Internat. J. Math. & Math. Sci.
VOL. 20 NO. 4 (1997) 617-620

A NEW LOOK AT MEANS ON TOPOLOGICAL SPACES

PETER HILTON

Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364 USA
and
Department of Mathematical Sciences
SUNY, Binghamton
Binghamton, NY 13902-6000 USA

(Received June 5, 1997)

ABSTRACT. We use methods of algebraic topology to study when a connected topological space admits an n-mean map.

KEY WORDS AND PHRASES: Means, topological spaces, comeans
1991 AMS SUBJECT CLASSIFICATION CODES: 55PXX

1. INTRODUCTION

Carathéodory and Aumann (see [1],[2]) were among the pioneers who first considered the question of what path-connected regions X in $\mathbb{R}^m$ or $\mathbb{C}^m$ could support an ***n*-mean**, that is, a map $\mu : X^n \to X$ satisfying

(i) $\mu\triangle = 1$; $X \to X$, where $\triangle$ is the diagonal map $\triangle : X \to X^n$; and

(ii) $\mu\sigma = \mu : X^n \to X$, where $\sigma \in S_n$, the symmetric group on n letters, acting on X^n by permuting components One of their main concerns was to find out if the existence of such an n-mean, $n \geq 2$, implied that X was simply connected

In 1954, Beno Eckmann [4] attacked the question with the tools of algebraic topology He supposed X to be a polyhedron and only required conditions (i), (ii) above up to homotopy One of his principal conclusions was that if X is compact and admits a (homotopy) n-mean for all n, then X is contractible

In 1962, Eckmann, together with Tudor Ganea and the author, returned to the study of n-means in a more general setting (see [5]). Thus the n-mean defined in [4] was a morphism in the category $\mathcal{T}_h$ of based connected CW-complexes and based homotopy classes of based maps In this generality one was able to exploit the idea of mean-preserving functors. Thus if $\mathcal{C}$, $\mathcal{D}$ are categories with products and $F : \mathcal{C} \to \mathcal{D}$ is a product-preserving functor, then $F\mu$ is an n-mean in $\mathcal{D}$ for any n-mean μ in $\mathcal{C}$ Moreover, one could also examine the dual question of the existence of n-*comeans*

It turns out that the concept of P-*local objects* and P-*localization*, where P is a family of primes, and the results related to these concepts in the categories $\mathcal{T}_h$ and $\mathcal{N}$, the category of nilpotent groups (see [6]), enable one to simplify many arguments in [5] and to extend the results of that paper

2. MEANS IN THE CATEGORY OF GROUPS

Let $\mathcal{G}$ be the category of groups Let n be an integer, $n \geq 2$, and let P be the family of primes p such that $p \nmid n$ We then prove

THEOREM 2.1. The group G admits an n-mean μ in $\mathcal{G} \Leftrightarrow G$ is commutative and P-local In that case, if we write G additively, μ is given by

$$\mu(g_1, g_2, \cdots, g_n) = \frac{1}{n}(g_1 + g_2 + \cdots + g_n). \tag{2 1}$$

PROOF. Note first that if G is commutative, then G is P-local if and only if G admits unique division by n It is then plain that (2 1) defines an n-mean on G

Conversely, let μ be an n-mean on G For $g, h \in G$ (at this stage, we write G multiplicatively), set $\mu(g, e, \cdots, e) = \gamma$, $\mu(h, e, \cdots, e) = \delta$ Then, by condition (ii),

$$\mu(e, g, \cdots, e) = \cdots = \mu(e, e, \cdots, g) = \gamma,$$

so that, by condition (i),

$$g = \mu(g, g, \cdots, g) = \gamma^n.$$

Similarly, $h = \delta^n$ But $\mu(g, h, \cdots, e) = \gamma\delta$, $\mu(h, g, \cdots, e) = \delta\gamma$, and $\mu(g, h, \cdots, e) = \mu(h, g, \cdots, e)$ Thus γ commutes with δ, so that g commutes with h and G is commutative To show that G is P-local it remains to show that n^{th} roots are unique in G. But, again using properties (i) and (ii), we conclude that $\mu(g^n, e, \cdots, e) = \mu(g, g, \cdots, g) = g$, so that g is determined by g^n Thus G is commutative and P-local and, writing additively, we have

$$\mu(g_1, g_2, \cdots, g_n) = \sum_{i=1}^{n} (g_i, 0, \cdots, 0) = \sum_{i=1}^{n} \frac{1}{n} g_i = \frac{1}{n}(g_1 + g_2 + \cdots + g_n).$$

□

COROLLARY 2.2. Let G be a group and let $n_1 \geq 2$, $n_2 \geq 2$ be integers Then G admits an $n_1 n_2$-mean if and only if G admits an n_1-mean and an n_2-mean

3. MEANS IN THE CATEGORY $\mathcal{T}_h$

Let X be a connected CW-complex with base point. We prove, with n, P as in Section 2,

THEOREM 3.1. Suppose X admits an n-mean $\mu : X^n \to X$ in $\mathcal{T}_h$ Then X is a P-local commutative H-space

PROOF. We regard the i^{th} homotopy group π_i as defining a product-preserving functor from $\mathcal{T}_h$ to $\mathcal{G}$ Then $\mu_* = \pi_i \mu : (\pi_i X)^n \to \pi_i X$ is an n-mean in $\mathcal{G}$ It follows that $\pi_i X$ is commutative (this is only significant for $i = 1$) and P-local and that μ_* has the form (2 1).

Let $i_1 : X \to X^n$ be the obvious embedding. Then $(\mu i_1)_*$ is the endomorphism $g \mapsto \frac{1}{n} g$ of the commutative P-local group $\pi_i X$ It follows that $(\mu i_1)_*$ is an automorphism for all i, so that μi_1 is a self-homotopy-equivalence of X Let $\rho : X \to X$ be homotopy inverse to μi_1. Let $i_{12}' : X^2 \to X^n$ be the obvious embedding and let $m = \rho \mu i_{12} : X^2 \to X$. Then it is easy to see that m is a commutative H-structure on X We conclude that X is a P-local commutative$H$-space □

From Theorem 2.1 we deduce, more easily than in [5],

THEOREM 3.2. If a compact, connected polyhedron X admits an n-mean for some $n \geq 2$, then X is contractible.

PROOF. Since the homotopy groups of X are P-local, so are the homology groups $H_i X, i \geq 1$ (see [6]). Now Browder has shown [3] that a compact, connected polyhedron X which is an H-space satisfies Poincaré duality. Thus, if X is not contractible, there exists a positive dimension N which contains the universal class giving rise to the duality isomorphism $H_i(X) \simeq H^{N-i}(X)$. In particular, $H_N X = \mathbb{Z}$, but this is absurd, since $\mathbb{Z}$ is not divisible by n □

REMARK 1. We have not invoked commutativity of the H-structure in this argument If we do so, we may apply a theorem of Hubbuck showing that X would be equivalent to a product of circles, which is also impossible for a non-contractible P-local space

REMARK 2. Theorem 3.2 is delicate The n-solenoid is compact and admits an n-mean but is not a polyhedron The Eilenberg-MacLane space $K(\mathbb{Q}, m)$ is a polyhedron and admits an n-mean for every n, but is not compact

We have not proved—and doubt the truth of—the converse of Theorem 3 1 However, one may readily prove

THEOREM 3.3. If X is a P-local, connected, commutative, associative H-space, then X admits a unique homomorphic n-mean. Further, if the connected H-space (X, m) admits a homomorphic n-mean, then (X, m) is commutative and associative.

The case $n = 2$ admits a very neat and precise statement. If $\mu : X^2 \to X$ is a 2-mean on X, we define ρ as in the proof of Theorem 3.1 as homotopy inverse to μi_1, and $m = \rho\mu$ is a commutative H-structure on the P-local space X, where P is the family of odd primes Conversely, if $m : X^2 \to X$ is a commutative H-structure on the P-local space X, we define τ to be homotopy inverse to $m\triangle : X \to X$ (notice that $m\triangle$ induces doubling on the homotopy groups of X and is therefore a self-homotopy-equivalence). Then $\mu = \tau m$ is a 2-mean on X.

THEOREM 3.4. The function $\mu \mapsto \rho\mu$ sets up a one-one correspondence between 2-means on the P-local connected CW-complex X and commutative H-structures on X

PROOF. If $m = \rho\mu$, then $\mu\triangle = \rho\mu\triangle = \rho$, so τ, defined above, is homotopy inverse to ρ and $\tau m = \mu$ If $\tau\mu = \mu$, then $\tau = ui_1$ so, again, ρ is homotopy inverse to τ and $\rho\mu = m$ Thus the function $m \mapsto \tau m$ is inverse to the function $\mu \mapsto \rho\mu$

4. THE DUAL STORY

Whereas the product in a familiar category (like $\mathcal{T}_h, \mathcal{G}$) takes a familiar form essentially independent of the category, the form of the coproduct depends very much on the category in question The three categories which will come into question here are $\mathcal{T}_h$, $\mathcal{G}$, and $\mathcal{A}b$, the category of abelian groups

Let $\mathcal{C}$ be a category admitting finite coproducts, we will write $C \vee D$ for the coproduct of C and D in $\mathcal{C}$ and C_n for the coproduct of n copies of C in $\mathcal{C}$. Obviously, the symmetric group S_n acts on C_n, we will write $\nabla : C_n \to C$ for the codiagonal, which is the morphism that coincides with the identity on each copy of C in C_n Then an ***n*-comean** on C is a morphism $\mu : C \to C_n$ such that (i) $\nabla\mu = 1 : C \to C$, and (ii) $\sigma\mu = \mu$, for all $\sigma \in S_n$ We prove

THEOREM 4.1. In $\mathcal{G}$ only the trivial group admits an n-comean, $n \geq 2$

PROOF. Let G be a non-trivial group and let $g \in G$, $g \neq e$ If $\mu : G \to G_n$ is an n-comean, $n \geq 2$, then it follows from (i) that $\mu g \neq e$ Now G_n is the free product of n copies of G, so a non-trivial element of G_n is uniquely expressible as $h_{i_1} h_{i_2} \cdots h_{i_k}$, where $G_{(i)}$ is the i^{th} copy of G in G_n, $h_i \in G_{(i)}$, $h_i \neq e$, and $i_q \neq i_{q+1}$, $q = 1, 2, \cdots, k-1$ Such an element is obviously moved under any permutation σ which moves i_1, so that condition (ii) is violated. □

THEOREM 4.2. In $\mathcal{A}b$, the abelian group A admits an n-comean, $n \geq 2$, if and only if it admits an n-mean In that case $\mu : A \to A_n$ is given by

$$\mu(a) = \left(\frac{a}{n}, \frac{a}{n}, \cdots, \frac{a}{n}\right). \tag{4 1}$$

PROOF. We note first that, in $\mathcal{A}b$, $C \vee D = C \oplus D$, so that $A_n = A^n$ If A admits an n-mean, then, by Theorem 2.1, it is clear that (4.1) is an n-comean. Suppose conversely that $\mu : A \to A_n$ is an n-comean It is then plain from (ii) that $\mu(a) = (\alpha, \alpha, \cdots, \alpha)$ for some $\alpha \in A$ such that, by (i), $n\alpha = a$. It remains to show that division by n is unique in A. But

$$\mu(na) = (n\alpha, n\alpha, \cdots, n\alpha) = (a, a, \cdots, a),$$

so that a is determined by na □

REMARK. Note that the situations for means and comeans are very different Means in $\mathcal{G}$ coincide with means in $\mathcal{A}b$, on the other hand, there are no non-trivial comeans in $\mathcal{G}$ but there are non-trivial comeans in $\mathcal{A}b$, and, moreover, the objects in $\mathcal{A}b$ admitting n-comeans coincide with those admitting n-means

We now study n-comeans in $\mathcal{T}_h$. Using the same notation as in Theorem 3.1, we prove

THEOREM 4.3. Suppose X is a connected CW-complex admitting an n-comean $\mu : X \to X_n$ in $\mathcal{T}_h$, $n \geq 2$ Then X is a simply connected P-local commutative H'-space

PROOF. Now X_n is just a bouquet of n copies of X Since $\pi_1 : \mathcal{T}_h \to \mathcal{G}$ is coproduct-preserving, $\pi_1\mu$ is an n-comean on the fundamental group $\pi_1 X$, so that, by Theorem 4 1, X is simply connected Now the homology groups H_i, $i \geq 1$, are coproduct-preserving functors $\mathcal{T}_h \to \mathcal{A}b$, so that, by Theorems 2 1 and 4 2, the homology groups $H_i X$ are the P-local. Since X is simply connected, this implies that X is P-local Finally we adopt a line of reasoning entirely analogous to that in the proof of Theorem 3 1 to conclude that X admits a commutative H'-structure $m : X \to X_2$ (Notice that, since X is simply connected, a map $f : X \to X$ inducing homology isomorphisms is a homotopy equivalence.) □

Notice that there are straightforward and valid duals of Theorems 3.3 and 3 4 On the other hand, Theorem 3.2 does *not* dualize. For example, the Moore space $M(\mathbb{Z}/2, m)$, $m \geq 2$, characterized as the unique simply connected homotopy type with $H_2 = \mathbb{Z}/2$, $H_i = 0$, $i \geq 3$, is a compact $(m+1)$-dimensional polyhedron which admits an n-comean for every odd n

REFERENCES

[1] AUMANN, G , *Math. Annalen* **111** (1935), 713-730

[2] AUMANN, G , *Math. Annalen* **119** (1943), 210-215

[3] BROWDER, W , On torsion in H-spaces, *Ann. Math.* (2) **74** (1961), 24-51

[4] ECKMANN, B., Räume mit Mittelbildung, *Comm. Math. Helv.* **28** (1954), 329-340

[5] ECKMANN, B., GANEA, T and HILTON, T., Generalized means, *Studies in Mathematical Analysis and Related Topics*, Stanford University Press (1962), 82-92.

[6] HILTON, P., MISLIN, G. and ROITBERG, J., *Localization of nilpotent groups and spaces*, Mathematical Studies 15, North Holland (1975)