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HAMILTONIAN-CONNECTED GRAPHS AND THEIR STRONG CLOSURES

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ABSTRACT. Let G be a simple graph of order at least three. We show that G is Hamiltonian-connected if and only if its strong closure is Hamiltonian-connected. We also give an efficient algorithm to compute the strong closure of G.

KEY WORDS AND PHRASES. Hamiltonian-connected graph, strong closure, degree sequence.
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1. INTRODUCTION.

Let $G = (V, E)$ be a simple graph, $n = |V| (\geq 3)$ and $m = |E|$. G is called Hamiltonian-connected if every two vertices of G are connected by a Hamiltonian path. If G is Hamiltonian-connected and $n \geq 4$, then $m \geq \frac{1}{2}(3n+1)$ (see [2], p. 61).

In this paper, we define the strong closure $sc(G)$ of a simple graph G. We also show that G is Hamiltonian-connected if and only if its strong closure $sc(G)$ is Hamiltonian-connected (Theorem 2.3). It follows immediately that if $sc(G)$ is a complete graph, then G is Hamiltonian-connected (Corollary 2.4). As in the case of Hamiltonian graphs, there is no characterization of Hamiltonian-connected graphs. If we compute the strong closure of G and find it is complete, then G is Hamiltonian-connected. As another application, a result of O. Ore also follows from Corollary 2.4 (see Corollary 2.5).

In section 3, we give an efficient algorithm to compute the strong closure $sc(G)$ of any simple graph G. This algorithm can be executed in $O(n|K|)$ time, where $|K'| = \frac{1}{2}(n^2 - n - 2m)$.

2. HAMILTONIAN-CONNECTED GRAPHS.

For each vertex v of G, let $D(v) = \{u \in V(G): u \text{ is adjacent to } v\}$. Then $d(v) = |D(v)|$ is the degree of v in G.

We have the main result of this paper.

THEOREM 2.1. Suppose that u is not adjacent to v in G and $d(u) + d(v) \geq n+1$. Then G is Hamiltonian-connected if and only if $G + (u, v)$ is Hamiltonian-connected.

PROOF. Suppose that $G + (u, v)$ is Hamiltonian-connected, but G is not. Since G is not Hamiltonian-connected, there exist two vertices x and y such that there is no Hamiltonian $x - y$ path in G. Since $G + (u, v)$ is Hamiltonian-connected, there is a Hamiltonian $u - v$ path in $G + (u, v)$ and hence in G. Therefore it follows that $(x, y) \neq (u, v)$. Let $P = \{w_1, w_2, \cdots, w_n\}$ be a Hamiltonian $x - y$ path in $G + (u, v)$, where $x = w_1$ and $y = w_n$.

CASE 1. Assume that $x \neq u$ and $y \neq v$. Since P is a Hamiltonian $x-y$ path in $G+(u,v)$ and P is not a Hamiltonian $x-y$ path in $G,(u,v)$ must be an edge of P in $G+(u,v)$. Therefore $u = w_k$ and $v = w_{k+1}$ for some $1 < k < n-1$. ($k \neq n-1$; for otherwise $k+1 = n$ and $v = w_n = y$). Since (u,v) is not an edge of $G, u, v \notin D(u)$ and $u, v \notin D(v)$. Suppose $w_t \in D(u)$, where $t \neq k-1$ and $t \neq n$. \ Since $u(=w_k)$ and $v(=w_{k+1})$ are not in $D(u)$, it follows that $t \neq k$ and $t \neq k+1$. We show that $w_{t+1} \notin D(v)$. Suppose that this is not true, then w_{t+1} is adjacent to v. If $t < k-1$, then the path $(=w_1)$, $w_2, \cdots, w_t, w_k(=u)$, w_{k-1}, $w_{k-2}, \cdots, w_{t+1}$, $v(=w_{k+1})$, $w_{k+2}, \cdots$, $y(=w_n)$ is a Hamiltonian $x-y$ path in G. If $t > k+1$, then the path $x(=w_1), w_2, \cdots, w_k(=u), w_t, w_{t-1}, \cdots, w_{k+1}(=v), w_{t+1}, w_{t+2}, \cdots, y(=w_n)$ is a Hamiltonian $x-y$ path in G. This is impossible. Therefore, $w_{t+1} \notin D(v)$. Since $t \neq k-1$ and $t \neq n$, it follows that there are at least $d(u)-2$ vertices to which v is not adjacent. Since $u, v \notin D(v)$, we have

$$d(v) \leq (n-2) - (d(u)-2) = n - d(u).$$

Therefore $d(u) + d(v) \leq n$, which is a contradiction. Therefore Case 1 is impossible.

CASE 2. Assume that $v = y(=w_n)$. Since $(x,y) \neq (u,v)$, it follows that $u \neq x$ and so $u = w_{n-1}$. Let $w_t \in D(u)$, where $t \neq n-2$. Then by the same argument as in Case 1, $w_{t+1} \notin D(v)$. Hence

$$d(v) \leq (n-2) - (d(u)-1) = n - d(u) - 1$$

and so $d(u) + d(v) \leq n-1$, which is impossible. Therefore G is Hamiltonian-connected. The converse of the theorem is clearly true. This completes the proof of the theorem.

Theorem 2.1 motivates the following definition.

The *strong closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n+1$ until no such pair remains. We denote the strong closure of G by $sc(G)$.

REMARK. The closure $c(G)$ of G is defined and studied in [2] and [4]. It is useful in the study of Hamiltonian graphs. The definition of $sc(G)$ is similar to that of $c(G)$.

LEMMA 2.2. $sc(G)$ is well-defined.

PROOF. This follows from the proof of ([2], p. 56, Lemma 4.4.2).

THEOREM 2.3. A graph is Hamiltonian-connected if and only if its strong closure is Hamiltonian-connected.

PROOF. This follows immediately from Theorem 2.1 and Lemma 2.2.

Theorem 2.3 gives some interesting results.

COROLLARY 2.4. If $sc(G)$ is a complete graph, then G is Hamiltonian-connected.

PROOF. If $sc(G)$ is complete, then it is Hamiltonian-connected and so by Theorem 2.3, G is also Hamiltonian-connected.

The following result was obtained by O. Ore (see [1], p. 136, Theorem 11.3 or [5]).

COROLLARY 2.5. If $d(u) + d(v) \geq n+1$ for every pair of nonadjacent vertices u and v, then G is Hamiltonian-connected.

PROOF. Since $d(u) + d(v) \geq n+1$ for every pair of nonadjacent vertices u and v, it follows that $Sc(G)$ is a complete graph. Therefore by Corollary 2.4, G is Hamiltonian-connected.

If G has vertices $v_1, v_2, \cdots, v_n$, the sequence $(d(v_1), d(v_2), \cdots, d(v_n))$ is called a degree sequence of G. The following result is similar to a result obtained by Chvátal (see [2], p 57, Theorem 4.5).

COROLLARY 2.6. Let $(d_1, d_2, \cdots, d_n)$ be a degree sequence of G such that $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose that there is no value of p less than $\frac{1}{2}(n+1)$ for which $d_{p-1} \leq p$ and $d_{n-p} < n-(p-1)$. Then G is Hamiltonian-connected.

PROOF. By a similar argument as in the proof of ([2], p. 57, Theorem 4.5), we can show that $sc(G)$ is a complete graph. Therefore by Corollary 2.4, G is Hamiltonian-connected.

3. AN ALGORITHM FOR FINDING STRONG CLOSURE.

In this section, we give an algorithm to find $sc(G)$. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$.

STEP 1. For $1 \leq i < j \leq n$, let

$$f(i,j) = \begin{cases} d(u_i) + d(v_j), & \text{if } u_i \notin D(v_j) \\ 0 & \text{, if} u_i \in D(v_j) \end{cases}$$

STEP 2. Choose $f(I,J) = max\ \{f(i,j): 1 \leq i < j \leq n\}$.

If $f(I,J) < n+1$, then go to Step 4.

STEP 3. $f(I,J) \leftarrow 0$.

If $f(p,I) \neq 0$, then $f(p,I) \leftarrow f(p,I) + 1 (1 \leq p < I)$.

If $f(I,p) \neq 0$, then $f(I,p) \leftarrow f(I,p) + 1 (I < p \leq n)$.

If $f(q,J) \neq 0$, then $f(q,J) \leftarrow f(q,J) + 1 (1 \leq q < J)$.

If $f(J,q) \neq 0$, then $f(J,q) \leftarrow f(J,q) + 1 (J < q \leq n)$.

Go to Step 2.

STEP 4. Form $sc(G)$ by joining u_i to u_j, if $f(i,j) = 0$ $(1 \leq i < j \leq n)$.

Let G be represented by an adjacency matrix. Steps 1 and 4 can be implemented in $O(n^2)$ time. Clearly, Step 3 runs in $O(n)$ time. Let $K = \{(i,j): f(i,j) \neq 0 \quad 1 \leq i < j \leq n\}$. Then

$$|K| = 1 + 2 + \cdots + (n-1) - m = \frac{1}{2}(n^2 - n - 2m).$$

By using F-heaps data structure [3], find $max\{f(i,j)\}$ takes $O(log_2 |K|) = O(log_2 n)$ time. Hence Steps 2 and 3 take $O(|K|(n + log_2 n)) = O(n|K|)$. Thus overall we have an $O(n|K|)$ algorithm.

LEMMA 3.1. If $n \geq 4$ and $d(u) \leq 2$ in G, then $d(u) \leq 2$ in $sc(G)$.

PROOF. Let v be a vertex of G which is not adjacent to u. Then $d(v) \leq n-2$. Hence $d(u) + d(v) \leq 2 + (n-2) = n$ and so Lemma 3.1 is true.

Lemma 3.1 allows us not to consider u in the computation of $sc(G)$ if $d(u) \leq 2$ in G.

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