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LIMIT SETS IN PRODUCT OF SEMI-DYNAMICAL SYSTEMS

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Abstract. Continuing the study of the properties of Poisson stability and distality [4], we mention the conditions under which $\Omega_x(x) = \Pi\Omega_\alpha(x_\alpha), \alpha \in I$ and thus, the product of Poisson stable motions remains Poisson stable in the product system.

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1. Introduction. We deal mainly with the product of w-limit sets in the product space of semi-dynamical systems (s.d.s.). In [1], Prem Bajaj has shown that the product of semi-dynamical systems is a semi-dynamical system. He has also shown that $\Pi\Omega_\alpha(x_\alpha), \alpha \in I$ contains the w-limit set $\Omega_x(x)$ of x in the product system. In general, equality does not hold in the above. Indeed $\Omega_x(x)$ may be empty. He has given two theorems: one in which $\Omega_x(x)$ is nonempty and the other indicating a case of equality viz. Theorems 2.3 and 2.4.

In this paper, continuing the study of the properties of Poisson stability and distality [4], we mention the conditions under which $\Omega_x(x) = \Pi\Omega_\alpha(x_\alpha)$, $\alpha \in I$, $x = \{x_\alpha\}$ and therefore, the product of Poisson stable motions, under these conditions, is Poisson stable.

2. Definitions and notations

Definition 2.1. A continuous mapping $\pi : X \times \mathbb{R}^+ \to X$ on a topological space X is said to define a semi-dynamical system (X,π) if $\pi(x,0) = x$ and $\pi(\pi(x,t),s) = \pi(x,t+s)$ for every $x \in X$ and $t,s \in \mathbb{R}^+$. ($\mathbb{R}^+$ denotes the set of nonnegative reals.)

Definition 2.2. Let (X_α,π_α), $\alpha \in I$ be a family of dynamical systems. Let $X = \Pi X_\alpha$ be the product space. Let $x \in X$ and $x = \{x_\alpha\}$. Define a map π from $X \times \mathbb{R}$ into X by $\pi(x_\alpha t) = (x_\alpha t)$, $\alpha \in I$, then (X,π) is a dynamical system. The dynamical system (X,π), obtained above, is called the direct product or the product of the family (X_α,π_α), $\alpha \in I$.

We take the usual definitions of positive limit set Ω_x, positive distal, positive Poisson stable, and positive Lagrange stable motions. As usual, we drop the word positive and we use the notations of [1, 4].

3. Main results

Proposition 3.1. *Let (X_α,π_α), $\alpha \in I$, be a family of {Lagrange stable} {distal} s.d.s.*

and (X,π) *the product s.d.s. Let* $x \in X$ *and* $x = \{x_\alpha\}$, *then* (X,π) *is {Lagrange stable} {distal}.*

Proposition 3.2. *If a Lagrange stable motion is Poisson stable and distal, then* $\mathrm{Cl}\Upsilon(x) = \Upsilon(x) = \Omega_x$.

Proof. The proof follows from [4, Thm. 2.1]. □

Theorem 3.3. *Let* (X_α,π_α), $\alpha \in I$, *be a family of dynamical systems and* (X,π) *the product of the dynamical systems. Let* $x \in X$ *and* $x = \{x_\alpha\}$. *Then* $\Omega_x(x) \subseteq \Pi\Omega_\alpha(x_\alpha)$, *where* $\Omega_\alpha(x_\alpha)$ *is the positive limit set of* x_α *in the dynamical systems* (X_α,π_α). *(The two* π*'s have distinct meanings according to the context.)*

Since, in general, the equality does not hold and Ω_x may be empty, the Poisson stability in the constituent dynamical system may be lost from the product of the dynamical systems. Here, we find the conditions under which $\Omega_x(x) = \Pi\Omega_\alpha(x_\alpha)$, $\alpha \in I$ and thus, the product of Poisson stable motions remains Poisson stable in the product system.

Theorem 3.4. *If a compact motion is Poisson stable and distal, then it is a compact recurrent motion.*

Proof. Let the motion $\pi(x,t)$ be Poisson stable and distal, then its trajectory $\Upsilon(x)$ is closed. Therefore,

$$\Upsilon(x) = \mathrm{Cl}\Upsilon(x) = \Omega_x. \tag{3.1}$$

As the motion is compact, each of the above sets is compact and minimal and thus, by Birkhoff recurrence theorem, $\pi(x,t)$ is compact and recurrent. □

Theorem 3.5. *Let* (X,π) *be a semi-dynamical system. Let* π *be a Lagrange stable, then* π *is distal if and only if, for every net* t_i *in* $\mathbb{R}^+$, *the phase space*

$$X = \{z \in X : xt_j \longrightarrow z \text{ for some } x \in X \text{ and some subnet } t_j \text{ of } t_i\} \tag{3.2}$$

[2, Thm. 2.6].

Theorem 3.6. *Let* (X,π) *be Lagrange stable and distal s.d.s. then every net in the trajectory* $\Upsilon(x)$ *of the Poisson stable motion* $\pi(x,t)$ *is a Cauchy net.*

Proof. Let $\Upsilon(x)$ be the trajectory of the Poisson stable motion $\pi(x,t)$ in s.d.s. (X,π) which is Lagrange stable and distal. Let xt_n be a net in $\Upsilon(x)$ which is compact (Proposition 3.2). Therefore, xt_n has a subnet, say xt_m with $xt_m \to z$, i.e., z is a cluster point of xt_n. Hence, xt_n is a Cauchy net. □

Theorem 3.7. *Let* (x_α,π_α), $\alpha \in I$, *be a family of Lagrange stable and distal s.d.s. and* (X,π) *be the product s.d.s. Let* $x \in X$ *and* $x = \{x_\alpha\}$. *A motion* $\pi(x,t)$ *is Poisson stable in* (X,π) *if and only if* $\pi_\alpha(x_\alpha,t)$ *is Poisson stable in* (X_α,π_α) *for each* $\alpha \in I$.

Proof. Let (x_α,π_α), $\alpha \in I$, be a Lagrange stable and distal s.d.s. Let $\pi(x_\alpha,t) = x_\alpha t$ be a Poisson stable motion in (X_α,π_α), $\alpha \in I$, then its trajectory $\Upsilon_\alpha(x_\alpha)$ is compact and the net $x_\alpha t_n$, $\alpha \in I$, is a Cauchy net in $\Upsilon_\alpha(x_\alpha)$ (Theorem 3.6). Now, the Cauchy nets $x_\alpha t_n, \alpha \in I$ yield the Cauchy net xt_n in $\Upsilon(x)$ in (X,π) [3, p. 194]. As the product of compact sets is a compact set, $\Upsilon(x)$ is compact and xt_n is a net in compact $\Upsilon(x)$. Thus, it has a subnet $xt_m \to z$, i.e., z is a cluster point of xt_n. Hence, xt_n is frequently in every neighborhood U of z. Given a neighborhood U of z for every $i \in A$, there is a $j \in A$, $i \geq J$ such that $xt_i \in U$ however $t_i \to +\infty$. Hence, $\pi(x,t)$ is Poisson stable. The converse follows from [3, Thm. 25, p. 194] which states that a net in the product is a Cauchy net if and only if its projection into each coordinate space is a Cauchy net. □

Theorem 3.8. *Let (X_α,π_α), $\alpha \in I$, be a family of Lagrange stable distal s.d.s. Let $x \in X$, $x = \{x_\alpha\}$, and (X,π) the product s.d.s. Let $\Upsilon_\alpha(x_\alpha)$, $\alpha \in I$, be the product of trajectries. Then $\Pi\Upsilon_\alpha(x_\alpha) = \Upsilon(x)$. Moreover,*

$$\Pi\Omega_\alpha(x_\alpha) = \Omega_x(x). \tag{3.3}$$

Proof. Since each $\Upsilon_\alpha(x_\alpha)$, $\alpha \in I$, is closed and compact,

$$\mathrm{Cl}\,\Pi\Upsilon_\alpha(x_\alpha) = \Pi\,\mathrm{Cl}\,\Upsilon_\alpha(x_\alpha) = \mathrm{Cl}\,\Upsilon(x), \tag{3.4}$$

$$\Pi\Upsilon_\alpha(x_\alpha) = \Upsilon(x). \tag{3.5}$$

Moreover,

$$\Pi\Omega_\alpha(x_\alpha) = \Omega_x(x). \tag{3.6}$$

□

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