$ and hence $\{v_k\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_o,r_1/2))}$. Since x_o is arbitrary, it follows that $v\in C^{2+\alpha}(\Omega)$ and is a solution of $(P+)$.

Case 2. $p(x)\in C(\Omega)$. We have $v_k \xrightarrow{s-C(B(x_o,r_1))} v$ and, consequently, $\Delta v_k = pv_k^\gamma - \nabla v_k|^q \xrightarrow{s-C(B(x_o,r_1))} p(x)v^\gamma - |\nabla v|^q \equiv z$. That the Laplacian is a closed linear operator implies that $v\in D(\Delta)$, $\Delta v = z$. Since x_o is arbitrary, we have that v is a classical solution of $(P+)$. □

2.1.2. $\Omega = \mathbf{R}^N$

Theorem 3. *Let $\Omega=\mathbf{R}^N$. Assume that (1.3), (1.4), and condition (P) hold. Then equation $(P+)$ has a positive entire large solution.*

Proof. By Theorem 2, for $k=1,2,\ldots$, the boundary blow-up problem

$$\begin{gathered}\Delta v_k + |\nabla v_k|^q = p(x)v_k^\gamma, \quad |x|<k,\\ v_k(x)\longrightarrow\infty, \quad \text{as } |x|\longrightarrow k\end{gathered} \tag{2.27}$$

has a classical solution. By the maximum principle, it is clear that

$$v_1\ge v_2\ge\cdots\ge v_k\ge v_{k+1}\ge\cdots \tag{2.28}$$

in $\mathbf{R}^N$. Let $v(x)=\lim_{k\to\infty}v_k(x)$, $x\in\mathbf{R}^N$. We claim that v is the desired solution. To prove this, we consider the related problem

$$\begin{gathered}\Delta u_k = p(x)u_k^\gamma, \quad |x|<k,\\ u_k(x)\longrightarrow\infty, \quad \text{as } |x|\longrightarrow k.\end{gathered} \tag{2.29}$$

It is shown in [10] that (2.29) has a unique positive solution for each k, and that

$$u_1\ge u_2\ge\cdots\ge u_k\ge u_{k+1}\ge\cdots\ge w>0 \tag{2.30}$$

for some $w\to\infty$ as $|x|\to\infty$. It follows easily from the maximum principle that $v_k\ge u_k$ for $k=1,2\ldots$. Thus, $v(x)\to\infty$ as $|x|\to\infty$. By a similar argument as (C3) in Theorem 2, we have that v is the desired solution. □

2.2. Equation $(P-)$

2.2.1. Ω **is a bounded domain.** The following theorem is our main result for problem $(P-)$ on a bounded domain. Much of the proof is similar to that of Theorem 2 and is, therefore, only outlined.

Theorem 4. *Suppose that* $\gamma > 1$, $q > 0$, *condition* (P) *holds, and* Ω *is a bounded domain in* $\mathbf{R}^N$, $N \geq 3$, *with smooth boundary. Then equation* $(P-)$ *has a large nonnegative solution in* Ω.

Proof. The only significant difference between this proof and that of Theorem 2 lies in obtaining an upper bound M for the sequence $\{v_k\}$ near x_0. There, Lemma 1 is needed, but for $(P-)$ the proof is much easier. Indeed, it is easy to prove that $v_k(x) \leq w(x)$ for all $x \in \Omega$, where w is a solution of (See [10, Thm. 1]).

$$\Delta w = p(x) w^{\gamma}, \quad x < \Omega,$$
$$w(x) \longrightarrow \infty, \quad x \longrightarrow \partial\Omega. \tag{2.31}$$

□

2.2.2. $\Omega = \mathbf{R}^N$**.** Our main result for equation $(P-)$ is the following theorem.

Theorem 5. *Let* $\Omega = \mathbf{R}^N$, $\gamma > 1$, *and assume that condition* (P) *holds.*

If there exist positive numbers β *and* R *such that* $0 \leq p(x) \leq Mr^{-2-\beta}$, *whenever* $r \equiv |x| > R$, *then equation* $(P-)$ *has a nonnegative entire large solution provided that* $\max\{\gamma, q\} > 2$ *if* $q \geq 1$, *and* $\gamma \leq 1 + \beta(1-q)/(2-q)$ *if* $q < 1$.

To prepare for proving this theorem, we now state and prove three lemmas.

Lemma 6. *Let* M *be any nonnegative number and* β *any positive number.*

Then, for R *sufficiently large, there is a nonnegative solution of the problem*

$$w'' + \frac{N-1}{r} w' - |w'|^q \geq M r^{-2-\beta} w^{\gamma}, \quad r \geq R,$$
$$w(r) \to \infty, \quad r \to \infty \tag{2.32}$$

provided that $\gamma > 1$ *if* $q > 2$, *and* $\gamma \leq 1 + \beta(1-q)/(2-q)$ *if* $q < 1$.

Proof. Denote $L(w) \equiv w'' + (N - 1/r)w' - |w'|^q - Mr^{-2-\beta} w^{\gamma}$. Let α be a positive real number whose value will be made precise later. We have

$$\begin{aligned} L(r^{\alpha}) &= \alpha(\alpha-1)r^{\alpha-2} + \frac{N-1}{r}\alpha r^{\alpha-1} - \alpha^q r^{\alpha q - q} - Mr^{-2-\beta} r^{\alpha\gamma} \\ &= \alpha(\alpha+N-2)r^{\alpha-2} - \alpha^q r^{(\alpha-1)q} - Mr^{-2-\beta} r^{\alpha\gamma} \\ &= r^{-(2+\beta)+\alpha\gamma}\left[\alpha(\alpha+N-2)r^{\beta+\alpha(1-\gamma)} - \alpha^q r^{(\alpha-1)q+2+\beta-\alpha\gamma} - M\right] \\ &\equiv r^{-(2+\beta)+\alpha\gamma}\left[\alpha(\alpha+N-2)r^{\alpha_1} - \alpha^q r^{\alpha_2} - M\right]. \end{aligned} \tag{2.33}$$

Requiring that $\alpha_1 \geq 0$ and $\alpha_1 \geq \alpha_2$ yield

$$\alpha \leq \frac{\beta}{\gamma - 1} \tag{2.34}$$

and

$$\alpha \geq \frac{2-q}{1-q} \quad \text{for } q < 1, \quad \text{or} \quad \alpha \leq \frac{q-2}{q-1} \quad \text{for } q > 2, \tag{2.35}$$

respectively. Thus, for $q < 1$, we require that

$$\gamma \le 1 + \beta\left(\frac{1-q}{2-q}\right). \tag{2.36}$$

In this case, we take $\alpha = (1-q)/(2-q)$. For $q > 2$, we take $\gamma > 1$ and $\alpha \le \min\{\beta/(\gamma-1), (q-2)/(q-1)\}$. Hence, we may choose R large so that $L(r^\alpha) \ge 0$. Consequently, $w(r) \equiv r^\alpha$, where α is as defined above, is the desired solution. □

Lemma 7. *Let M and β be any positive numbers, and let $\phi(r) \equiv 2^{2+\beta}Mr^{-2-\beta}$. If there is a nonnegative solution u of*

$$\begin{gathered}\Delta u - |\nabla u|^s \ge \phi(r)u^\gamma, \quad |x| \ge R,\\ u(x) \to \infty, \quad |x| \to \infty\end{gathered} \tag{2.37}$$

that satisfies $u \ge r^{(s-2)/(s-1)}$ for some $s > 2$, then there is a nonnegative solution of

$$\begin{gathered}\Delta w - |\nabla w|^q \ge \phi(r)w^\gamma, \quad |x| \ge R,\\ w(x) \to \infty, \quad |x| \to \infty,\end{gathered} \tag{2.38}$$

where $1 \le q \le 2$, provided that $\gamma > 2 + \beta$.

Proof. Let $w = cu^\alpha$, where u is a nonnegative solution of (2.37), $0 < c$, $\alpha < 1$ are to be determined later. We have

$$\begin{aligned} L(w) &\equiv \Delta w - |\nabla w|^q - \phi(r)w^\gamma\\ &\ge c\alpha u^{\alpha-1}\left(|\nabla u|^s + \phi(r)u^\gamma\right)\\ &\quad - \left[c\alpha(1-\alpha)u^{\alpha-2}|\nabla u|^2 + (c\alpha)^q u^{(\alpha-1)q}|\nabla u|^q + \phi(r)c^\gamma u^{\alpha\gamma}\right]\\ &\ge c\alpha u^{\alpha-1}|\nabla u|^s + \left(c\alpha u^{\alpha+\gamma-1} - c^\gamma u^{\alpha\gamma}\right)\phi(r)\\ &\quad - c\alpha\left[(1-\alpha)u^{\alpha-2}|\nabla u|^2 + u^{(\alpha-1)}|\nabla u|^q\right]. \end{aligned} \tag{2.39}$$

We choose $0 < c$, $\alpha < 1$ so that $L(w) \ge 0$. Since $u \to \infty$, we may choose $R > 0$ so that $u \ge 1$.

Hence, we need to consider only two cases:

(a) $|u| \ge 1$ and $|\nabla u| \ge 1$;

(b) $|u| \ge 1$ and $|\nabla u| < 1$.

For case (a), it can be easily shown that $L(w) \ge 0$ by appropriately choosing c and α. For (b), it suffices to have

$$\begin{gathered}\tfrac{1}{3}c\alpha u^{\alpha+\gamma-1} \ge c^\gamma u^{\alpha\gamma},\\ \tfrac{1}{3}\phi(r)u^{\alpha+\gamma-1} \ge c\alpha(1-\alpha)u^{\alpha-2},\\ \tfrac{1}{3}\phi(r)u^{\alpha+\gamma-1} \ge c\alpha u^{\alpha-1}.\end{gathered} \tag{2.40}$$

Thus, we need only to require that

$$\tfrac{1}{3}\phi(r)u^\gamma \ge c\alpha. \tag{2.41}$$

Since $\phi(r) = 2^{2+\beta}Mr^{-2-\beta}$ and $u(x) \ge r^{(s-2)/(s-1)}$, it is sufficient to have

$$\tfrac{1}{3}2^{2+\beta}Mr^{(-2-\beta)+(\gamma)(s-2)/(s-1)} \ge c\alpha. \tag{2.42}$$

This is true since we can choose s large enough so that

$$(-2-\beta)+\gamma\left(\frac{s-2}{s-1}\right)>0. \tag{2.43}$$

Hence, we have $L(w)\geq 0$. □

Lemma 8. *Suppose that $\beta>0$ and $0\leq p(x)\leq 2^{2+\beta}Mr^{-2-\beta}$ for large M. Let w be a nonnegative solution of*

$$\begin{gathered}\Delta w-|\nabla w|^q\geq 2^{2+\beta}M(1+r)^{-2-\beta}w^{\gamma},\quad |x|\geq R\\ w(x)\to\infty,\quad |x|\to\infty\end{gathered} \tag{2.44}$$

for some $R\geq 1$. Let $M_o=\max_{|x|=R}w(x)$. Then, any solution of

$$\begin{gathered}\Delta v_k-|\nabla v_k|^q=p(x)v_k^{\gamma},\quad |x|<k\\ v_k(x)\to\infty,\quad |x|\to k\end{gathered} \tag{2.45}$$

must satisfy

$$v_k(x)\geq w(x)-M_o\quad \text{for } R\leq |x|<k, \tag{2.46}$$

for any $k>R$.

Proof. Suppose that the conclusion is false. That is, suppose that

$$\max_{R\leq x\leq k}\left[w(x)-M_o-v_k(x)\right]=w(x_o)-M_o-v_k(x_o)>0 \tag{2.47}$$

for some x_o. Since $w(x)-M_o-v_k(x)\leq -v_k(x)\leq 0$ if $|x|=R$ and $w(x)-M_o-v_k(x)\to-\infty$ as $|x|\to k$, we know that $R<|x_o|<k$. Hence, $\Delta(w-M_o-v_k)\leq 0$ at x_o and $\nabla(w-M_o-v_k)=0$ at x_o. Thus,

$$\begin{aligned}0&\geq\Delta(w-M_o-v_k)=\Delta w-\Delta v_k\\ &\geq|\nabla w|^q+(1+r)^{-2-\beta}M2^{2+\beta}w^{\gamma}-|\nabla v_k|^q-pv_k^{\gamma}\\ &=2^{2+\beta}(1+r)^{-2-\beta}Mw^{\gamma}-pv_k^{\gamma}\\ &>\left[2^{2+\beta}M(1+r)^{-2-\beta}-p\right]v_k^{\gamma}.\end{aligned} \tag{2.48}$$

Hence, we get

$$0>\left[2^{2+\beta}M(1+r)^{-2-\beta}-p\right]v_k^{\gamma}. \tag{2.49}$$

However, since $p(x)\leq Mr^{-2-\beta}$ and $r\geq R\geq 1$, we get

$$p(x)\leq M\left(\frac{1+r}{r}\right)^{2+\beta}(1+r)^{-2-\beta}\leq M2^{2+\beta}(1+r)^{-2-k}. \tag{2.50}$$

Hence, we get

$$2^{2+\beta}M(1+r)^{-2-\beta}-p\geq 0, \tag{2.51}$$

which contradicts (2.49). Hence, $w(x)-M_o-v_k(x)\leq 0$ in $R\leq|x|\leq k$. □

We now prove Theorem 5.

Proof. By Theorem 4, we have that, for $k = 1, 2, \ldots$, the boundary blow-up problem

$$\Delta v_k - |\nabla v_k|^q = p(x) v_k^\gamma, \quad |x| < k, \qquad v_k(x) \longrightarrow \infty, \quad \text{as } |x| \to k \tag{2.52}$$

has a nonnegative classical solution. By the maximum principle, it is clear that

$$v_1 \geq v_2 \geq \cdots \geq v_k \geq v_{k+1} \geq \cdots \tag{2.53}$$

in $\mathbf{R}^N$. Let $v(x) = \lim_{k\to\infty} v_k(x)$, $x \in \mathbf{R}^N$, where we assume that $v_k = \infty$ for $|x| > k$ for all k. Let $v(x) = \lim_{k\to\infty} v_k(x)$, $x \in \mathbf{R}^N$. We claim that v is the desired solution. To prove this, we need only to prove that v satisfies $(P-)$ and that $v \to \infty$ as $|x| \to \infty$. To prove the second statement, it suffices to find a nonnegative lower bound, say $w(x)$, for the sequence $\{v_k\}_1^\infty$ such that $w \to \infty$ as $|x| \to \infty$. Also, by another standard regularity argument (see [10]), we can show that v is smooth and is a classical solution of $(P-)$.

If $q > 2$ or $q < 1$, let w be a solution of (2.44) which we know to exist by Lemma 6, where M is a constant. Then, by Lemma 11, $v_k(x) \geq w(x) - M_o$, where $M_o = \max_{|x|=1} w(x)$ for $1 \leq |x| \leq R$.

Thus, $v(x) \geq w(x) - M_o$ for $1 \leq |x| \leq R$. Hence, $v(x) \to \infty$ as $|x| \to \infty$. We conclude that v is a solution of $(P-)$. In particular, if $q > 2$, then $v \geq w = r^{(q-2)/(q-1)}$.

Assume that $1 \leq q \leq 2$. By Lemma 6, there is a solution, say u, to equation (2.37), where $s > 2$. Now, let w be a solution of (2.38). It is clear that w solves (2.44) and hence $v \geq w - M_o$, where M_o is defined as in Lemma 8. Again, we get $v \to \infty$ as $|x| \to \infty$, and is a classical solution of $(P-)$. □

3. Nonexistence results

Theorem 9. *Suppose that condition (P) and (1.4) hold. If $0 \leq \gamma \leq 1$, then $(P+)$ has no positive entire large solution in $\mathbf{R}^N$.*

Proof. We first show that (1.4) implies that

$$\int_0^\infty r^{1-N} \int_0^r s^{N-1} \phi(s)\, ds\, dr < \infty. \tag{3.1}$$

Indeed, for any $R > 0$, we have

$$\begin{aligned}
\int_0^R r^{1-N} \int_0^r s^{N-1} \phi(s)\, ds\, dr
&= \frac{1}{2-N} \int_0^R \frac{d}{dr}\left(r^{2-N}\right) \int_0^r s^{N-1} \phi(s)\, ds\, dr \\
&= \frac{1}{2-N} r^{2-N} \int_0^r s^{N-1} \phi(s)\, ds \,\Big|_0^R - \frac{1}{2-N} \int_0^R r^{2-N} r^{N-1} \phi(r)\, dr \\
&= \frac{1}{2-N} R^{2-N} \int_0^R s^{N-1} \phi(s)\, ds - \frac{1}{2-N} \int_0^R r \phi(r)\, dr \\
&\leq \frac{1}{N-2} \int_0^\infty r \phi(r)\, dr < \infty.
\end{aligned} \tag{3.2}$$

By (3.1), we can choose r_o large enough so that

$$\beta \equiv \int_{r_o}^{\infty} t^{1-N} \int_0^t s^{N-1} \phi(s)\, ds\, dt < 1. \tag{3.3}$$

Now, suppose that $(P+)$ has a positive large solution $u(x)$. Using technique similar to that described in [5], we define

$$\bar{u}(r) \equiv \frac{1}{v_o(s^{N-1}r)} \int_{|x|=r} u(x)\, d\sigma_r \equiv \int_{|x|=r} u(x)\, d\sigma, \tag{3.4}$$

where $v_o(s^{N-1}r)$ is the volume of the $(N-1)$-dimensional sphere and σ_r is the measure on the sphere. We have

$$\begin{aligned} \Delta\bar{u} &= \bar{u}'' + \frac{N-1}{r}\bar{u}' = \int_{|x|=r} \Delta u\, d\sigma \\ &= \int_{|x|=r} \left[p(x)u^{\gamma} - |\nabla u|^q\right] d\sigma \le \phi(r) \int_{|x|=r} u^{\gamma}\, d\sigma \\ &\le \phi(r) \left[\int_{|x|=r} u\, d\sigma\right]^{\gamma} \quad \text{(By Jensen's Inequality)} \\ &= \phi(r)\bar{u}^{\gamma}(r). \end{aligned} \tag{3.5}$$

Thus, we have

$$\bar{u}'' + \frac{N-1}{r}\bar{u}' \le \phi(r)\, \bar{u}^{\gamma}(r). \tag{3.6}$$

Integrating the above inequality yields

$$\begin{aligned} \bar{u}(r) &\le \bar{u}(r_o) + r_o^{N-1}\bar{u}'(r_o) \frac{r^{2-N}}{2-N}\Bigg|_{r_o}^{r} + \int_{r_o}^{r} t^{1-N} \int_{r_o}^{t} s^{N-1}\phi(s)\bar{u}^{\gamma}(s)\, ds\, dt \\ &\le \bar{u}(r_o) + \frac{1}{N-2} r_o \bar{u}'(r_o) + \int_{r_o}^{r} t^{1-N} \int_{r_o}^{t} s^{N-1}\phi(s)\bar{u}^{\gamma}(s)\, ds\, dt \\ &\le \bar{u}(r_o) + \frac{1}{N-2} r_o \bar{u}'(r_o) + \left(\max_{r_o \le r \le R} \bar{u}(r)\right)^{\gamma} \beta \quad \text{for } r_o \le r \le R \\ &\equiv A + \left(\max_{r_o \le r \le R} \bar{u}(r)\right)^{\gamma} \beta. \end{aligned} \tag{3.7}$$

Let $h(R) = \max_{r_o \le r \le R} \bar{u}(r)$. Since $0 \le \gamma \le 1$, then the last inequality gives

$$h(R) \le A + (h(R))^{\gamma}\beta \le A + \beta(1 + h(R)) \quad \forall R \ge r_o, \tag{3.8}$$

or equivalently,

$$0 \le \bar{u}(R) \le h(R) \le \frac{A+\beta}{1-\beta}. \tag{3.9}$$

Thus, $\bar{u}$ is bounded and hence cannot be a large solution. Consequently, u cannot be a large solution. □

Combining Theorems 3 and 9, we get the following corollary.

Corollary 10. *Assume that conditions (P) and (1.4) hold. Let $\gamma > q$, then equation $(P+)$ has a positive entire large solution if and only if $\gamma > 1$.*

Theorem 11. *Suppose that* Ω *is bounded and* $0 \le \gamma \le 1$. *Then* $(P+)$ *has no positive large solution in* Ω.

Proof. Suppose that $(P+)$ has a positive large solution $u(x)$ in Ω. Assume, without loss of generality, that $0 \in \Omega$. Let B be a ball of radius R centered at 0 and containing Ω such that $\partial B \cap \partial\Omega \neq \emptyset$. Let $M = \max_{x\in\bar{\Omega}} p(x)$.

Let v_n be the unique solution of

$$\begin{aligned} \Delta v_n &= M v_n^{\gamma}, \quad x \in B, \\ v_n &= n, \quad x \in \partial B. \end{aligned} \tag{3.10}$$

Clearly, $v_n \le u$ in B. (We assume that $u = \infty$ in $B \backslash \Omega$). As shown in Lemma 1, v_n is radially symmetric and thus satisfies

$$\begin{aligned} v_n'' + \frac{N-1}{r} v_n' &= M v_n^{\gamma}, \quad x \in B, \\ v_n &= n, \quad x \in \partial B. \end{aligned} \tag{3.11}$$

Hence, we have

$$\left(r^{N-1} v_n'\right)' = M r^{N-1} v_n^{\gamma}. \tag{3.12}$$

Integrating this inequality yields

$$v_n'(r) = M r^{1-N} \int_0^r s^{N-1} v_n^{\gamma}(s)\, ds \ge 0. \tag{3.13}$$

Choose $0 < r_o < R$ so that

$$\frac{R^2 - r_o^2}{2N} \le \frac{1}{2M}. \tag{3.14}$$

Integrating (3.13), using (3.14), and noting that v_n is monotonically increasing gives

$$\begin{aligned} v_n(r) &= v_n(r_o) + M \int_{r_o}^r t^{1-N} \int_0^t s^{N-1} v_n^{\gamma}(s)\, ds\, dt \\ &\le v_n(r_o) + M v_n^{\gamma}(r) \left[\frac{r^2 - r_o^2}{2N}\right] \\ &\le v_n(r_o) + \tfrac{1}{2} v_n^{\gamma}(r). \end{aligned} \tag{3.15}$$

Since $0 \le \gamma \le 1$, we have $v_n^{\gamma} \le 1 + v_n$, which gives

$$v_n(r) \le v_n(r_o) + \tfrac{1}{2} + \tfrac{1}{2} v_n. \tag{3.16}$$

Thus, we get

$$v_n(r) \le 1 + 2 v_n(r_o) \quad \text{for } \ r_o \le r \le R. \tag{3.17}$$

In particular, we have

$$v_n(R) \le 1 + 2 v_n(r_o), \tag{3.18}$$

that is

$$v_n(r_o) \ge \frac{n-1}{2}. \tag{3.19}$$

Thus, for each $x_o \in \Omega$ such that $r_o = |x_o|$ satisfies (3.14), we get

$$u(x_o) \geq v_n(r_o) \geq \frac{n-1}{2} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

Hence, $u(x)$ does not exist. □

Corollary 12. *Suppose that $\gamma > q$ and Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 3)$. Let $p(x)$ satisfy condition (P). Then equation $(P+)$ has a positive large solution if and only if $\gamma > 1$.*

4. Condition on p is nearly optimal. We have shown in Theorem 2 that if the nonnegative function p is such that each of its zero points is enclosed by a bounded surface of nonzero points, then equation $(P+)$ has a large positive solution. In this section we show that, if the condition does not hold in the sense that p vanishes in an "outer ring" of the domain, then equation $(P+)$ has no positive large solution.

Theorem 13. *Suppose that $g(x,0) = 0$ $\forall x \in \Omega$. If there exists a domain $D \subseteq \Omega$ such that $\bar{D} \subset \Omega$ and $g(x,t) = 0$ $\forall x \in \Omega \backslash D$, $\forall t \geq 0$, then there is no positive large solution of*

$$\Delta u + |\nabla u|^q = g(x,u), \quad x \in \Omega. \tag{4.1}$$

Note that this includes the case $g(x,u) = p(x)u^\gamma$, $\gamma \geq 0$, and $p(x) = 0$ in $\Omega \backslash D$.

Proof. Suppose that such a solution u exists. Let w be the unique positive solution of

$$\begin{aligned} \Delta w &= 0, \quad x \in \Omega \backslash D, \\ w &= 0, \quad x \in \partial D, \\ w &= 1, \quad x \in \partial \Omega. \end{aligned} \tag{4.2}$$

It should be noted that u satisfies

$$\begin{aligned} \Delta u &\leq 0, \quad x \in \Omega \backslash D, \\ u &= \infty, \quad x \in \partial \Omega, \\ u &\geq 0, \quad x \in \partial D. \end{aligned} \tag{4.3}$$

By the maximum principle, we get $kw \leq u$ for any $k > 0$ since kw satisfies

$$\begin{aligned} \Delta(kw) &= 0, \quad x \in \Omega \backslash D, \\ kw &= k, \quad x \in \partial \Omega, \\ kw &= 0, \quad x \in \partial D. \end{aligned} \tag{4.4}$$

Therefore, for any $x_o \in \Omega \backslash D$, we have $w(x_o) > 0$ and $kw(x_o) \leq u(x_o)$ $\forall k > 0$. Hence, $u(x_o) = \infty$, which is a contradiction. □

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