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CONTROL SUBGROUPS AND BIRATIONAL EXTENSIONS OF GRADED RINGS

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ABSTRACT. In this paper, we establish the relation between the concept of control subgroups and the class of graded birational algebras. Actually, we prove that if $R = \bigoplus_{\sigma\in G} R_\sigma$ is a strongly G-graded ring and $H \lhd G$, then the embedding $i : R^{(H)} \hookrightarrow R$, where $R^{(H)} = \bigoplus_{\sigma\in H} R_\sigma$, is a Zariski extension if and only if H controls the filter $\mathcal{L}(R-P)$ for every prime ideal P in an open set of the Zariski topology on R. This enables us to relate certain ideals of R and $R^{(H)}$ up to radical.

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1. Introduction. The study of birational extension of rings was, in some sense, started by F. van Oystaeyen in 1978, [11]. The main motivation in introducing the class of birational algebras was to generalize the notion of Zariski central rings which behaves very well with respect to localizations at prime ideals [10]. Birational algebras received some interest on one hand because all the semiprime PI rings are birational algebras over their centers and on the other hand because the birationality properties determine interesting classes within the classes of fully left bounded Noetherian rings, HNP rings, regular and biregular rings, and Von Neuman rings, cf. [7, 11].

In Passman's book [8], some use has been made of control subgroups of ideals of group rings, in particular, in the study of radicals of group rings. The case where a subgroup H of a group G left controls some basis of a Gabriel filter of left ideals of a group ring $F[G]$ has been investigated intensively and extensively in [5]. The notions of control subgroups of submodules of graded modules as well as for other objects related to graded objects have been introduced in [3] with the aim of studying localizations and filtrations of strongly graded rings, cf. [3, 4].

All of this has prompted us to establish the relation between the concept of control subgroups and the class of graded birational algebras. After introducing the basic notions and definitions in Section 2, we prove, in Section 3, that if $R = \bigoplus_{\sigma\in G} R_\sigma$ is a strongly G-graded ring and $H \lhd G$, then an extension $R^{(H)} \hookrightarrow R$, where $R^{(H)} = \bigoplus_{\sigma\in H} R_\sigma$, is a Zariski extension if and only if H controls the filter $\mathcal{L}(R-P)$, for every prime ideal P in an open set of the Zariski topology. This enables us to relate certain ideals of R and $R^{(H)}$ up to radical. The main results are contained in Theorem 3.2, Corollary 3.3, and Theorem 3.8.

2. Preliminaries. All the rings considered are associative with unit. For definitions and properties of birational extensions of rings, we refer to [11]. A ring homomorphism $f : A \to B$ is said to be an extension in the sense of C. Procesi [9] if $B = f(A)C_B(f(A))$, where $C_B(f(A)) = \{b \in B, ba = ab \text{ for all } a \in f(A)\}$. In this paper, we consider only the inclusions $A \hookrightarrow B$, so the condition reduces to $B = A \cdot C_B(A)$. For any ring A, the set of prime ideals $X = \mathrm{Spec}(A)$ may be equipped with the Zariski topology given by the open sets $X(I) = \{P \in \mathrm{Spec}(A), I \not\subset P\}$ associated with (two-sided) ideals I of A. If I is an ideal of A, then, by $\mathrm{rad}(I)$, we denote the intersection $\cap\{P; P \in \mathrm{Spec}(A)$ and $I \subset P\}$. An extension $f : A \to B$ is a birational extension if there exist nonempty open sets $U \subset Y = \mathrm{Spec}(B)$ and $V \subset X = \mathrm{Spec}(A)$ such that

(1) if $P \in Y$ is such that $f^{-1}(P) \in V$, then $P \in U$;

(2) the correspondence $P \to f^{-1}(P)$ induces a topological isomorphism $U \cong V$.

Moreover, this birational extension $f : A \to B$ is called a Zariski extension if, to an ideal J of B, there corresponds an ideal $J' \subset J$ of A such that $X(I' \cdot J') \cong Y(I \cdot J)$ under f^{-1}.

If P is a prime ideal of a ring A, then, by $\mathscr{L}(A-P)$, we denote the symmetric filter

$$\mathscr{L}(A-P) = \{I \subset A, I \text{ is a left ideal of } A \text{ with } AsA \subset I \text{ for an } s \in A-P\}. \tag{2.1}$$

The kernel functor induced by $\mathscr{L}(A-P)$ is denoted by σ. For details on torsion theory and localization theory, we refer to [1, 2].

Throughout, G is a finite group with neutral element e, H is a subgroup of G, and the ring $R = \bigoplus_{\sigma \in G} R_\sigma$ is a strongly G-graded ring. For details on graded rings and modules, we refer to [6]. All modules are left modules. If M is a G-graded R-module, then, by $M^{(H)}$, we denote the R_e-module $\bigoplus_{h \in H} M_h$. If $m = \sum_{\sigma \in G} m_\sigma \in M$, then $\mathrm{Supp}(m)$ denotes the subset $\{\sigma \in G, m_\sigma \neq 0\}$. There is a canonical R_e-linear map $\pi_H : M \to M^{(H)}$ given by $\pi_H(\sum_{\sigma \in G} m_\sigma) = \sum_{\sigma \in H} m_\sigma$. It has been proved in [3] that, for any R-submodule N of M, the following assertions are equivalent

(1) $\pi_H(N) \subset N$.

(2) $\pi_H(N) = N \cap M^{(H)}$.

(3) $N = R \cdot \pi_H(N)$.

If one of these conditions holds, then we say that H left controls N in M. The control subgroup of N in M is the smallest subgroup H of G left controlling N in M. Thus, the control subgroup $C_M(N)$ of N in M is the intersection in G of all the subgroups controlling N in M [3].

3. Control subgoups and birational extensions. Throughout this section, H is a normal subgroup of G and R is a strongly G-graded ring. By R-mod, we denote the category of left R-modules.

Definition 2.1. We say that the subgroup H of G (left) control a Gabriel filter $\mathscr{L}$ of left ideals of R if, for every $L \in \mathscr{L}$, there exists (a left ideal) an ideal $I \in \mathscr{L}$ such that $I \subset L$ and I being (left) controlled by H, i.e., $I = R \cdot (I \cap R^{(H)})$. A Gabriel filter $\mathscr{L}'$ of left ideals of $R^{(H)}$ is said to be G-invariant if $I' \in \mathscr{L}'$ implies $R_g I' R_{g^{-1}} \in \mathscr{L}'$ for every $g \in G$, cf. [3].

Theorem 3.2. *Suppose that the embedding $i : R^{(H)} \hookrightarrow R$ is an extension. If H controls $\mathscr{L}(R-P)$ for all P in a nonempty open set U of* Spec (R)*, then $i : R^{(H)} \hookrightarrow R$ is a birational extension.*

Proof. Let $X = \operatorname{Spec} R^{(H)}$, $Y = \operatorname{Spec} R$, and $U = Y(I)$ for some ideal I of R. In fact, the ideal I can be chosen such that I is controlled by H. Indeed, since $I \in \mathcal{L}(R-P)$, for every $P \in Y(I)$, then there exists an ideal $I_P \subset I$ such that $I_P \in \mathcal{L}(R-P)$ and I_P is controlled by H. Thus, the ideal $J = \sum_{P\in Y(I)} I_P \subset I$ is controlled by H. Moreover, $J \in \mathcal{L}(R-P)$ and $Y(I) = Y(J)$. If V is the open set $X(\pi_H(J))$ of $\operatorname{Spec}(R^{(H)})$, then $P \cap R^{(H)} \in V$ for every $P \in Y(J)$. Hence, we can define a map $\psi : U \to V$ as follows:

$$\psi(P) = i^{-1}(P) = P \cap R^{(H)}. \tag{3.1}$$

Since H controls J, then, for every $P \in Y$ with $i^{-1}(P) = P \cap R^{(H)} \in V$ we have $J \not\subset P$ or, equivalently, $P \in U = Y(J)$. Thus, it only remains to prove that ψ is a homeomorphism.

Let $P \in Y(J)$ and $Q \in Y$ such that $Q \neq P$ and $Q \cap R^{(H)} = P \cap R^{(H)}$. We consider the following two cases

(i) If $Q \not\subset P$, then $Q \in \mathcal{L}(R-P)$. Hence, there exists an ideal $Q' \in \mathcal{L}(R-P)$ such that $Q' \subset Q$ and H controls Q'. Thus,

$$Q' = R \cdot \left(Q' \cap R^{(H)}\right) \subset R \cdot \left(Q \cap R^{(H)}\right) = R \cdot \left(P \cap R^{(H)}\right) \subset P. \tag{3.2}$$

On the other hand, $Q' \in \mathcal{L}(R-P)$ implies $Q' \not\subset P$ and we have a contradiction.

(ii) If Q is a proper subset of P, then $Q \in U$. Thus, H controls $\mathcal{L}(R-Q)$ and $P \in \mathcal{L}(R-Q)$.

Since $P \not\subset Q$, case (ii) reduces to case (i) and we obtain the same contradiction. Therefore, ψ is injective.

To prove that ψ is surjective, let $P \in V$. Since R is G-graded, then $(R\cdot P) \cap R^{(H)} = P$. Indeed, every $x \in (R \cdot P) \cap R^{(H)}$ can be decomposed as

$$x = \sum_{i=1}^{n} a^{(i)} b^{(i)}; \quad a^{(i)} \in R,\, b^{(i)} \in P,\, i = 1,2,\dots,n. \tag{3.3}$$

Moreover, if $S = \operatorname{Supp} a^{(i)}$ and $T = \operatorname{Supp} b^{(i)}$, $i = 1,2,\dots,n$, then

$$a^{(i)} = \sum_{s\in S} a_s^{(i)}, \qquad b^{(i)} = \sum_{t\in T} b_t^{(i)}; \quad a_s^{(i)} \in R_s,\, b_t^{(i)} \in R_t. \tag{3.4}$$

Since $x \in R^{(H)}$ and $T \subset H$, the G-gradation of R implies that $S \subset H$. Thus, $a^{(i)} \in R^{(H)}$, $i = 1,2,\dots,n$. This entails that $x \in P$. Hence, $(R\cdot P) \cap R^{(H)} \subset P$. Clearly, $P \subset (R\cdot P) \cap R^{(H)}$. Thus, $(R \cdot p) \cap R^{(H)} = P$. Since i is an extension, then $R \cdot P$ is an ideal of R. If L is the maximal ideal of R such that $L \cap R^{(H)} = P$, then it is not hard to prove that L is a prime ideal of R. Since J is controlled by H, then $L \in Y(J)$. It follows that L is the $\psi-$preimage of P. Hence, ψ is surjective.

Now, let W be an open set in $Y(J)$. Thus, $W = Y(F)$, where $F \subset J$ is an ideal of R. It follows that there exists an ideal $F' \subset F$ of R such that H controls F' and $Y(F) = Y(F')$. It is not hard to prove that $\psi(W) = X(\pi_H(F'))$. Since $\pi_H(F') \subset \pi_H(F) \subset \pi_H(J)$, then $X(\pi_H(F'))$ is open in $X(\pi_H(J))$. Therefore, ψ is an open map.

Finally, we show that ψ is continuous. If O is an open set in $X(\pi_H(J))$, then there exists an ideal q of $R^{(H)}$ such that $q \subset \pi_H(J)$ and $O = X(q)$. Clearly, $\psi^{-1}(O) = Y(R \cdot q) \subset Y(J)$. Thus, $\psi^{-1}(O)$ is open in $Y(J)$. Hence, ψ is continuous. □

Corollary 3.3. *If $i : R^{(H)} \hookrightarrow R$ is an extension and H controls $\mathcal{L}(R-P)$ for all*

$P \in Y(J) \subset \mathrm{Spec}(R)$, then

(i) *If $L \subset J$ is an ideal of R, then* $\mathrm{rad}(L) = \mathrm{rad}(R \cdot (L \cap R^{(H)}))$.

(ii) *For any ideal L of R,* $\mathrm{rad}(J \cdot L) = \mathrm{rad}(J \cdot (L \cap R^{(H)}) \cdot R)$.

(iii) *If $P \in \mathrm{Spec}(R)$ and $q = P \cap R^{(H)}$, then* $\mathrm{rad}(J \cdot P) = \mathrm{rad}(J \cdot q)$.

Furthermore, $i : R^{(H)} \hookrightarrow R$ is a Zariski extension.

Proof. The proof is a slight modification of the proof of [5, Cor. 3.3]. □

Proposition 3.4. *Suppose that the embedding $i : R^{(H)} \hookrightarrow R$ is an extension. If P is a prime ideal of R such that H left controls the filter $\mathcal{L}(R-P)$, then H controls $\mathcal{L}(R-P)$.*

Proof. Let $L \in \mathcal{L}(R-P)$. By assumption, there exists a left ideal $I \in \mathcal{L}(R-P)$ such that $I \subset L$ and H left controls I. It follows that there exists an $s \in R-P$ such that $RsR \subset I$. If Y is a left transversal for H in G, then $R = \bigoplus_{y \in Y} R_y R^{(H)}$. Hence, there exists a finite subset Y' of Y such that $s = \sum_{y \in Y'} r_y s_y$, where, for every $y \in Y'$, $r_y \in R_y$ and $s_y \in R^{(H)}$. Since $s \notin P$, then there is $y_0 \in Y'$ with $s_{y_0} \in R-P$. Moreover,

$$R^{(H)} s_{y_0} R^{(H)} \subset \pi_H \left(R^{(H)} R_{y_0^{-1}} s R^{(H)} \right) \subset I. \tag{3.5}$$

Since i is an extension, then $J = RR^{(H)} s_{y_0} R^{(H)} \subset I$ is an ideal of R. Thus, $J = Rs_{y_0}R \in \mathcal{L}(R-P)$ and $J \subset L$. Since H controls J, the result follows. □

Proposition 3.5. *Suppose that P is a prime ideal of R. If $q = P \cap R^{(H)}$, then $\mathcal{L}(R^{(H)} - q)$ is G-invariant.*

Proof. By definition, if $I \in \mathcal{L}(R^{(H)} - q)$, then there exists an $s \in R^{(H)} - q$ such that $R^{(H)} s R^{(H)} \subset I$. It follows from the strong gradation of R that

$$R^{(H)} R_g s R_{g^{-1}} R^{(H)} = R_g R^{(H)} s R^{(H)} R_{g^{-1}} \subset R_g I R_{g^{-1}}. \tag{3.6}$$

Thus, $R_g I R_{g^{-1}} \in \mathcal{L}(R^{(H)} - q)$ or, equivalently, $\mathcal{L}(R^{(H)} - q)$ is G-invariant. □

Remark 2.6. With assumptions as those in Proposition 3.4, if $q = P \cap R^{(H)}$ and $L = \{I' \subset R^{(H)}$, there exists an $I \in \mathcal{L}(R-P)$ such that $I' = I \cap R^{(H)}\}$, then one can easily check that L is nothing but $\mathcal{L}(R^{(H)} - q)$.

Proposition 3.7. *With assumptions as those in Proposition 3.4, if σ' denotes the Kernel functor associated with $\mathcal{L}(R^{(H)} - q)$, then $Q_\sigma(R) \cong Q_{\sigma'}(R)$ as $R^{(H)}$-modules, where $Q_\sigma(-)$, resp. $Q_{\sigma'}(-)$ denote the localization functor in R-mod, resp. $R^{(H)}$-mod corresponding to σ, resp. σ'.*

Proof. A direct consequence of the previous remark and [3, Thm. 4.5]. □

Theorem 3.8. *If $i : R^{(H)} \hookrightarrow R$ is a Zariski extension, then H controls $\mathcal{L}(R-P)$ for all P in a nonempty open set U of* $\mathrm{Spec}(R)$.

Proof. Let $U = Y(J)$ be an open set of $Y = \mathrm{Spec}(R)$ and $V = X(J')$ be the corresponding open set of $X = \mathrm{Spec}(R^{(H)})$ such that $Y(J) \cong X(J')$ under i^{-1}. Clearly, $P \in Y(J)$ implies the existence of an $s \in J-P$ such that $RsR \subset J$. Hence, $J \in \mathcal{L}(R-P)$. If $P \in Y(J)$ and $I' \in \mathcal{L}(R-P)$, then there exists an ideal $I \in \mathcal{L}(R-P)$ such that $I \subset I'$. Thus, the primitivity of P implies $J \cdot I \in \mathcal{L}(R-P)$. Moreover, we obtain, from [11, Prop. 1.2], that $Y(J\cdot I)=Y(J'\cdot(I\cap R^{(H)})\cdot R)$ or, equivalently, $\mathrm{rad}(J\cdot I)=\mathrm{rad}(J'\cdot(I\cap R^{(H)})\cdot R)$. Since $J\cdot I\subset\mathrm{rad}(J\cdot I)$ and $J\cdot I\in\mathscr{L}(R-P)$, then $\mathrm{rad}(J\cdot I)\in\mathscr{L}(R-P)$. Furthermore, the kernel functor σ induced by $\mathscr{L}(R-P)$ is radical (cf. [11, Thm. 2.4]). Therefore, $\mathrm{rad}(J'\cdot(I\cap R^{(H)})\cdot R)\in\mathscr{L}(R-P)$ implies $L=J'\cdot(I\cap R^{(H)})\cdot R\in\mathscr{L}(R-P)$. Because i is an extension, L is an ideal of R. Obviously, H controls L and $L\subset(I\cap R^{(H)})\cdot R\subset I\subset I'$. Thus, H controls $\mathscr{L}(R-P)$ and the assertion follows. □

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