Internat. J. Math. & Math. Sci.
VOL. 20 NO. 2 (1997) 339-346

OPTIMAL CONTROL OF NONSMOOTH SYSTEM GOVERNED BY QUASI-LINEAR ELLIPTIC EQUATIONS

GONG LIUTANG
FEI PUSHENG
Department of Mathematics
Wuhan University
Wuhan 430072
CHINA

(Received June 20, 1995 and in revised form September 20, 1995)

ABSTRACT. In this paper, we discuss a class of optimal control problems of nonsmooth systems governed by quasi-linear elliptic partial differential equations, give the existence of the problem. Through the smoothness and the approximation of the original problem, we get the necessary condition, which can be considered as the Euler-Lagrange condition under quasi-linear case.

KEY WORDS AND PHRASES: Optimal control, Quasi-linear equation, Necessary condition.
1991 AMS SUBJECT CLASSIFICATION CODES: 49K20, 49B15.

1. INTRODUCTION

Let Ω be a bounded domain of R^n with Lipschitz continuous boundary Γ. We consider the Dirichlet problem

$$-\sum_{i=1}^{n}(a_i(y_{x_i}))_{x_i} = u \quad \text{in} \quad \Omega \tag{1.1a}$$

$$y = 0 \quad \text{on} \quad \Gamma \tag{1.1b}$$

where $u \in L^2(\Omega)$, $a_i : R \mapsto R$ are local Lipschitz functions, and $\forall y,\ z \in R$, satisfy

$$a_i(y)y \geq \omega|y|^p + C \quad \omega > 0, \quad p \geq 2, \quad C > 0 \tag{1.2}$$

$$(a_i(y) - a_i(z))(y - z) \geq \eta|y - z|^2 \quad \eta > 0 \tag{1.3}$$

$$|\dot{a}_i(y)| \leq C(|y|^{p-2} + 1) \quad p \geq 2, \quad C \geq 0 \quad \text{a.e. on } R \tag{1.4}$$

where denote $\dot{a}_i(y)$ the derivative of $a_i(y)$.

Consider the following optimal control problem:

$$(ocp) \quad \min_{u \in L^2(\Omega)} J(y,u) = G(y) + \Phi(u)$$

where y is given by (1.1), $G(y)$ is a lower semicontinuous convex function, $\Phi(u)$ is a convex function satisfy

(1) $\Phi(u)$ is continuously Frechet differentiable in $L^2(\Omega)$

(2) $\Phi(u)$ is coercive in $L^2(\Omega)$, i.e.

$$\lim_{\|u\|_{L^2(\Omega)} \to \infty} \Phi(u) = \infty.$$

Consider the problem (ocp), Casas and Fernandez [5] studied the special case when $a_i (1 < i < n)$ are continuously differentiable and

$$J(u) = \frac{1}{2}\int_\Omega |y - y_d|^2 dx + \frac{1}{2}\int_\Omega |u(x)|^2 dx,$$

got the necessary condition, Barbu [4] studied the nonsmooth linear systems and got the Euler-Lagrange condition, etc But for the nonsmooth quasi-linear systems, there are no conclusions yet In this paper, we discuss this problem, give the existence and get the necessary condition of it through the smoothness of $a_i(x)$, $G(y)$ and the approximation of (ocp). Also this way can be used to study the boundary control problem

2. THE EXISTENCE OF SOLUTION OF (ocp)

To get the existence of (ocp), we need to study the Dirichlet problem (1.1) first From Ladyzhenskaya [2], we get.

LEMMA 2.1. Let $u \in L^2(\Omega)$ satisfy $\|u\|_{L^2(\Omega)} \leq M$, and the Dirichlet problem (1.1) satisfy (1.2), (1 3), (1 4) Then·

1) There exists a unique $y_u \in W_0^{1,p} \cap L^2(\Omega)$ be a solution of (1 1) and a constant $C > 0$ depending only on M, ω, η, c, such that

$$\|y_u\|_{W_0^{1,p}\cap L^2(\Omega)} \leq C.$$

2) $\forall u_m \in L^2(\Omega)(m \in N)$, (1.1) has solutions $y_{u_m} \in Y$, assume:

$$u_m \rightharpoonup u \quad \text{weakly in} \quad L^2(\Omega) \quad \text{as} \quad m \to \infty.$$

Then,

$$y_{u_m} \to y_u \quad \text{strongly in} \quad Y.$$

In this paper we denote Y to be $W_0^1(\Omega) \cap L^2(\Omega)$

From the second part of Lemma 2.1 we can define an operator $\Theta : L^2(\Omega) \mapsto Y$ as·

$$\Theta(u) = y_u.$$

Applying Lemma 2.1 we can get the existence theorem of (ocp).

THEOREM 2.2. There exists at least one solution of the optimal control problem (ocp), we denote it $[\overline{y}, \overline{u}]$.

PROOF. Suppose that $\{u_i\}$ is a minimizing sequence of (ocp), because of the coercivity of $J(y, u)$, we get the boundness of $\|u_i\|_{L^2(\Omega)}$, then there exists a subsequence u_i (denoted the same way) and $\overline{u} \in L^2(\Omega)$, such that

$$u_i \rightharpoonup \overline{u}, \quad \text{weakly in} \quad L^2(\Omega) \quad \text{as} \quad i \to \infty.$$

From Lemma 2.1 we get $y_{u_i} \to y_{\overline{u}}$ (denoted $\overline{y}$) strongly in Y, and the lower semicontinuity of $J(y, u)$ shows

$$J(\overline{y}, \overline{u}) = \liminf_{i\to\infty} J(y_{u_i}, u_i) \leq J(y, u), \forall\, u \in L^2(\Omega)$$

i.e

$$[\overline{y}, \overline{u}] \quad \text{is a solution of } (ocp).$$

3. NECESSARY CONDITION OF THE SOLUTION OF (ocp)

Before studying the necessary condition of (ocp), we need the definition of the Generalized Gradient and the Yosida regularization of the Lipschitz function (Tiba D. [10]).

DEFINITION 3.1. If $F(x)$ is a local Lipschitz function, its Generalized Gradient denoted $DF(x)$ is the convex hull of the set of cluster points for the sequences $grad(x + h_i)$, where $h_i \to 0$ are chosen such that $grad(x + h_i)$ exist, i.e.

$$DF(x) = conv\{w \in R^n, \exists\, h_i \to 0, \exists\, grad\, F(x + h_i) \to w\}.$$

DEFINITION 3.2. $G(y) : X \mapsto R$, its Yosida regularization denoted G^ϵ is

$$G^\epsilon(y) = \inf_{x \in X} \left\{ \frac{|x-y|^2}{2\epsilon} + G(x) \right\}.$$

Now we study the necessary condition through three steps

3.1 THE SMOOTHNESS OF $a_i(x)$

We define

$$a_i^\epsilon(x) = \int_{-\infty}^{+\infty} a_i(x - \epsilon\tau)\rho(\tau)d\tau.$$

where $\rho(x)$ is a $C_0^\infty(\Omega)$ function satisfy·

(1) $$\rho(\tau) = \rho(-\tau), \int_{-\infty}^{+\infty} \rho(\tau)d\tau = 1.$$

(2) $$\begin{cases} \rho(\tau) > 0 & \text{if } |\tau| \le 1 \\ \rho(\tau) = 0 & \text{if } |\tau| > 1. \end{cases}$$

Obviously, $a_i^\epsilon(x)$ is a C_0^∞ function and $a_i^\epsilon(x) \to a_i(x)$ uniformly as $\epsilon \to 0$, an elementary calculation shows that.

LEMMA 3.1. $a_i^\epsilon(x)$ satisfy (1.2), (1.3) and (1.4) everywhere (the constant C may be changed).

PROOF. We only prove (1 4); the others are similar

$$\begin{aligned} \dot{a}_i^\epsilon(y) &= \int_{-\infty}^{+\infty} \dot{a}_i(y - \epsilon\tau)\rho(\tau)d\tau \\ &\overset{y-\epsilon\tau=z}{=} \int_{-\infty}^{+\infty} \dot{a}_i(z)\rho\Big(\frac{y-z}{\epsilon}\Big)\frac{1}{\epsilon}dz \\ &\le C\int_{-\infty}^{+\infty} (1+|z|^{p-2})\rho\left(\frac{y-z}{\epsilon}\right)\frac{1}{\epsilon}d\tau \\ &\le C + C|y|^{p-2}. \end{aligned}$$

The smoothness of $a_i(x)$ change equation (1.1) into

$$-\sum_{i=1}^{n} (a_i^\epsilon(y_{x_i}))_{x_i} = u \quad \text{in} \quad \Omega \tag{1.1a)*}$$

$$y = 0 \quad \text{on} \quad \Gamma \tag{1.1b)*}$$

The same as before, arbitrary $u \in L^2(\Omega)$, $(1.1)^*$ has a unique solution $y_u^\epsilon \in Y$, we also define $\Theta_\epsilon : L^2(\Omega) \mapsto Y$ as

$$\Theta_\epsilon(u) = y_u^\epsilon.$$

Compared with the original operator Θ, we have:

THEOREM 3.2. $\forall f, f^\epsilon \in L^2(\Omega)$, let $y = \Theta(f)$, $y^\epsilon = \Theta_\epsilon(f^\epsilon)$, assume

$$f^\epsilon \rightharpoonup f \quad \text{weakly in} \quad L^2(\Omega), \quad \text{as} \quad \epsilon \to 0.$$

Then,

$$y^\epsilon \to y \quad \text{strongly in} \quad H_0^1(\Omega), \quad \text{as} \quad \epsilon \to 0.$$

PROOF. For arbitrary $\phi \in W_0^{1,p}(\Omega)$, multiply (1.1), $(1.1)^*$ by it, and integrating by parts we have

$$\sum_{i=1}^{n} \int_\Omega a_i^\epsilon(y_{x_i}^\epsilon)\phi_{x_i}dx = \int_\Omega f^\epsilon \phi dx. \tag{3 1}$$

$$\sum_{i=1}^{n}\int_{\Omega} a_i(y_{x_i})\phi_{x_i}dx = \int_{\Omega} f\phi dx. \tag{3 2}$$

Let $\phi = y^{\epsilon} - y$, subtract (3.1) and (3.2), we get

$$\sum_{i=1}^{n}\int_{\Omega}\left(a_i^{\epsilon}(y_{x_i}^{\epsilon}) - a_i(y_{x_i})\right)\left(y_{x_i}^{\epsilon} - y_{x_i}\right)dx = \int_{\Omega}(f^{\epsilon} - f)(y^{\epsilon} - y)dx. \tag{3 3}$$

In view of (1.3), we have

$$\eta\int_{\Omega}\left|y_{x_i}^{\epsilon} - y_{x_i}\right|^2 dx \le \int_{\Omega}(f^{\epsilon} - f)(y^{\epsilon} - y)dx + \int_{\Omega}\left(a_i^{\epsilon}(y_{x_i}) - a_i(y_{x_i})\right)\left(y_{x_i}^{\epsilon} - y_{x_i}\right)dx. \tag{3.4}$$

Since $f^{\epsilon} \rightharpoonup f$ in $L^2(\Omega)$, the first term of right of (3.4) goes to zero as $\epsilon \to 0$. The second part, because of the boundness of $\|y\|_Y$ and $\|y^{\epsilon}\|_Y$, and $a_i^{\epsilon}(y) \to a_i(y)$ uniformly, we can obtain it goes to zero also, i e.

$$\int_{\Omega}\left|y_{x_i}^{\epsilon} - y_{x_i}\right|^2 dx \to 0, \quad \text{as} \quad \epsilon \to 0.$$

By the equivalence norm of $\|\cdot\|_{H_0^1(\Omega)}$ and $|\cdot|_{H_0^1(\Omega)}$, and the Sobolev imbedding theorem, we have

$$\|y^{\epsilon} - y\|_{H_0^1(\Omega)} \to 0 \quad \text{as} \quad \epsilon \to 0,$$

i.e

$$y^{\epsilon} \to y \quad \text{strongly in} \quad H_0^1(\Omega).$$

COROLLARY 3.3. $\forall f \in L^2(\Omega)$, let $y^{\epsilon} = \Theta_{\epsilon}(f)$, $y = \Theta(f)$, then, there exists a constant $c > 0$ satisfying

$$\|y^{\epsilon} - y\|_{L^2(\Omega)} \le C_{\epsilon}.$$

In view of Θ_{ϵ}, we have

THEOREM 3.4. Let Θ_{ϵ} be defined as before, then,

1) Θ_{ϵ} is Gateaux differentiable in $L^2(\Omega)$.

2) For arbitrary $f, g \in L^2(\Omega)$, denote $r = \nabla\Theta^{\epsilon}(f)g$, then r is a unique solution of the following Dirichlet problem

$$-\sum_{i=1}^{n}\left(a_i^{\epsilon}(y_{x_i})\right)_{x_i} = g \quad \text{in} \quad \Omega \tag{3.5a}$$

$$r = 0 \quad \text{on} \quad \Gamma' \tag{3.5b}$$

PROOF. $\forall f, g \in L^2(\Omega)$, $\lambda > 0$, denote $y^{\lambda} = \Theta_{\epsilon}(f + \lambda g)$, $y = \Theta_{\epsilon}(f)$, from Corollary 3.3, we have

$$y^{\lambda} \to y \quad \text{strongly in} \quad L^2(\Omega) \quad \text{as} \quad \lambda \to 0.$$

Next we prove the theorem through 3 steps.

STEP 1. The sequence $\left\{\frac{y^{\lambda} - y}{\lambda}\right\}$ is bounded in $H_0^1(\Omega)$.

Similarly to (3.1), multiply (1.1)* by ϕ, integrating by parts, we get

$$\sum_{i=1}^{n}\int_{\Omega} a_i^{\epsilon}(y_{x_i}^{\lambda})\phi_{x_i}dx = \int_{\Omega}(f + \lambda g)\phi dx$$

$$\sum_{i=1}^{n}\int_{\Omega} a_i^{\epsilon}(y_{x_i})\phi_{x_i}dx = \int_{\Omega} f\phi dx.$$

Let $\phi = y^{\lambda} - y$, and subtract them we get

$$\sum_{i=1}^{n} \int_{\Omega} \left(a_i^\epsilon(y_{x_i}^\lambda) - a_i^\epsilon(y_{x_i})\right)\left(y^\lambda - y\right)_{x_i} dx = \lambda \int_{\Omega} g(y^\lambda - y)dx. \tag{3.6}$$

Thinking about (1 3), we derive

$$\eta \int_{\Omega} \left|y_{x_i}^\lambda - y_{x_i}\right|^2 dx \leq \lambda \|g\|_{L^2(\Omega)} \|y^\lambda - y\|_{L^2(\Omega)}$$
$$\leq C\|g\|_{L^2(\Omega)} \|y^\lambda - y\|_{H_0^1(\Omega)}.$$

Because of the equivalence of the norm $\|u\|_{H_0^1(\Omega)}$ and $|u|_{H_0^1(\Omega)}$, we know

$$\left\| \frac{y^\lambda - y}{\lambda} \right\|_{H_0^1(\Omega)} \leq C\|g\|_{L^2(\Omega)} \tag{3.7}$$

i e

$$\left\{ \frac{y^\lambda - y}{\lambda} \right\} \quad \text{is bounded in} \quad H_0^1(\Omega).$$

STEP 2. We denote $z^\lambda = \frac{y^\lambda - y}{\lambda}$, we will get a subsequence and prove the boundness of $m_i^\lambda z_{x_i}^\lambda$ in $L^s(\Omega)$, $s \in \left(1, \frac{p}{p-2}\right)$

From (3.7), we get a subsequence z^λ (denote in the same way) such that $z^\lambda \rightarrow r$, weakly in $H_0^1(\Omega)$.

By the mean-value formula, we have

$$\left(a_i^\epsilon(y_{x_i}^\lambda) - a_i^\epsilon(y_{x_i})\right) = m_i^\lambda (y_{x_i}^\lambda - y_{x_i})$$

where m_i^λ depend on $y_{x_i}^\lambda$, y_{x_i} and $\dot{a}_i^\epsilon$, and furthermore from (1.4) we know $m_i^\lambda \in L^{\frac{p}{p-2}}(\Omega)$

Reconsider (3.6) we know $\int_\Omega m_i^\lambda |z_{x_i}^\lambda|^2 dx$ is bounded with respect to $\lambda > 0$

By the inequality of Young we have

$$|m_i^\lambda z_{x_i}^\lambda| \leq |m_i^\lambda| \left(1 + |z_{x_i}^\lambda|^{1+\nu}\right), \quad \nu > 0$$

$$|m_{x_i}^\lambda|\, |z_{x_i}^\lambda|^{1+\nu} = |m_i^\lambda|^{1-\mu} \left(|m_i^\lambda|^\mu |z_{x_i}^\lambda|^{1+\nu}\right) \leq \frac{1}{\alpha} |m_i^\lambda|^{(1-\mu)\alpha} + \frac{1}{\beta}\left(|m_i^\lambda|^\mu |z_{x_i}^\lambda|^{1+\nu}\right)^\beta.$$

Here, $\mu = (1+\nu)/2 < 1$, $1 < (1-\mu)\alpha < \frac{p}{p-2}$, $\alpha > 2$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

So $\|m_i^\lambda z_{x_i}^\lambda\|_{L^s(\Omega)}$, $s \in \left(1, \frac{p}{p-2}\right)$ is bounded.

STEP 3. $z \rightarrow r$ in $L^2(\Omega)$, r satisfy the Dirichlet problem (3.5). $z \rightarrow r$ in $L^2(\Omega)$ is obvious from $z \rightarrow r$ in $H_0^1(\Omega)$

From Step 2 we have

$$m_i^\lambda z_{x_i}^\lambda \rightharpoonup h \quad \text{weakly in} \quad L^s(\Omega) \quad \text{as} \quad \lambda \rightarrow 0.$$

Next we will prove $h = \dot{a}_i^\epsilon(y_{x_i}) r_{x_i}$, a.e. on Ω

Since

$$\frac{\left(a_i^\epsilon(y_{x_i}^\lambda) - a_i^\epsilon(y_{x_i})\right)}{\lambda} = \frac{\left(a_i^\epsilon(y_{x_i}^\lambda) - a_i^\epsilon(y_{x_i})\right)}{y_{x_i}^\lambda - y_{x_i}} z_{x_i}^\lambda = m_i^\lambda z_{x_i}^\epsilon$$

and

$$z_{x_i}^\lambda \rightharpoonup r_{x_i} \quad \text{weakly in} \quad L^2(\Omega). \tag{3.8}$$

From Lemma 2 1 we get $y_{x_i}^\lambda \rightarrow y_{x_i}$ in $L^2(\Omega)$. By Egorov Theorem we know: $\forall \sigma > 0, \exists\, \Omega_\sigma \subset \Omega$, such that $m(\Omega - \Omega_\sigma) < \sigma$, and $y_{x_i}^\lambda \rightarrow y_{x_i}$ uniformly in $L^2(\Omega)$.

Then,

$$\frac{a_i^\epsilon(y_{x_i}^\lambda) - a_i^\epsilon(y_{x_i})}{y_{x_i}^\lambda - y_{x_i}} \to \dot{a}_i^\epsilon(y_{x_i}) \quad \text{strongly in} \quad L^2(\Omega). \tag{3 9}$$

(3 8) and (3.9) show

$$h = \dot{a}_i^\epsilon(y_{x_i})r_{x_i} \quad \text{a.e} \quad \Omega.$$

Going to (3 1) we get

$$\sum_{i=1}^n \int_\Omega (\dot{a}_i^\epsilon(y_{x_i})r_{x_i})r_{x_i}dx = \int_\Omega gr\,dx.$$

Integrating by parts, we obtain

$$-\sum_{i=1}^n (\dot{a}_i^\epsilon(y_{x_i})r_{x_i})_{x_i} = g \quad \text{in} \quad \Omega$$
$$r = 0 \quad \text{on} \quad \Gamma$$

The uniqueness is obvious from Corollary 3.3

For arbitrary $f \in L^2(\Omega)$, we define $(\nabla\Theta_\epsilon(f))^* : L^2(\Omega) \mapsto L^2(\Omega)$ be the adjoint operator of $\nabla\Theta_\epsilon(f)$, from Theorem 3.4 we have

COROLLARY 3.5. For arbitrary $p,\ q \in L^2(\Omega)$, denote $p = (\nabla\Theta_\epsilon(f))^* q$

Then, $p,\ q$ satisfy the following linear boundary-value equation

$$\sum_{i=1}^n (\dot{a}_i^\epsilon(y_{x_i})p_{x_i})_{x_i} = -q \quad \text{in} \quad \Omega \tag{3.10a}$$
$$p = 0 \quad \text{on} \quad \Gamma \tag{3.10b}$$

PROOF. Only multiply this equation by $r = \nabla\Theta_\epsilon(f)g$, integrating by parts and applying Theorem 3 4 we may get it.

3.2 THE APPROXIMATION OF (*ocp*)

We define

$$(ocpa) \quad \min_{u\in L^2(\Omega)} J^\epsilon(y,u) = G^\epsilon(y) + \Phi(u) + \frac{1}{2}\|u - \overline{u}\|^2_{L^2(\Omega)}$$

where y is the solution of $(1.1)^*$ and $\overline{u}$ is the solution of (ocp), G^ϵ is the Yosida regularization of G.

Obviously $J^\epsilon(y,u)$ is coercive, similarly to (ocp) we know there exists at least one solution of $(ocpa)$, we denote it $[y^\epsilon, u^\epsilon]$, the relation between $[y^\epsilon, u^\epsilon]$ and $[\overline{y}, \overline{u}]$ is:

THEOREM 3.6. Suppose $[y^\epsilon, u^\epsilon]$ is a solution of $(ocpa)$, then, there exist $p^\epsilon,\ q^\epsilon \in L^2(\Omega)$, such that

$$\sum_{i=1}^n \left(\dot{a}_i^\epsilon(y_{x_i})p^\epsilon_{x_i}\right)_{x_i} = -q^\epsilon \quad \text{in} \quad \Omega \tag{3 11a}$$
$$p^\epsilon = 0 \quad \text{on} \quad \Gamma \tag{3.11b}$$

Here

$$q^\epsilon = \partial G^\epsilon(y^\epsilon), p^\epsilon = \nabla\Phi(u^\epsilon) + u^\epsilon - \overline{u} \tag{3 12}$$

and moreover

$$y^\epsilon \to \overline{y},\ u^\epsilon \to \overline{u} \quad \text{in} \quad L^2(\Omega) \tag{3.13}$$

$$p^\epsilon \to \overline{p},\ q^\epsilon \to \overline{q} \quad \text{in} \quad L^2(\Omega) \tag{3 14}$$

where $[\overline{y}, \overline{u}]$ is a solution of (ocp), and

$$\overline{p} = \nabla\Phi(\overline{u}),\ \overline{q} \in \partial G(\overline{y}). \tag{3.15}$$

PROOF. Because $[y^\epsilon, u^\epsilon]$ is a solution of $(ocpa)$ we have

$$(\nabla J^{\epsilon}(y^{\epsilon}, u^{\epsilon}), v) = 0, \quad \forall v \in L^2(\Omega)$$

i e

$$(\partial G^{\epsilon}(y^{\epsilon}), \nabla\Theta_{\epsilon}(u^{\epsilon})v) + (\nabla\Phi(u^{\epsilon}) + u^{\epsilon} - \overline{u}, v) = 0.$$

Define

$$p^{\epsilon} = \nabla\Phi(u^{\epsilon}) + u^{\epsilon} - \overline{u}, \;\; q^{\epsilon} = \partial G^{\epsilon}(y^{\epsilon}).$$

Obviously,

$$p^{\epsilon} = -(\nabla\Theta_{\epsilon}(u^{\epsilon}))^{*}q^{\epsilon}.$$

Apply Corollary 3 5, we get p^{ϵ}, q^{ϵ} satisfy (3 11)

Next we prove $y^{\epsilon} \to \overline{y}$, $u^{\epsilon} \to \overline{u}$, in $L^2(\Omega)$. Because $[y^{\epsilon}, u^{\epsilon}]$ is a solution of $(ocpa)$, we derive

$$J^{\epsilon}(y^{\epsilon}, u^{\epsilon}) \le J^{\epsilon}(\Theta_{\epsilon}(\overline{u}), \overline{u}) = G^{\epsilon}(\Theta_{\epsilon}(\overline{u})) + \Phi(\overline{u}) \le G(\overline{y}) + \Phi(\overline{u}) + \frac{|\Theta_{\epsilon}(\overline{u}) - \Theta(\overline{u})|}{2\epsilon}$$

i.e

$$\limsup_{\epsilon\to 0} J^{\epsilon}(y^{\epsilon}, u^{\epsilon}) \le G(\overline{y}) + \Phi(\overline{u}).$$

So $J^{\epsilon}(y,u)$ is bounded, and the coercivity of $J^{\epsilon}(y,u)$ shows that $\|u^{\epsilon}\|_{L^2(\Omega)}$ is bounded Then there exists a subsequence of u^{ϵ} and $u_0 \in L^2(\Omega)$ such that

$$u^{\epsilon} \rightharpoonup u_0, \quad \text{weakly in} \quad L^2(\Omega)$$

and moreover

$$|y^{\epsilon} - y_0| = |\Theta_{\epsilon}(u^{\epsilon}) - \Theta(u_0)| \le |\Theta_{\epsilon}(u^{\epsilon}) - \Theta_{\epsilon}(u_0)| + |\Theta_{\epsilon}(u_0) - \Theta(u_0)|.$$

From Theorem 3.2 and Corollary 3.3 we know

$$\|y^{\epsilon} - y_0\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad \epsilon \to 0$$

i.e.

$$y^{\epsilon} \to y_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \epsilon \to 0.$$

Since the lower semicontinuity of $J^{\epsilon}(y,u)$ we get

$$G(y_0) + \Phi(u_0) + \frac{1}{2}\|u_0 - \overline{u}\| \le G(\overline{y}) + \Phi(\overline{u}).$$

But $[\overline{y}, \overline{u}]$ is the solution of (ocp), so $\overline{u} = u_0$, $\overline{y} = y_0$.

And from

$$G^{\epsilon}(y^{\epsilon}) + \Phi(u^{\epsilon}) + \frac{1}{2}\|u^{\epsilon} - \overline{u}\|^2_{L^2(\Omega)} \le G^{\epsilon}(\overline{y}) + \Phi(\overline{u})$$

we get

$$u^{\epsilon} \to \overline{u} \quad \text{strongly in} \quad L^2(\Omega).$$

Because Φ is continuously Frechet differentiable, we get

$$p^{\epsilon} \to \overline{p} \quad \text{strongly in} \quad L^2(\Omega)$$

and

$$\overline{p} = \nabla\Phi(\overline{u}).$$

Obviously,

$$q^{\epsilon} \rightharpoonup \overline{q} \quad \text{weakly in} \quad L^2(\Omega).$$

From Tiba D. [10] we have

$$\overline{q} \in \partial G(\overline{y}).$$

3.3 NECESSARY CONDITION OF (ocp)

Through the discussion before we have

THEOREM 3.7. The Dirichlet problem (1.1) satisfies (1.2), (1 3), (1 4), and (ocp) defined as before Suppose $[\overline{y}, \overline{u}]$ is the solution of (ocp).

Then, there exist $\overline{p}, \overline{q} \in L^2(\Omega)$ satisfy the following Dirichlet problem

$$\sum_{i=1}^{n} \left(D_{a_i}(\overline{y}_{x_i})\overline{p}_{x_i}\right)_{x_i} \ni \overline{q} \quad \text{in} \quad \Omega \tag{3.16a}$$

$$\overline{p} = 0 \quad \text{on} \quad \Gamma. \tag{3 16b}$$

Here,

$$\overline{p} = \nabla\Phi(\overline{u}), \quad \overline{q} \in \partial G(\overline{y}). \tag{3 17}$$

PROOF. Multiply (3 11) by p^ϵ, we get the boundness of $\|p^\epsilon\|_{H_0^1(\Omega)}$ Furthermore, $\dot{a}_i^\epsilon(y_{x_i}^\epsilon)\,|p_{x_i}^\epsilon|^2$ is bounded in $L^1(\Omega)$

Since $a_i^\epsilon(y_{x_i}^\epsilon)$ is bounded in $L^{\frac{p}{p-2}}(\Omega)$, the same as the proof of Theorem 3.4, using the Young inequality we have

$$\dot{a}_i^\epsilon(y_{x_i}^\epsilon)p_{x_i}^\epsilon \rightharpoonup h^i \quad \text{in} \quad L^s(\Omega), \quad \forall s \in \left(1, \frac{p}{p-2}\right)$$

and

$$y_{x_i}^\epsilon \rightarrow \overline{y}_{x_i} \quad \text{in} \quad L^2(\Omega)$$

$$\dot{a}_i^\epsilon(y_{x_i}^\epsilon) \rightharpoonup \nu^* \quad \text{in} \quad L^{\frac{p}{p-2}}(\Omega).$$

So

$$\nu^* \in Da_i(\overline{y}_{x_i})$$

$$h^i \in Da_i(\overline{y}_{x_i})\overline{p}_{x_i} \quad \text{a.e.} \quad \Omega.$$

Let $\epsilon \rightarrow 0$ in (3.11), we know that $\overline{p}$, $\overline{q}$ satisfy (3.16).

The others are obvious

ACKNOWLEDGMENT. The authors wish to thank Prof I. Lasiecka for his suggestions which made the paper more readable. This project was supported by the National Natural Science Foundation of China.

REFERENCES

[1] ADAMS, R.A., *Sobolev Spaces*, Academic Press, New York, 1975.

[2] LADYZHENSKAYA, O.A., *Linear and Quasi-linear Elliptic Equation,* Academic Press, New York/London, 1968.

[3] BARBU, V., Optimal Control of Variational Inequalities, *Research Notes in Mathematics*, **100**, Pitman London, 1984.

[4] BARBU, V., Necessary Condition for Nonconvex Distributed Control Problems Governed by Elliptic Variational Inequalities, *J. of Mathematical Analysis and Application*, **80** (1981), 566-597.

[5] CASAS, E. and FERNANDEZ, L.A., Distributed Control of System Governed by a General Class of Quasilinear Elliptic Equations, *J. of Differential Equations*, **104** (1993), 20-47.

[6] LIONS, J.L., *Optimal Control System Governed by Partial Differential Equation*, Springer Berlin, 1971.

[7] LIONS, J.L and STAMPACCHIA, G., Variational Inequalities, *SIAM J. on Applied Mathematics*, **15** (1967), 493-519.

[8] MIGNOT, F. and PUEL, J.P., Optimal Control in Some Variational Inequalities, *SIAM J. Control and Optimization*, **22** (1986), 467-477

[9] OMATA, S., *Distributed Parameter System Theory and Application*, Oxford, 1989.

[10] ITBA, D., Optimal Control System of Nonsmooth Distributed Parameter System, *Lecture Notes in Mathematics*, **1459**, Springer-Verlag, 1990