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ON AN INVERSE TO THE HÖLDER INEQUALITY

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ABSTRACT. An extension is given for the inverse to Hölder's inequality obtained recently by Zhuang.

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Recently Zhuang [1] proved the following inverse of the arithmetico-geometric inequality.

THEOREM A. Let $0 < a \le x \le A$, $0 < b \le y \le B$, $1/p + 1/q = 1$, $p > 1$; then

$$\frac{x}{p} + \frac{y}{q} \le \max\left\{\frac{A/p + b/q}{A^{1/p}b^{1/q}}, \frac{a/p + B/q}{a^{1/p}B^{1/q}}\right\} x^{1/p}y^{1/q}, \tag{1}$$

or

$$x + y \le \max\left\{\frac{A + b}{A^{1/p}b^{1/q}}, \frac{a + B}{a^{1/p}B^{1/q}}\right\} x^{1/p}y^{1/q}, \tag{2}$$

the sign of equality in (1) and (2) holds if and only if either $(x,y) = (a,B)$ or $(x,y) = (A,b)$. Moreover, if $a \ge B$, then

$$\frac{a/p + B/q}{a^{1/p}B^{1/q}} x^{1/p}y^{1/q} \le \frac{x}{p} + \frac{y}{q} \le \frac{A/p + b/q}{A^{1/p}b^{1/q}} x^{1/p}y^{1/q}, \tag{3}$$

the sign of equality on the right-hand side of (3) holds if and only if $(x,y) = (A,b)$, and the sign of equality on the left-hand side of (3) holds if and only if $(x,y) = (a,B)$. The sign of inequality in (3) is reversed if $b \ge A$.

This enables us to formulate the following theorem.

THEOREM 1. Suppose x, y, a, b, A, B, p, q are as in Theorem A and $\alpha, \beta > 0$. Then

$$\alpha x^p + \beta y^q \le \max(C, D)xy, \tag{4}$$

where

$$C = (\alpha A^p + \beta b^q)/(Ab), \qquad D = (\alpha a^p + \beta B^q)/(aB). \tag{5}$$

Equality occurs if and only if either $(x,y) = (a,B)$ or $(x,y) = (A,b)$. Moreover, if $\alpha p a^p \ge \beta q B^q$, then

$$Cxy \le \alpha x^p + \beta y^q \le Dxy, \tag{6}$$

with equality on the right-hand side if and only if $(x,y) = (A,b)$ and on the left if and only if $(x,y) = (a,B)$. The inequalities in (6) are reversed if $\alpha p A^p \le \beta q b^q$.

PROOF. Inequalities (4) and (6) follow from (1) and (3) under the substitutions

$$x \to \alpha p x^p, \quad y \to \beta q y^q, \quad a \to \alpha p a^b, \quad b \to \beta q b^q, \quad A \to \alpha p A^p, \quad B \to \beta q B^q.$$

REMARK. Theorem 1 gives (1) and (2) together, (1) resulting from the substitutions $\alpha = 1/p$, $x \to x^{1/p}$, $A = A^{1/p}$, $a \to a^{1/p}$ and corresponding relations for β, y etc. with q in place of p, while (2) results from similar substitutions with $\alpha = 1 = \beta$.

The following result now gives an extension of the inverse to Hölder's inequality obtained in [1]. We suppose that all the integrals involved exist.

THEOREM 2. Let the functions f, g satisfy $0 < a \le f(x) \le A$, $0 < b \le g(x) \le B$ for almost all $x \in X$ with respect to a measure μ. Suppose $\alpha, \beta, p, q, C, D$ are as in Theorem 1. Then

$$\left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/q} \le (\alpha\beta)^{-1/p}(\beta q)^{-1/q} \max(C, D) \int_X fg\, d\mu \tag{7}$$

and equality holds if and only if

$$\mu(E_1 \cup F_1) = \mu(X)$$

and

$$\mu(E_1) = \frac{(\alpha p A^p - \beta q b^q)\mu(X)}{\alpha p(A^p - a^p) + \beta q(B^q - \beta^q)},$$

where

$$\begin{aligned} E_1 &= \{x \in X : f(x) = a, g(x) = B\}, \\ F_1 &= \{x \in X : f(x) = A, g(x) = b\}. \end{aligned}$$

Moreover, if $\alpha p a^p \ge \beta q B^q$, then

$$\left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/q} \le (\alpha p)^{-1/p}(\beta q)^{-1/q} D \int_X fg\, d\mu, \tag{8}$$

with equality only if $(f, g) = (a, B)$ a.e. on X and $\alpha p a^p = \beta q B^q$, and if $\alpha p A^p \le \beta q b^q$, then

$$\left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/q} \le (\alpha p)^{-1/p}(\beta q)^{-1/q} C \int_X fg\, d\mu, \tag{9}$$

with equality only if $(f, g) = (A, b)$ a.e. on X and $\alpha p A^p = \beta q b^q$.

PROOF. The first statement was proved in [1]. A simple proof of the remainder of the theorem was given for the case $\alpha = 1/p$, $\beta = 1/q$ in [2]. We give a similar simple proof for the general case.

$$\begin{aligned} \max(C, D) \int_X fg\, d\mu &= \int_X \max(C, D) fg\, d\mu \\ &\ge \int_X (\alpha f^p + \beta g^q) d\mu \\ &= \frac{1}{p}(\alpha p) \int_X f^p d\mu + \frac{1}{q}(\beta q) \int_X g^q d\mu \\ &\ge (\alpha p)^{1/p}(pq)^{1/q} \left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/q}, \end{aligned}$$

by the arithmetico-geometric inequality.

The equality conditions result from those in Theorem 1 and the arithmetico-geometric inequality.

Similarly we can prove (8). Using the second inequality in (6) we have

$$
\begin{aligned}
D\int_X fg\,d\mu &= \int_X Dfg\,d\mu \\
&\geq \int_X (\alpha f^p + \beta g^q)d\mu \\
&= \frac{1}{p}(\alpha p)\int_X f^p d\mu + \frac{1}{q}(\beta q)\int_X g^q d\mu \\
&\geq (\alpha p)^{1/p}(pq)^{1/q}\left(\int_X f^p d\mu\right)^{1/p}\left(\int_X g^q d\mu\right)^{1/q}.
\end{aligned}
$$

Relation (9) follows similarly.

REMARK. The simplest cases of (8) and (9) occur for $\alpha = 1/p$, $\beta = 1/q$. Then we have that if $a^p \geq B^q$, then

$$\left(\int_X f^p d\mu\right)^{1/p}\left(\int_X g^q d\mu\right)^{1/q} \leq D_1 \int_X fg\,d\mu$$

and if $A^p \leq b^q$, then

$$\left(\int_X f^p d\mu\right)^{1/p}\left(\int_X g^q d\mu\right)^{1/q} \leq C_1 \int_X fg\,d\mu,$$

where

$$
\begin{aligned}
D_1 &= \left(\tfrac{1}{p}a^p + \tfrac{1}{q}B^q\right)/(aB), \\
C_1 &= \left(\tfrac{1}{p}A^p + \tfrac{1}{q}b^q\right)/(Ab).
\end{aligned}
$$

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