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RELATED FIXED POINT THEOREMS ON TWO COMPLETE AND COMPACT METRIC SPACES

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ABSTRACT. A new related fixed point theorem on two complete metric spaces is obtained A generalization is given for two compact metric spaces

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The following related fixed point theorem was proved in [1].

THEOREM 1.1. Let (X, d) and (Y, ρ) be complete metric spaces, let T be a continuous mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities

$$d(STx, STx') \leq c\max\{d(x,x'), d(x, STx), d(x', STx'), \rho(Tx, Tx')\},$$
$$\rho(TSy, TSy') \leq c\max\{\rho(y,y'), \rho(y, TSy), \rho(y', TSy'), d(Sy, Sy')\}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y Further, $Tz = w$ and $Sw = z$.

We now prove the following related fixed point theorem.

THEOREM 1.2. Let (X, d) and (Y, ρ) be complete metric spaces, let T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities

$$\begin{aligned} d(Sy, Sy')d(STx, STx') \leq c\max\{&d(Sy, Sy')\rho(Tx, Tx'), d(x', Sy)\rho(y', Tx), \\ &d(x,x')d(Sy, Sy'), d(Sy, STx)d(Sy', STx')\} \end{aligned} \tag{1}$$

$$\begin{aligned} \rho(Tx, Tx')\rho(TSy, TSy') \leq c\max\{&d(Sy, Sy')\rho(Tx, Tx'), d(x', Sy)\rho(y', Tx), \\ &\rho(y,y')\rho(Tx, Tx'), \rho(Tx, TSy)\rho(Tx', TSy')\} \end{aligned} \tag{2}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. If either T or S is continuous then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

PROOF. Let x be an arbitrary point in X. We define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$(ST)^n x = x_n, \quad T(ST)^{n-1}x = y_n$$

for $n = 1, 2, \ldots$.

We will assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n, otherwise, if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n, we could put $x_n = z$ and $y_n = w$.

Applying inequality (1) we get

$$\begin{aligned} d(x_{n-1}, x_n)d(x_n, x_{n+1}) &= d(Sy_{n-1}, Sy_n)d(STx_{n-1}, STx_n) \\ &\leq c\max\{d(x_{n-1}, x_n)\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)\rho(y_n, y_n), \\ &\qquad [d(x_{n-1}, x_n)]^2, d(x_{n-1}, x_n)d(x_n, x_{n+1})\} \end{aligned} \tag{3}$$

from which it follows that

$$d(x_n, x_{n+1}) \leq c\max\{\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)\}.$$

Applying inequality (2) we get

$$\begin{aligned} [\rho(y_n, y_{n+1})]^2 &= \rho(Tx_{n-1}, Tx_n)\rho(TSy_{n-1}, TSy_n) \\ &\leq c\max\{d(x_{n-1}, x_n)\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)\rho(y_n, y_n), \\ &\qquad \rho(y_{n-1}, y_n)\rho(y_n, y_{n+1}), \rho(y_n, y_n)\rho(y_{n+1}, y_{n+1})\} \end{aligned} \tag{4}$$

from which it follows that

$$\rho(y_n, y_{n+1}) \leq c\max\{d(x_{n-1,}x_n), \rho(y_{n-1}, y_n)\}.$$

It now follows easily by induction that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq c^n \max\{d(x, x_1), \rho(y_1, y_2)\} \\ \rho(y_n, y_{n+1}) &\leq c^{n-1} \max\{d(x, x_1), \rho(y_1, y_2)\} \end{aligned}$$

for $n = 1, 2, \ldots$. Since $c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X with w in Y

Applying inequality (1) we have

$$\begin{aligned} d(Sw, x_n)d(STz, x_{n+1}) &= d(Sw, Sy_n)d(STz, STx_n) \\ &\leq c\max\{d(Sw, x_n)\rho(Tz, y_{n+1}), d(x_n, Sw)\rho(y_n, Tz), d(z, x_n)d(Sw, x_n), \\ &\qquad d(Sw, STz)d(x_n, x_{n+1})\}. \end{aligned}$$

Letting n tend to infinity, we have

$$d(Sw, z)d(STz, z) \leq cd(Sw, z)\rho(Tz, w)$$

and so either

$$Sw = z \tag{5}$$

or

$$d(STz, z) \leq c\rho(Tz, w). \tag{6}$$

Applying inequality (2) we have

$$\begin{aligned} \rho(Tz, y_{n+1})\rho(TSw, y_{n+1}) &= \rho(Tz, Tx_n)\rho(TSw, TSy_n) \\ &\leq c\max\{d(Sw, x_n)\rho(Tz, y_{n+1}), d(x_n, Sw)\rho(y_n, Tz), \rho(w, y_n)\rho(Tz, y_{n+1}), \\ &\qquad \rho(Tz, TSw)\rho(y_{n+1}, y_{n+1})\}. \end{aligned}$$

Letting n tend to infinity, we have

$$\rho(Tz, w)\rho(TSw, w) \leq cd(z, Sw)\rho(Tz, w)$$

and so either

$$Tz = w \tag{7}$$

or

$$\rho(TSw, w) \leq cd(z, Sw). \tag{8}$$

If T is continuous, then

$$w = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} Tx_n = Tz.$$

If inequality (6) holds, then it implies that

$$z = STz = Sw,$$

and so equation (5) will necessarily hold. We then have

$$TSw = Tz = w.$$

If S is continuous, then

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Sy_n = Sw.$$

If inequality (8) holds, then it implies that

$$w = TSw = Tz$$

and so equation (7) will necessarily hold. We then have

$$STz = Sw = z.$$

To prove uniqueness, suppose that ST has a second fixed point z' and TS has a second fixed point w' Then applying inequality (1) we have

$$[d(z,z')]^2 = [d(STz, STz')]^2 \leq c\max\left\{d(z,z')\rho(Tz,Tz'), [d(z,z')]^2\right\},$$

which implies that

$$d(z,z') \leq c\rho(Tz,Tz'). \tag{9}$$

Further, applying inequality (2) we have

$$[\rho(Tz,Tz')]^2 = \rho(Tz,Tz')\rho(TSTz,TSTz') \leq c\max\left\{d(z,z')\rho(Tz,Tz'), [\rho(Tz,Tz')]^2\right\},$$

which implies that

$$\rho(Tz,Tz') \leq cd(z,z'). \tag{10}$$

It now follows from inequalities (9) and (10) that

$$d(z,z') \leq c\rho(Tz',w) \leq c^2 d(z,z')$$

and so $z = z'$ since $c < 1$, proving the uniqueness of the fixed point z of ST.

Now $TSw' = w'$ implies that $STSw' = Sw'$ and so $Sw' = z$ Thus

$$w = TSw = TSz = TSw' = w',$$

proving that w is the unique fixed point of TS. This completes the proof of the theorem.

COROLLARY 1.3. Let (X, d) be a complete metric space and let T be a continuous mapping of X onto X satisfying the inequality

$$\begin{aligned} d(Ty,Ty')d(T^2x,T^2x') \leq c\max\{&d(Ty,Ty')d(tx,Tx'), d(x',Ty)d(y',Tx), \\ &d(x,x')d(Ty,Ty'), d(Ty,T^2x)d(Ty',T^2x')\} \end{aligned}$$

for all x, x', y, y' in X, where $0 \leq c < 1$. Then T has a unique fixed point z in X.

PROOF. It follows from the theorem with $(X,d) = (Y,\rho)$ and $S = T$ that T^2 has a unique fixed point z. Then $T^2(Tz) = T(T^2z) = Tz$ and so we see that Tz is also a fixed point of T^2 Since the fixed point is unique, we must have $Tz = z$.

We now prove a fixed point theorem for compact metric spaces.

THEOREM 1.4. Let (X,d) and (Y,ρ) be compact metric spaces, let T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities

$$d(Sy,Sy')d(STx,STx') < \max\{d(Sy,Sy')\rho(Tx,Tx'), d(x',Sy)\rho(y',Tx), d(x,x')d(Sy,Sy'), d(Sy,STx)d(Sy',STx')\} \quad (11)$$

$$\rho(Tx,Tx')\rho(TSy,TSy') < \max\{d(Sy,Sy')\rho(Tx,Tx'), d(x',Sy)\rho(y',Tx), \rho(y,y')\rho(Tx,Tx'), \rho(Tx,TSy)\rho(Tx',TSy')\} \quad (12)$$

for all x, x' in X and y, y' in Y. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

PROOF. Suppose first of all that there exists no $a < 1$ such that

$$d(Sy,STSy)d(STx,STSTx) \leq a\max\{d(Sy,STSy)\rho(Tx,TSTx), d(STx,Sy)\rho(TSy,Tx), d(x,STx)d(Sy,STSy), d(Sy,STx)d(STSy,STSTx)\} \quad (13)$$

for all x in X and y in Y. Then there exist sequences $\{x_n\}$ in X and $\{y_n\}$ in Y such that

$$d(Sy_n,STSy_n)d(STx_n,STSTx_n) > (1-n^{-1})\max\{d(Sy_n,STSy_n)\rho(Tx_n,TSTx_n), d(STx_n,Sy_n)\rho(TSy_n,Tx_n), d(x_n,STx_n)d(Sy_n,STSy_n), d(Sy_n,STx_n)d(STSy_n,STSTx_n)\} \quad (14)$$

for $n = 1, 2, \ldots$. Since X and Y are compact, and by relabelling if necessary, we may suppose that the sequence $\{x_n\}$ converges to z' in X and the sequence $\{y_n\}$ converges to w' in Y. Letting n tend to infinity in inequality (16), it follows that

$$d(Sw',STSw')d(STz',STSTz') \geq \max\{d(Sw',STSw')\rho(Tz',TSTz'), d(STz',Sw')\rho(TSw',Tz'), d(z',STz')d(Sw',STSw'), d(Sw',STz')d(STSw',STSTz')\}. \quad (15)$$

This is only possible if the right hand side of inequality (17) is zero. It follows that either $STz' = STSTz'$ or $Sw' = STSw'$.

If $STz' = STSTz'$, then $STz' = z$ is a fixed point of ST and it follows that $Tz = w$ is a fixed point of TS

If $Sw' = STSw'$, then $Sw' = z$ is a fixed point of ST and it again follows that $Tz = w$ is a fixed point of TS.

Now suppose that there exists no $b < 1$ such that

$$\rho(Tx,TSTx)\rho(TSy,TSTSy) \leq b\max\{d(Sy,STSy)\rho(Tx,TSTx), d(STx,Sy)\rho(TSy,Tx), \rho(y,TSy)\rho(Tx,TSTx), \rho(Tx,TSy)\rho(TSTx,TSTSy)\} \quad (16)$$

for all x in X and y in Y. Then it follows similarly that ST has a fixed point z and TS has a fixed point w.

Finally, suppose that there exist $a, b < 1$ satisfying inequalities (15) and (18) Then with $c = \max\{a,b\}$, it follows that if the sequences $\{x_n\}$ and $\{y_n\}$ are defined as in the proof of Theorem 2, inequalities (3) and (4) will hold. It then follows as in the proof of Theorem 2 that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y. Since ST and TS are continuous, it now follows that z is a fixed point of ST and w is a fixed point of TS.

To prove uniqueness, suppose that ST has a second distinct common fixed point z' Then applying inequality (13) we have

$$[d(z,z')]^2 = [d(STz,STz')]^2 < \max\left\{d(z,z')\rho(Tz,Tz'),[d(z,z')]^2\right\},$$

which implies that

$$d(z,z') < \rho(Tz,Tz'). \tag{17}$$

Further, applying inequality (14) we have

$$[\rho(Tz,Tz')]^2 = \rho(Tz,Tz')\rho(TSTz,TSTz') < \max\left\{d(z,z')\rho(Tz,Tz'),[\rho(Tz,Tz')]^2\right\},$$

which that

$$\rho(Tz,Tz') < d(z,z'). \tag{18}$$

It now follows from inequalities (19) and (20) that

$$d(z,z') < \rho(Tz',Tz) < d(z,z'),$$

a contradiction and so the fixed point z must be unique.

The uniqueness of w is proved similarly. This completes the proof of the theorem

COROLLARY 1.5. Let (X,d) be a compact metric space and let T be a continuous mapping of X into X satisfying the inequality

$$d(Ty,Ty')d(T^2x,T^2x') < \max\{d(Ty,Ty')d(Tx,Tx'),d(x',Ty)d(y',Tx), d(x,x')d(Ty,Ty'),d(Ty,T^2x)d(Ty',T^2x')\}$$

for all x,x',y,y' in X for which the right hand side of the inequality is positive. Then T has a unique fixed point z in X.

REFERENCES

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