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WEIGHTED NORM INEQUALITIES FOR THE $\mathcal{H}$-TRANSFORMATION

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ABSTRACT. In this paper we establish weighted norm inequalities for an integral transform whose kernel is a Fox function.

KEY WORDS AND PHRASES: Fox function, integral transformation, weighted norm inequalities.
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1. INTRODUCTION

The transformations we will investigate in this paper are the ones called $\mathcal{H}$-transformations. These transformations are defined by

$$\mathcal{H}(f)(x) = \int_0^\infty \mathfrak{H}_{p,q}^{m,n}\left(xt\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right)f(t)dt, \quad f\in C_0, \tag{1}$$

where $\mathfrak{H}_{p,q}^{m,n}$ denotes the Fox function ([9]) and as usual C_0 represents the class of complex valued functions on $(0,\infty)$ which are continuous and compactly supported. In the last years, the $\mathcal{H}$-transformation has been studied by several authors (see [6], [7], [14] and [18]) and it reduces to important integral transforms (Laplace, Hankel, Meijer, Hardy, ...) by specifying the involved parameters. In a previous paper [5] the authors (simultaneously to A. A. Kilbas, M. Saigo and S. A. Shlapakov [15], [16] and [17]), investigated the behavior of transformation (1) in certain weighted L_p spaces introduced by P. G. Rooney [21].

Weighted Fourier transform norm inequalities have been exhaustively studied (see [2], [3], [4], [10], [13], [20], amongst others). Inspired by the above works our aim in this paper is to give conditions on a positive Borel measure Ω on $(0,\infty)$, and on a measurable nonnegative function v on $(0,\infty)$ which are sufficient in order that the inequality

$$\left\{\int_0^\infty |\mathcal{H}(f)(x)|^s d\Omega(x)\right\}^{\frac{1}{s}} \le C\left\{\int_0^\infty v(x)|f(x)|^r dx\right\}^{\frac{1}{r}}, \quad f\in C_0, \tag{2}$$

holds where $1\le r,\ s\le\infty$ and C is a suitable positive constant. Also we analyze some special cases of (2). Moreover we establish some properties on Ω and v that are implied by (2).

We now introduce some notations that will be used throughout this paper. We need consider some parameters related to the $\mathfrak{H}$-function. Let $m,\ n,\ p,\ q\in\mathbb{N}$ being $0\le m\le s$, $0\le n\le r$ and $r+s\ge 1$. Assume that a_j, $j=1,\dots,r$ and b_j, $j=1,\dots,s$, are real numbers and α_j, $j=1,\dots,r$, and β_j, $j=1,\dots,s$, are positive real numbers. We define

$$\alpha = \begin{cases}\max\left\{-\frac{b_j}{\beta_j},\ j=1,\dots,m\right\} & ,\ \text{for}\quad m>0\\ -\infty & ,\ \text{for}\quad m=0\end{cases}$$

$$\beta = \begin{cases} \min\left\{\frac{1-a_j}{\alpha_j},\ j=1,\dots,n\right\} & , \text{ for } n>0 \\ +\infty & , \text{ for } n=0 \end{cases}$$

$$\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$$

$$\nu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$$

$$\xi = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j$$

$$\eta = \prod_{j=1}^{p}\alpha_j^{-\alpha_j}\prod_{j=1}^{q}\beta_j^{\beta_j}.$$

Also we remember a result that was established in [5] and that will be very useful in the sequel

THEOREM A (Corollary 1 of [5]). If $\alpha < \gamma < \beta$ and if either

(a) $\xi > 0$ or

(b) $\xi = 0, \mu \neq 0$ and $\nu + \mu\gamma - \frac{1}{2}(q-p) < -1$

holds, then the function $\mathfrak{H}$ is defined by

$$\mathfrak{H}(x) = \mathfrak{H}_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} x^{-s}h(s)ds \tag{3}$$

for every $x > 0$, where

$$h(s) = h_{p,q}^{m,n}\left(\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix} \,\middle|\, s\right) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}.$$

Here the empty products as usual are understood as 1 Moreover

$$|\mathfrak{H}(x)| \le C_\gamma x^{-\gamma} \tag{4}$$

for every $x>0$, C_γ being a positive constant. Furthermore if $\alpha<\gamma<\beta$, $\xi=\mu=0$ and $\nu-\frac{1}{2}(q-p)<-1$ then (3) and (4) hold for every $x>0$ except for $x=\eta$. ■

In view of the above considerations we will assume in the sequel that our parameters satisfy one of the following four conditions, namely

(i) $\xi > 0$

(ii) $\xi = 0, \mu > 0$ and $\beta < -\frac{1}{\mu}\left[\nu + 1 + \frac{1}{2}(p-q)\right]$

(iii) $\xi = 0, \mu < 0$ and $\alpha > -\frac{1}{\mu}\left[\nu + 1 + \frac{1}{2}(p-q)\right]$

(iv) $\xi = 0, \mu = 0$ and $\nu + \frac{1}{2}(p-q) < -1$

Throughout this paper for every $1 \le r \le \infty$ we denote by r' the conjugate of r (that is, $r' = \frac{r}{r-1}$) Also when some of the exponents in our weighted inequality are infinite said inequality must be understood in the obvious form.

2. WEIGHTED NORM INEQUALITIES FOR THE $\mathcal{H}$-TRANSFORM

We shall firstly give sufficient conditions on a positive function v on $(0,\infty)$ and on a positive Borel measure Ω on $(0,\infty)$ in order that the inequality

$$\left\{\int_0^\infty |\mathcal{H}(f)(x)|^s d\Omega\right\}^{\frac{1}{s}} \le C\left\{\int_0^\infty v(x)|f(x)|^r dx\right\}^{\frac{1}{r}}, \quad f \in C_0,$$

holds, where $1 \le r,\ s < \infty$ and C denotes a certain positive constant. When either $r=\infty$ or $s=\infty$ inequality (2) takes the obvious form. The employed procedure here is inspired by the one used by J. J Benedetto and H. P. Heinig ([2] and [3]) in their studies about Fourier transforms

PROPOSITION 1. Assume that Ω is a positive Borel measure on $(0,\infty)$ and that v is a nonnegative measurable function on $(0,\infty)$ belonging to $L^1_{loc}(0,\infty)$.

If $1 \le r \le s \le \infty$ and there exist $\alpha < a$, $b < \beta$ such that

$$B_1 = \sup_{x>0}\left\{\int_0^x t^{-as}d\Omega(t)\right\}^{\frac{1}{s}}\left\{\int_0^{\frac{1}{x}} t^{-ar'}v(t)^{1-r'}dt\right\}^{\frac{1}{r'}} < \infty$$

and

$$B_2 = \sup_{x>0}\left\{\int_x^\infty t^{-bs}d\Omega(t)\right\}^{\frac{1}{s}}\left\{\int_{\frac{1}{x}}^\infty t^{-br'}v(t)^{1-r'}dt\right\}^{\frac{1}{r'}} < \infty,$$

then (2) holds for every $f \in C_0$.

Also if $1 \le s < r < \infty$ and there exist $\alpha < a$, $b < \beta$ such that

$$B_1' = \int_0^\infty\left\{\int_0^{\frac{1}{x}} z^{-as}d\Omega(z)\right\}^{\frac{h}{s}}\left\{\int_0^x z^{-ar}v(z)^{1-r'}dz\right\}^{\frac{h}{s'}} x^{-ar'}v(x)^{1-r'}dx < \infty$$

and

$$B_2' = \int_0^\infty\left\{\int_{\frac{1}{x}}^\infty z^{-bs}d\Omega(z)\right\}^{\frac{h}{s}}\left\{\int_x^\infty z^{-br}v(z)^{1-r'}dz\right\}^{\frac{h}{s'}} x^{-br'}v(x)^{1-r'}dx < \infty$$

where $\frac{1}{h} = \frac{1}{s} - \frac{1}{r}$, then (2) holds for every $f \in C_0$

PROOF. First we consider the case $1 < r \le s < \infty$

Let $f \in C_0$. By virtue of (4) for every $\alpha < a$, $b < \beta$ there exists $C_{a,b} > 0$ such that

$$|\mathcal{H}(f)(x)| \le C_{a,b}\left\{\int_0^{\frac{1}{x}}(xt)^{-a}|f(t)|dt + \int_{\frac{1}{x}}^\infty (xt)^{-b}|f(t)|dt\right\}, \quad x > 0.$$

By using the Minkowski inequality we obtain

$$\left\{\int_0^\infty |\mathcal{H}(f)(x)|^s d\Omega(x)\right\}^{\frac{1}{s}} \le C_{a,b}\left[\left\{\int_0^\infty\left\{\int_0^{\frac{1}{x}} t^{-a}|f(t)|dt\right\}^s x^{-as}d\Omega(x)\right\}^{\frac{1}{s}} + \left\{\int_0^\infty\left\{\int_{\frac{1}{x}}^\infty t^{-b}|f(t)|dt\right\}^s x^{-bs}d\Omega(x)\right\}^{\frac{1}{s}}\right] = C_{a,b}(J_1 + J_2). \tag{5}$$

A straightforward change of variable leads to

$$J_1 = \left\{\int_0^\infty\left\{\int_0^{\frac{1}{x}} t^{-a}|f(t)|dt\right\}^s x^{-as}d\Omega(x)\right\}^{\frac{1}{s}} = \left\{\int_0^\infty\left\{\int_x^\infty h(u)du\right\}^s x^{-as}d\Omega(x)\right\}^{\frac{1}{s}}$$

where $h(u) = u^{a-2}\left|f\left(\frac{1}{u}\right)\right|$, $u > 0$

Therefore from Theorem 4 (1.3.1) [19] one infers

$$J_1 \le C_1\left\{\int_0^\infty h(t)^r v_1(t)dt\right\}^{\frac{1}{r}} = C_1\left\{\int_0^\infty |f(t)|^r v(t)dt\right\}^{\frac{1}{r}} \tag{6}$$

with $C_1 > 0$ and $v_1(t) = v\left(\frac{1}{t}\right)t^{2r-2-ar}$, $t > 0$, provided that $B_1 < \infty$.

On the other hand, we have

$$J_2 = \left\{\int_0^\infty\left\{\int_{\frac{1}{x}}^\infty t^{-b}|f(t)|dt\right\}^s x^{-bs}d\Omega(x)\right\}^{\frac{1}{s}} = \left\{\int_0^\infty\left\{\int_0^x g(t)dt\right\}^s x^{-bs}d\Omega(x)\right\}^{\frac{1}{s}}$$

where $g(t) = t^{b-2}\left|f\left(\frac{1}{t}\right)\right|$, $t > 0$. Then by invoking Theorem 1 (1.3.1) [19] it follows

$$J_2 \leq C_2 \left\{ \int_0^\infty g(t)^r v_2(t) dt \right\}^{\frac{1}{r}} = C_2 \left\{ \int_0^\infty |f(t)|^r v(t) dt \right\}^{\frac{1}{r}} \tag{7}$$

where $v_2(t) = v\left(\frac{1}{t}\right) t^{2r-2-br}$, $t > 0$, when $B_2 < \infty$

By combining (5), (6) and (7) we can immediately deduce (2)

When either $r = \infty$ or $s = \infty$ the proof can be made in a similar way

In the case $1 \leq s < r \leq \infty$ (2) can be established as the above case by invoking the Theorem 2 (1 3.2) [19]. ■

In the sequel we present some special cases of inequality (2). The following results are related to known weighted norm inequalities for other integral transforms due to P Heywood and P G Rooney ([11], [12]), N E. Aguilera and E O Harboure [1], B Muckenhoupt [20] and S A Emara and H P Heinig [8]

A generalization of Theorem 2.1 of [12] is the following

PROPOSITION 2. Let $\alpha < 1 - \eta < \beta$ and $1 \leq s \leq \infty$. Then

$$\left\{ \int_0^\infty |x^{1-\eta} \mathcal{H}(f)(x)|^s \frac{dx}{x} \right\}^{\frac{1}{s}} \leq C \int_0^\infty x^{\eta-1} |f(x)| dx, \quad f \in C_0, \tag{8}$$

for certain $C > 0$.

PROOF. This result, that also can be proved in a similar way to Theorem 2 1 of [12], is a consequence of Proposition 1 In effect if $1 - \eta < a < \beta$ we have

$$\left\{ \int_x^\infty t^{s(1-\eta-a)-1} dt \right\}^{\frac{1}{s}} \left\| t^{-a-\eta+1} \chi_{[\frac{1}{x},\infty)}(t) \right\|_{\infty, t^{\eta-1}dt} = (s(1-\eta-a))^{\frac{1}{s}}, \ x > 0 \text{ and } 1 \leq s < \infty$$

where $\| \|_{\infty, t^{\eta-1}dt}$ denotes the essential supremum respect to the measure $t^{\eta-1}dt$ and χ_E represents as usual the characteristic function associated to the measure set E.

In a similar way we can see that if $\alpha < b < 1 - \eta$ and $1 \leq s < \infty$. Then

$$\sup_{x>0} \left\{ \int_0^x t^{s(1-\eta-b)-1} dt \right\}^{\frac{1}{s}} \left\| t^{-b-\eta+1} \chi_{(0,\frac{1}{x})}(t) \right\|_{\infty, t^{\eta-1}dt} < \infty.$$

Hence according to Proposition 1 (8) holds for every $1 \leq s < \infty$

When $s = \infty$ the result can be proved analogously ■

We now investigate the inequality (2) when $d\Omega = u(x)dx$ being u is a measurable nonnegative function on $(0, \infty)$, $v = 1$ and $r = s$.

PROPOSITION 3. Let $1 \leq r \leq 2$, $\alpha < 0$ and $\frac{1}{2} < \beta$. If u is a locally integrable nonnegative function on $(0, \infty)$ for which there exists a constant $M > 0$ such that for every measurable set E $\int_E u(x)dx \leq M|E|^{r-1}$ is satisfied, then

$$\int_0^\infty u(x) |\mathcal{H}(f)(x)|^r dx \leq C \int_0^\infty |f(x)|^r dx, \quad f \in C_0, \tag{9}$$

for a certain $C > 0$.

PROOF. Our proof is essentially the same one given in Theorem 1 of [1]. Let $1 < r < 2$ we define the operator

$$(Tf)(x) = \begin{cases} u^{-\frac{b}{2}}(x) \mathcal{H}(f)(x) & , \text{ if } u(x) \neq 0 \\ 0 & , \text{ if } u(x) = 0 \end{cases}, \ f \in C_0$$

where $b = \frac{2}{2-r}$

Since $\alpha < 0 < \beta$, then by (4) $\mathfrak{H}$ is a bounded function on $(0, \infty)$ Hence, according to Theorem 2 of [1] we obtain

$$\int_{\{x\,|Tf(x)|>\lambda\}} u^b(x)dx \le \int_{\left\{x\, u^{\frac{b}{2}}(x)\le\frac{C_1}{\lambda}\int_0^\infty|f(x)|dx\right\}} u^b(x)dx \le \frac{C_2}{\lambda}\int_0^\infty |f(x)|dx$$

where C_i, $i=1,2$, are positive constants Thus T is a weak type (1,1) operator, on measure spaces $((0,\infty),dx)$ and $((0,\infty),u^b(x)dx)$

Moreover by virtue of Proposition 3 of [5] $\mathcal{H}$ is a bounded operator from $L_2(0,\infty)$ into itself because $\alpha<\frac{1}{2}<\beta$ Therefore

$$\int_0^\infty |Tf(x)|^2u^b(x)dx \le C\int_0^\infty |f(x)|^2dx$$

with $C>0$, and T is a strong type (2,2) operator between the spaces under consideration

Now by the Marcinkiewicz interpolation theorem we obtain the desired result for $1<r<2$

Finally, note that if $r=1$ then $\int_0^\infty u(x)dx<\infty$ and (9) holds trivially because $\alpha<0<\beta$ and by (4) Moreover if $r=2$ then u is bounded function on $(0,\infty)$ and since $\alpha<\frac{1}{2}<\beta$ (4) leads to (9) ■

By proceeding as in §7 of [1] we can deduce from Proposition 3 conditions for a function v that imply inequality (2) holds when Ω is the Lebesgue measure on $(0,\infty)$ and $r=s$

We now give conditions for u that are deduced from (9)

PROPOSITION 4. Let $1\le r<\infty$. Assume that one of the following two conditions is satisfied·

(i) There exists $j_0\in\mathbb{N}$, $1\le j_0\le p$, such that $-\frac{a_{j_0}}{\alpha_{j_0}}>\max\{\alpha,1-\frac{1}{r}\}$ and

$$\inf_{0<x<1}\left|\mathfrak{H}^{m,n}_{p,q}\left(\begin{array}{c}(a_1',\alpha_1),\dots,(a_p',\alpha_p)\\ \\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{array}\middle|\, x\right)\right| = K_1>0 \tag{10}$$

where $a_{j_0}'=a_{j_0}+1$ and $a_j'=a_j$, $1\le j\le p$, $j\ne j_0$.

(ii) There exists $j_0\in\mathbb{N}$, $1\le j_0\le q$, such that $\frac{1-b_{j_0}}{\beta_{j_0}}>\max\{\beta,1-\frac{1}{r}\}$ and

$$\inf_{0<x<1}\left|\mathfrak{H}^{m,n}_{p,q}\left(\begin{array}{c}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ \\ (b_1',\beta_1),\dots,(b_q',\beta_q)\end{array}\middle|\, x\right)\right| = K_2>0$$

where $b_{j_0}'=b_{j_0}-1$ and $b_j'=b_j$, $1\le j\le q$, $j\ne j_0$

Then there exists a positive constant L such that

$$\int_0^a u(x)dx\le Ca^{1-r},\quad \text{holds for every}\quad a>0,. \tag{11}$$

provided that (9) holds.

PROOF. We will establish the result when (i) is satisfied with $n+1\le j_0\le p$ The proof in the other cases can be made in a similar way

It is easy to see that

$$\frac{d}{dx}\left[x^{-a_{j_0}}\mathfrak{H}^{m,n}_{p,q}\left(\begin{array}{c}(a_1',\alpha_1),\dots,(a_p',\alpha_p)\\ \\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{array}\middle|\, x^{\alpha_{j_0}}\right)\right]$$

$$= -x^{-(a_{j_0}+1)}\mathfrak{H}^{m,n}_{p,q}\left(\begin{array}{c}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),..,(b_q,\beta_q)\end{array}\middle|\, x^{\alpha_{j_0}}\right),\quad x>0 \tag{12}$$

being $a_{j_0}'=a_{j_0}+1$ and $a_j'=a_j$, $j=1,\dots,p$, $j\ne j_0$

For $a>0$ fixed, define

$$f_a(x) = \begin{cases} x^{-\frac{a_{j0}+a_{j0}}{\alpha_{j0}}} & , \ 0 < x \le \frac{1}{a} \\ 0 & , \ x > \frac{1}{a} . \end{cases}$$

By using (12) we can write

$$(\mathcal{H}f_a)(x) = \int_0^{\frac{1}{a}} t^{-\frac{a_{j0}+a_{j0}}{\alpha_{j0}}} \mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ \\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\middle| tx\right)dt$$

$$= \alpha_{j0}\, x^{\frac{a_{j0}}{\alpha_{j0}}} \int_0^{(\frac{x}{a})^{\frac{1}{\alpha_{j0}}}} \frac{d}{dv}\left[- v^{-a_{j0}} \mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1',\alpha_1),\dots,(a_p',\alpha_p)\\ \\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\middle| v^{\alpha_{j0}}\right)\right]dv$$

$$= -\alpha_{j0}\, a^{\frac{a_{j0}}{\alpha_{j0}}} \mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1',\alpha_1),\dots,(a_p',\alpha_p)\\ \\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\middle| \frac{x}{a}\right)$$

because

$$\lim_{v\to 0^+} v^{-a_{j0}} \mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1',\alpha_1),\dots,(a_p',\alpha_p)\\ \\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\middle| v^{\alpha_{j0}}\right) = 0. \tag{13}$$

Since $\alpha < -\frac{a_{j0}}{\alpha_{j0}}$ to see (13) it is sufficient to take into account (4) Hence, by virtue of (10)

$$\int_0^a u(x)dx \le K_1^{-r} \int_0^a u(x)\left|\mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1',\alpha_1),\dots,(a_p',\alpha_p)\\ \\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\middle| \frac{x}{a}\right)\right|^r dx$$
$$= \left(K_1\alpha_{j0}a^{\frac{a_{j0}}{\alpha_{j0}}}\right)^{-r} \int_0^a u(x)|\mathcal{H}(f_a)(x)|^r dx.$$

Similarly from (9) one deduces

$$\int_0^a u(x)dx \le C\left(K_2\alpha_{j0}a^{\frac{a_{j0}}{\alpha_{j0}}}\right)^{-r} \int_0^{\frac{1}{a}} x^{-\frac{(a_{j0}+a_{j0})r}{\alpha_{j0}}}dx = C(K_2\alpha_{j0})^{-r}a^{r-1}.$$

Thus the proof is finished ■

Note that if $r = 1$ (11) implies that u is integrable over $(0,\infty)$ When $r = 2$, u is bounded on $(0,\infty)$ provided that (11) holds Also if $r > 2$ and (11) is satisfied then $u = 0$, a.e $(0,\infty)$

B Muckenhoupt [20] investigated sufficient conditions for the measurable functions u and v that guarantee that the inequality (2), with $d\Omega(x) = u(x)dx$, holds when the $\mathcal{H}$-transformation is replaced by the Fourier transform Also he studied the converse problem proving that, in some cases, the above cited conditions are necessary Later P Heywood and P.G Rooney [11] analyzed weighted norm inequalities for the Hankel transformation in a similar way We now use an analogous procedure to extend the results in [11] to the $\mathcal{H}$-transformation (note that this transform reduces to the Hankel transformation when the parameters take on suitable values)

It will be used to recall some definitions of [11]. For every $\eta \in \mathbb{R}$, $1 \le r < \infty$ and for every v nonnegative measurable function on $(0,\infty)$, the space $\mathcal{L}_{\eta,v,r}$ is constituted by all those measurable functions f on $(0,\infty)$ such that

$$\|f\|_{\eta,v,r} = \left\{\int_0^\infty |x^\eta v(x)f(x)|^r \frac{dx}{x}\right\}^{\frac{1}{r}} < \infty.$$

The space $\mathcal{L}_{\eta,v,r}$ is a Banach space when it is endowed with the topology associated to the norm $\| \, \|_{\eta,v,r}$ Also, if u and v are nonnegative measurable functions on $(0,\infty)$ we say that $(u,v) \in A(r,s,\delta)$ with $\delta \in \mathbb{R}$ and $1 < r,\ s < \infty$ when there exist positive constants B and C for which·

$$\left[\int_{u(x)>B\omega} \{x^{\delta}u(x)\}^{s} \frac{dx}{x}\right]^{\frac{1}{s}} \left[\int_{v(x)<\omega} \left\{\frac{x^{\delta}}{v(x)}\right\}^{r'} \frac{dx}{x}\right]^{\frac{1}{r'}} \leq C$$

for every $\omega > 0$

In Propositions 4-8 [5] we established some conditions on the parameters involved in the $\mathfrak{H}$-function in order that the $\mathcal{H}$-transformation can be extended to the space $\mathcal{L}_{\eta,r}$ as a bounded operator from $\mathcal{L}_{\eta,r}$ into $\mathcal{L}_{1-\eta,s}$ In the following Proposition the above results are improved We prove that under suitable conditions the $\mathcal{H}$-transformation can be extended to $\mathcal{L}_{\eta,v,r}$ as a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$ We only stated the result corresponding Proposition 8 of [5] although similar results corresponding to Propositions 4-7 of [5] can be established.

PROPOSITION 5. Let $1 < r \leq s < \infty$, $\xi > 0$ and $\alpha < 1 - \eta < \beta$ Suppose that $(u, v) \in A(r, s, 1 - \eta - \sigma)$, with $\alpha < \sigma < \beta$. Then the $\mathcal{H}$-transformation can be extended to $\mathcal{L}_{\eta,v,r}$ as a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$

PROOF. This result can be proved as Theorem 1 of [11]. It is sufficient to take into account that $|\mathfrak{H}(x)| \leq C_\sigma x^{-\sigma}$, $x > 0$, with $\alpha < \sigma < \beta$ and for certain $C_\sigma > 0$ By using this inequality instead of (2 5) of [11] and Proposition 8 of [5] instead of Lemma 1 of [11] the proof of our result follows as the one of Theorem 1 of [11]

On the other hand this result can be proved also by invoking Proposition 1 because if $(u, v) \in A(r, s, 1 - \eta - \sigma)$ being $\alpha < 1 - \eta, \sigma < \beta$ then the conditions $B_i < \infty$, $i = 1, 2$, in Proposition 1 are satisfied when $d\Omega$ and v are replaced by $x^{(1-\eta-\sigma)s-1}u(x)^s dx$ and $x^{(1-\eta-\sigma)r-1}v(x)^r$, respectively. ■

Our next objective is to establish a partial converse to Proposition 5

LEMMA 1. Let $1 < r \leq s < \infty$ and $0 < \eta < 1$. Assume that u and v are nonnegative measurable functions on $(0, \infty)$ such that u is decreasing, $\lim_{x\to\infty} u(x) = 0$ and v is increasing. Also suppose that

$$\inf_{0<x<1} \mathfrak{H}_{p,q}^{m,n}\left(\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ \\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\middle| x\right) = C_1 > 0. \tag{14}$$

Then there exists a positive constant $B > 0$ for which

$$\sup\{x : u(x) > B\omega\} \cdot \sup\{x : v(x) < \omega\} \leq 1,$$

for every $\omega > 0$, provided that $\mathcal{H}$ is a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$

PROOF. This result will be proved when we see that if

$$\sup\{x : u(x) > B\omega\} \cdot \sup\{x : v(x) < \omega\} > 1,$$

for some $\omega > 0$, then B is less than a positive constant only depending on r, s and η, the lemma then holds with any larger value of B

Let B, $\omega > 0$. For simplicity denote

$$M = M(B, \omega) = \sup\{x : u(x) > B\omega\}.$$

Since $\lim_{x\to\infty} u(x) = 0$, $M(B, \omega) < \infty$. Assume now $M(B, \omega) \cdot \sup\{x : v(x) < \omega\} > 1$ and define the function

$$f(x) = \begin{cases} 1 & , \text{ if } 0 < x < \frac{1}{M} \\ 0 & , \text{ if } x > \frac{1}{M}. \end{cases}$$

It is clear that $f \in \mathcal{L}_{\eta,v,r}$ and one has

$$\|f\|_{\eta,v,r} = \left\{\int_0^{\frac{1}{M}} |x^\eta v(x)|^r \frac{dx}{x}\right\}^{\frac{1}{r}} \leq \left\{\int_0^{\frac{1}{M}} \omega^r x^{\eta r-1} dx\right\}^{\frac{1}{r}} = \frac{\omega}{M^\eta (\eta r)^{\frac{1}{r}}} \tag{15}$$

because $v(x) \leq \omega$, for every $x \in \left(0, \frac{1}{M}\right)$. Since $\mathcal{H}f \in \mathcal{L}_{1-\eta,u,s}$ then by virtue of (14) and since

$u(x) \geq \omega B$, for every $x \in (0, M)$ we can write

$$\|\mathcal{H}f\|_{1-\eta,u,s} \geq \left\{\int_0^M \left|x^{1-\eta}u(x)\int_0^{\frac{1}{M}} \mathfrak{H}(xt)dt\right|^s \frac{dx}{x}\right\}^{\frac{1}{s}}$$

$$\geq \frac{C_1}{M}\left\{\int_0^M \left|x^{1-\eta}u(x)\right|^s \frac{dx}{x}\right\}^{\frac{1}{s}} > \frac{C_1 B\omega}{M}\left\{\int_0^M x^{(1-\eta)s-1}dx\right\}^{\frac{1}{s}} = \frac{C_1 B\omega}{M^{\eta}(s(1-\eta))^{\frac{1}{s}}} \tag{16}$$

for a suitable $K > 0$.

Moreover for a certain $C > 0$

$$\|\mathcal{H}f\|_{1-\eta,u,s} \leq C\|f\|_{\eta,v,r}. \tag{17}$$

By combining (15), (16) and (17) one concludes that

$$B \leq \frac{C[s(1-\eta)]^{\frac{1}{s}}}{C_1(\eta r)^{\frac{1}{r}}}.$$

Note that the constant in the right hand side of the last inequality is positive since $0 < \eta < 1$ Thus the proof is complete. ■

PROPOSITION 6. Let $1 < r \leq s < \infty$ and $0 < \eta < 1$ Assume that u and v are measurable nonnegative functions on $(0, \infty)$ such that u is decreasing, $\lim_{x\to\infty} u(x) = 0$, v is increasing and $\int_{v(x)<\omega} \left\{\frac{x^{1-\eta}}{v(x)}\right\}^{r'} \frac{dx}{x} < \infty$, for every $\omega > 0$. Then $(u, v) \in A(r, s, 1-\eta)$ provided that $\mathcal{H}$ is a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$ and (14) holds

PROOF. We define for every $\omega > 0$ the function

$$f_\omega(x) = \begin{cases} x^{\frac{\eta r-1}{1-r}} v(x)^{-r'} & \text{if } 0 < v(x) < \omega \\ 0 & \text{otherwise} \end{cases}$$

It is not hard to show that

$$\|f_\omega\|_{\eta,v,r} = \left\{\int_{v(x)<\omega} \left\{\frac{x^{1-\eta}}{v(x)}\right\}^{r'} \frac{dx}{x}\right\}^{\frac{1}{r}}$$

and $f_\omega \in \mathcal{L}_{\eta,v,r}$, for every $\omega > 0$

But since $\mathcal{H}$ is a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$, there exists a positive constant $C > 0$ such that

$$\|\mathcal{H}f_\omega\|_{1-\eta,u,s} \leq C\|f_\omega\|_{\eta,v,r}, \quad \omega > 0.$$

Hence

$$\left\{\int_{u(x)>B\omega} \left|x^{1-\eta}u(x)\mathcal{H}(f_\omega)(x)\right|^s \frac{dx}{x}\right\}^{\frac{1}{s}} \leq \|\mathcal{H}f_\omega\|_{1-\eta,u,s} \leq C\left\{\int_{v(x)<\omega} \left\{\frac{x^{1-\eta}}{v(x)}\right\}^{r'} \frac{dx}{x}\right\}^{\frac{1}{r}}, \ \omega > 0 \tag{18}$$

where B denotes the constant given in Lemma 1.

Moreover, according to Lemma 1, if $\omega, x, t > 0$, $u(x) > B\omega$ and $v(t) < \omega$, then

$$xt \leq \sup\{x : u(x) > B\omega\}\sup\{t : v(t) < \omega\} \leq 1.$$

Hence (14) leads to

$$\left\{\int_{u(x)>B\omega} \left|x^{1-\eta}u(x)\mathcal{H}(f_\omega)(x)\right|^s \frac{dx}{x}\right\}^{\frac{1}{s}}$$

$$= \left\{\int_{u(x)>B\omega} \left|x^{1-\eta}u(x)\int_{v(t)<\omega} t^{\frac{\eta r-1}{1-r}}\mathfrak{H}(xt)v(t)^{-r'}dt\right|^s \frac{dx}{x}\right\}^{\frac{1}{s}}$$

$$\geq C_1\left\{\int_{u(x)>B\omega} \left|x^{1-\eta}u(x)\right|^s \frac{dx}{x}\right\}^{\frac{1}{s}} \int_{v(x)<\omega} \left\{\frac{t^{1-\eta}}{v(t)}\right\}^{r'} \frac{dt}{t}, \ \omega > 0. \tag{19}$$

By combining (18) and (19) we conclude that $(u, v) \in A(r, s, 1-\eta)$. ■

S A Emara and H P Heinig [8] established interpolation theorems (Theorems 1 and 2 of [8]) that they employed to study the behavior of the Hankel and K-transformations on weighted L_p-spaces We can use such interpolation theorems to obtain new weighted norm inequalities for the $\mathcal{H}$-transform The weight functions that appear in this inequality are in the class $F^*_{r,s}$ that we are going to define Let u and v be nonnegative measurable functions defined on $(0,\infty)$ and let u^* and $\left(\frac{1}{v}\right)^*$ be the equimeasurable decreasing rearrangements of u and $\frac{1}{v}$, respectively We say that $(u, v) \in F^*_{r,s}$ if

$$\sup_{\omega>0}\left\{\int_0^{\frac{1}{\omega}} u^*(t)^s dt\right\}^{\frac{1}{s}}\left\{\int_0^{\omega}\left[\left(\frac{1}{v}\right)^*(t)\right]^{r'} dt\right\}^{\frac{1}{r'}} < \infty \tag{20}$$

holds for every $1 < r \le s < \infty$, and when $1 < s < r < \infty$ the conditions

$$\int_0^\infty \left\{\left\{\int_0^{\frac{1}{x}} u^*(t)^s dt\right\}^{\frac{1}{s}}\left\{\int_0^x\left[\left(\frac{1}{v}\right)^*(t)\right]^{r'} dt\right\}^{\frac{1}{r'}}\right\}^h \left(\frac{1}{v}\right)^*(x)^{r'} dx < \infty \tag{21}$$

$$\int_0^\infty \left\{\left\{\int_{\frac{1}{x}}^\infty \left[t^{-\frac{1}{2}}u^*(t)\right]^s dt\right\}^{\frac{1}{s}}\left\{\int_x^\infty\left[t^{-\frac{1}{2}}\left(\frac{1}{v}\right)^*(t)\right]^{r'} dt\right\}^{\frac{1}{r'}}\right\}^h \left\{\left(\frac{1}{v}\right)^*(x)x^{-\frac{1}{2}}\right\}^{r'} dx < \infty \tag{22}$$

hold, where $\frac{1}{h} = \frac{1}{s} - \frac{1}{r}$. Moreover if (20), (21) and (22) hold when u^* and $\left(\frac{1}{v}\right)^*$ are replaced by u and $\frac{1}{v}$, respectively, then we write $(u, v) \in F_{r,s}$

PROPOSITION 7. Assume that $1 < r,\ s < \infty$, $\alpha < 0$ and $\frac{1}{2} < \beta$ Then

$$\left\{\int_0^\infty |u(x)\mathcal{H}(f)(x)|^s dx\right\}^{\frac{1}{s}} \le C\left\{\int_0^\infty |v(x)f(x)|^r dx\right\}^{\frac{1}{r}},\quad f \in C_0, \tag{23}$$

holds for a certain $C > 0$, provided that $(u, v) \in F^*_{r,s}$.

PROOF. Since $\alpha < 0 < \beta$, according to (4) we can write

$$\sup_{x>0}|\mathcal{H}f(x)| \le C\int_0^\infty |f(x)|dx,\quad f \in L_1(0,\infty)$$

for a certain $C > 0$, and then $\mathcal{H}$ is a bounded operator from $L_1(0,\infty)$ into $L_\infty(0,\infty)$

Moreover, $\mathcal{H}$ is a bounded operator from $L_2(0,\infty)$ into itself because $\alpha < \frac{1}{2} < \beta$ (Proposition 3 of [5])

Hence from Theorems 1 and 2 of [8] we can infer that the inequality (23) is satisfied ■

We now prove a result that is a (partial) converse to Proposition 7 Note that here no monotonicity assumptions on the weights need be made.

PROPOSITION 8. Let $1 < r \le s < \infty$ and let u and v be nonnegative measurable functions on $(0,\infty)$. Assume that (14) holds and that $\int_0^\omega v(x)^{-r'}dx < \infty$, for every $\omega > 0$. Then $(u, v) \in F_{r,s}$ when (23) is satisfied.

PROOF. Firstly we define for every $\omega > 0$ the function

$$f_\omega(x) = \begin{cases} v(x)^{-r'} & ,\ \text{if } 0 < x < \omega \\ 0 & ,\ \text{if } x > \omega. \end{cases}$$

From (14) one deduces

$$\int_0^\infty u(x)\,|\mathcal{H}(f_\omega)(x)|^s dx = \int_0^\infty \left|u(x)\int_0^\infty \mathfrak{H}(xt)f_\omega(t)dt\right|^s dx$$
$$\ge \int_0^{\frac{1}{\omega}} \left|u(x)\int_0^\omega \mathfrak{H}(xt)v(t)^{-r'}dt\right|^s dx \ge M\int_0^{\frac{1}{\omega}} u(x)^s dx\left\{\int_0^\omega v(t)^{-r'}dt\right\}^s,\quad \omega > 0$$

for a certain $M > 0$. Moreover,

$$\int_0^\infty |f_\omega(x)v(x)|^r dx = \int_0^\omega v(x)^{-r'} dx, \quad \omega > 0.$$

Since (23) holds we can write

$$\left\{ M \int_0^{\frac{1}{\omega}} u(x)^s dx \left\{ \int_0^\omega v(t)^{-r'} dt \right\}^s \right\}^{\frac{1}{s}} \leq C \left\{ \int_0^\omega v(t)^{-r'} dt \right\}^{\frac{1}{r}}, \quad \omega > 0.$$

Thus we conclude that $(u, v) \in F_{r,s}$. ■

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