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Research Article

The Penalty Method for a New System of Generalized Variational Inequalities

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We consider a new system of generalized variational inequalities (SGVI). Using the penalty methods, we prove the existence of solution of SGVI in Hilbert spaces. Our results extend and improve some known results.

1. Introduction

Throughout this work, let H be real Hilbert space with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let K be a nonempty closed and convex subset of H. Given nonlinear mappings $T_1(x,y), T_2(x,y) : H \times H \to H$, and $\omega_1, \omega_2 \in H$, we consider the following problem:

$$\begin{aligned} &\langle rT_1(x,y) - \omega_1, u - x\rangle \geq 0, \quad \forall u \in K,\\ &\langle sT_2(x,y) - \omega_2, u - y\rangle \geq 0, \quad \forall u \in K, \end{aligned} \tag{1.1}$$

which is called the system of generalized variational inequality problem (SGVI), where $r > 0$ and $s > 0$ are constants. An element $(x^*, y^*) \in K \times K$ is called a solution of the problem (1.1) if

$$\begin{aligned} &\langle rT_1(x^*,y^*) - \omega_1, u - x^*\rangle \geq 0, \quad \forall u \in K,\\ &\langle sT_2(x^*,y^*) - \omega_2, u - y^*\rangle \geq 0, \quad \forall u \in K. \end{aligned} \tag{1.2}$$

Special cases of problem (1.1) are as follows.

(1) If $T_1(x,y) = T_2(x,y) = Tx$, $s = 1$, $r = 0$, and $\omega_2 = \omega$, $\omega_1 = \theta$, then problem (1.1) reduces to the variational inequality

$$\langle Tx - \omega, u - x \rangle \geq 0, \quad \forall u \in K. \tag{1.3}$$

Problem (1.3) was introduced by Browder [1, 2] and studied by many authors (e.g., see [3–7]).

(2) If $T_1(x,y) = Tx + x - y$, $T_2(x,y) = Ty + y - x$, $T : H \to H$, and $\omega_1 = \omega_2 = \theta$, then problem (1.1) reduces to the system of variational inequality problem

$$\begin{aligned} &\langle Tx + x - y, u - x \rangle \geq 0, \quad \forall u \in K, \\ &\langle Ty + y - x, u - y \rangle \geq 0, \quad \forall u \in K. \end{aligned} \tag{1.4}$$

Problem (1.4) was introduced and studied by Verma [8].

(3) If $T_1(x,y) = T(x,y) + x - y$, $T_2(x,y) = T(y,x) + y - x$, and $\omega_1 = \omega_2 = \theta$, then problem (1.1) becomes the following system of nonlinear variational inequalities

$$\begin{aligned} &\langle T(x,y) + x - y, u - x \rangle \geq 0, \quad \forall u \in K, \\ &\langle T(y,x) + y - x, u - y \rangle \geq 0, \quad \forall u \in K, \end{aligned} \tag{1.5}$$

which was considered by Chang et al. [9].

Remark 1.1. For a suitable choice of T_1 and T_2, the problem (1.1) includes many kinds of known systems of variational inequalities as special case (see [4–10] and the references therein). In this work, by using the penalty method, we study the existence of solutions for SGVI.

2. Preliminaries

In the sequel, we give some definitions and lemmas. In what follows, $\to$ and $\rightharpoonup$ stand for strong and weak convergence, respectively.

Definition 2.1 (see [11, pages 96–105]). Let $T : H \to H$ be a mapping.

(i) The mapping T is said to be pseudo-monotone if $D(T)$ is closed convex set and its restrictions to finite-dimensional subspaces are demicontinuous, and for every sequence $\{x_n\} \subset D(T)$, $x_n \rightharpoonup x$ in H, the inequality $\limsup_{n\to\infty}\langle Tx_n, x_n - x \rangle \leq 0$ implies that

$$\langle Tx, x - y \rangle \leq \liminf_{n\to\infty}\langle Tx_n, x_n - y \rangle, \quad \forall y \in D(T). \tag{2.1}$$

(ii) The mapping T is said to have the generalized pseudo-monotone property if for any sequence $\{[x_n, Tx_n]\}$ with $x_n \rightharpoonup x$ in X and $Tx_n \rightharpoonup f$ in H such that

$$\limsup_{n\to\infty}\langle Tx_n, x_n - x \rangle \leq 0, \tag{2.2}$$

we have $f = Tx$ and $\langle Tx_n, x_n \rangle \to \langle f, x \rangle$ as $n \to \infty$.

(iii) The mapping T is said to be monotone if for any $x, y \in D(T)$, the inequality $\langle Tx - Ty, x - y\rangle \geq 0$ holds.

(iv) The monotone mapping T is said to be maximal if the inequality $\langle Tx - g, x - y\rangle \geq 0$ for all $[x, Tx] \in G(T)$(graph of T) implies $[y, g] \in G(T)$.

(v) The mapping T is said to be coercive if there exists a continuous increasing function $C : R_+ \to R_+$ with $C(r) \to \infty$ as $r \to \infty$ such that for $x \in D(T)$

$$\langle Tx, x\rangle \geq C(\|x\|)\|x\|. \tag{2.3}$$

Definition 2.2. Let $T : H \times H \to H$ be a mapping.

(i) The mapping T is said to be (ξ, η)-*Lipschitz* continuous if there exist constants $\xi > 0, \eta > 0$ such that

$$\|T(x_1, y_1) - T(x_2, y_2)\| \leq \xi\|x_1 - x_2\| + \eta\|y_1 - y_2\|, \quad \forall x_1, x_2, y_1, y_2 \in H. \tag{2.4}$$

(ii) The mapping T is said to be α-strongly monotone in the first argument if there exists $\alpha > 0$ such that for each fixed $y \in H$, we have

$$\langle T(x_1, y) - T(x_2, y), x_1 - x_2\rangle \geq \alpha\|x_1 - x_2\|^2, \quad \forall x_1, x_2 \in H. \tag{2.5}$$

(iii) The mapping T is said to be β-strongly monotone in the second argument if there exists $\beta > 0$ such that for each fixed $x \in H$, we have

$$\langle T(x, y_1) - T(x, y_2), y_1 - y_2\rangle \geq \beta\|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in H. \tag{2.6}$$

(iv) The mapping T is said to be coercive in the first (second) argument if there exists a continuous increasing function $C_1 : R_+ \to R_+ (C_2 : R_+ \to R_+)$ with $C_1(r) \to \infty (C_2(r) \to \infty)$ as $r \to \infty$ such that for each fixed $y \in H (x \in H)$, we have

$$\begin{gathered} \langle T(x, y), x\rangle \geq C_1(\|x\|)\|x\|, \quad \forall x \in H, \\ \langle T(x, y), y\rangle \geq C_2(\|y\|)\|y\|, \quad \forall y \in H. \end{gathered} \tag{2.7}$$

Definition 2.3 (see [11]). (i) A single-valued bounded demicontinuous monotone operator $P : H \to H$ with the property that $K = \ker P = N(P) = \{x \in H : Px = 0\}$ is said to be the penalty operator of $K \subset H$.

(ii) P is demicontinuous if $x_n \to x$ implies that $Px_n \rightharpoonup Px$.

(iii) P is hemicontinuous if $h \in H, t_n > 0, x + t_n h \in H, t_n \to 0$ implies that $P(x + t_n h) \rightharpoonup Px$, where $\rightharpoonup$ stands for weak* convergence in H.

Lemma 2.4 (see [11, page 267]). *Let $P_K : H \to K$ be the projection on K, then the mapping $P = I - P_K$ is a penalty operator of K.*

Lemma 2.5 (see [11, page 98]). *(i) Any maximal monotone mapping $T : H \to H$ with $D(T) = H$ is a pseudo-monotone one.*

(ii) Any pseudo-monotone mapping has the generalized pseudo-monotone property.

(iii) The monotone mapping T is maximal if and only if $R(I + \lambda T) = H$, $for\ all\ \lambda > 0$.

3. Main Results

We consider first the problem of finding $(x, y) \in H \times H$ such that

$$\begin{aligned} \omega_1 &= rT_1(x, y) + \frac{1}{\varepsilon}Px, \\ \omega_2 &= sT_2(x, y) + \frac{1}{\delta}Py, \end{aligned} \tag{3.1}$$

which is called the system of nonlinear variational equation, where $r > 0, s > 0, \varepsilon > 0, \delta > 0$ and $\omega_1, \omega_2 \in H$, $P = I - P_K$ is the penalty operator of K (see Lemma 2.4). We will prove the existence of solutions for problem (3.1).

Lemma 3.1. *Let H be real Hilbert space and $K \subset H$ be a nonempty closed convex subset of H. Let mapping $T_1 : H \times H \to H$ be (ξ_1, η_1)-Lipschitz continuous and α_1-strongly monotone in the first argument and mapping $T_2 : H \times H \to H$ be (ξ_2, η_2)-Lipschitz continuous and α_2-strongly monotone in the second argument. If*

$$\begin{aligned} \gamma_1 &= \sqrt{1 - 2\alpha_1 r + \xi_1^2 r^2} + s\xi_2 < 1, \\ \gamma_2 &= \sqrt{1 - 2\alpha_2 s + \eta_2^2 s^2} + r\eta_1 < 1, \end{aligned} \tag{3.2}$$

then for each $\varepsilon > 0, \delta > 0$, problem (3.1) *admits a solution* $(x_\varepsilon, y_\delta)$.

Proof. Since P is continuous monotone operator, P is hemicontinuous monotone operator. By Corollary 2.3 in [11], P is maximal monotone. So, problem (3.1) is equivalent to the following problem

$$\begin{aligned} x &= J_P^\varepsilon[x - rT_1(x, y) + \omega_1], \\ y &= J_P^\delta[y - sT_2(x, y) + \omega_2], \end{aligned} \tag{3.3}$$

where $J_P^\varepsilon = (I + (1/\varepsilon)P)^{-1}$, $J_P^\delta = (I + (1/\delta)P)^{-1}$, and I is the identity mapping on H.

For any given $x_0 \in H, y_0 \in H$, we can compute the sequences $\{x_n\}$ and $\{y_n\}$ by Picard iterative schemes:

$$x_{n+1} = J_P^\varepsilon(x_n - rT_1(x_n, y_n) + \omega_1), \quad n \geq 0, \tag{3.4}$$

$$y_n = J_P^\delta(x_n - sT_2(x_n, y_n) + \omega_2), \quad n \geq 0. \tag{3.5}$$

Since J_P^{ε} is nonexpansive [11] and T_1 is (ξ_1, η_1)-*Lipschitz* continuous and α_1-strongly monotone in the first argument, then by (3.4), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \left\| J_P^{\varepsilon}(x_n - rT_1(x_n, y_n) + \omega_1) - J_P^{\varepsilon}(x_{n-1} - rT_1(x_{n-1}, y_{n-1}) + \omega_1) \right\| \\ &\le \left\| x_n - x_{n-1} - r(T_1(x_n, y_n) - T_1(x_{n-1}, y_{n-1})) \right\| \\ &\le \left\| x_n - x_{n-1} - r(T_1(x_n, y_n) - T_1(x_{n-1}, y_n)) \right\| \\ &\quad + r\left\| T_1(x_{n-1}, y_n) - T_1(x_{n-1}, y_{n-1}) \right\| \\ &\le \sqrt{1 - 2\alpha_1 r + \xi_1^2 r^2}\|x_n - x_{n-1}\| + r\eta_1 \left\| y_n - y_{n-1} \right\|. \end{aligned} \tag{3.6}$$

Similarly, we get from (3.5)

$$\left\| y_{n+1} - y_n \right\| \le \sqrt{1 - 2\alpha_2 s + \eta_2^2 s^2}\left\| y_n - y_{n-1} \right\| + s\xi_2 \|x_n - x_{n-1}\|. \tag{3.7}$$

By (3.6) and (3.7), we have

$$\begin{aligned} \|x_{n+1} - x_n\| + \left\| y_{n+1} - y_n \right\| &\le \left(\sqrt{1 - 2\alpha_1 r + \xi_1^2 r^2} + s\xi_2 \right) \|x_n - x_{n-1}\| \\ &\quad + \left(\sqrt{1 - 2\alpha_2 s + \eta_2^2 s^2} + r\eta_1 \right) \left\| y_n - y_{n-1} \right\| \\ &= \gamma_1 \|x_n - x_{n-1}\| + \gamma_2 \left\| y_n - y_{n-1} \right\| \\ &\le \gamma\left(\|x_n - x_{n-1}\| + \left\| y_n - y_{n-1} \right\| \right), \end{aligned} \tag{3.8}$$

where $\gamma = \max(\gamma_1, \gamma_2)$. By (3.2), we know that $0 < \gamma < 1$, and (3.8) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus, there exist $x_\varepsilon \in H$, $y_\delta \in H$ such that $x_n \to x_\varepsilon, y_n \to y_\delta$(as $n \to \infty$). By continuity of J_P^{ε}, J_P^{δ}, T_1, and T_2 and algorithm (3.4) and (3.5), we know that $(x_\varepsilon, y_\delta)$ satisfies the following relation:

$$\begin{aligned} x_\varepsilon &= J_P^{\varepsilon}(x_\varepsilon - rT_1(x_\varepsilon, y_\delta) + \omega_1), \\ y_\delta &= J_P^{\delta}(y_\delta - sT_2(x_\varepsilon, y_\delta) + \omega_2). \end{aligned} \tag{3.9}$$

Therefore, $(x_\varepsilon, y_\delta)$ is a solution of problem (3.1). This completes the proof of Lemma 3.1. □

Remark 3.2. If T is (s,t)-*Lipschitz* continuous, then T is bounded, that is, T map bounded set to bounded set.

Now, we give our main results.

Theorem 3.3. *Let H be a real Hilbert space and $K \subset H$ be a nonempty closed convex subset with $\theta \in K$. Let $T_1, T_2 : H \times H \to H$ be the same as in Lemma 3.1. If the mapping T_1 is coercive with respect to the first argument and the mapping T_2 is coercive with respect to the second argument, then for any $\omega_1 \in H, \omega_2 \in H$, there exists one solution (x^*, y^*) of the SGVI* (1.1).

Proof. Let $\varepsilon > 0, \delta > 0$. Then, by Lemma 3.1, the problem (3.1) has a solution $(x_\varepsilon, y_\delta)$.

By the monotonicity of P, we have

$$\begin{aligned}\langle \omega_1, x_\varepsilon \rangle &= \langle T_1(x_\varepsilon, y_\delta), x_\varepsilon \rangle + \frac{1}{\varepsilon}\langle Px_\varepsilon, x_\varepsilon \rangle \\ &= \langle T_1(x_\varepsilon, y_\delta), x_\varepsilon \rangle + \frac{1}{\varepsilon}\langle Px_\varepsilon - P\theta, x_\varepsilon - \theta \rangle \\ &\geq \langle T_1(x_\varepsilon, y_\delta), x_\varepsilon \rangle \\ &\geq C_1(\|x_\varepsilon\|)\|x_\varepsilon\|,\end{aligned} \tag{3.10}$$

since T_1 is coercive with respect to the first argument. Hence $\{x_\varepsilon \mid \varepsilon > 0\}$ remains bounded as $\varepsilon \to 0$. Similarly, we have

$$\begin{aligned}\langle \omega_2, y_\delta \rangle &= \langle T_2(x_\varepsilon, y_\delta), y_\delta \rangle + \frac{1}{\delta}\langle Py_\delta, y_\delta \rangle \\ &\geq \langle T_2(x_\varepsilon, y_\delta), y_\delta \rangle \geq C_2(\|y_\delta\|)\|y_\delta\|.\end{aligned} \tag{3.11}$$

Hence $\{y_\delta \mid \delta > 0\}$ remains also bounded as $\delta \to 0$. The boundedness of T_1 and T_2 and the fact that

$$\begin{aligned}Px_\varepsilon &= \varepsilon(\omega_1 - rT_1(x_\varepsilon, y_\delta)) \\ Py_\delta &= \delta(\omega_2 - sT_2(x_\varepsilon, y_\delta))\end{aligned} \tag{3.12}$$

implies that $\|Px_\varepsilon\| \leq c$ and $\|Py_\delta\| \leq c$ for some constant $c > 0$. By reflexivity of H, we can choose sequences $\{x_n\}$ and $\{y_n\}$, $x_n = x_{\varepsilon_n}$, $y_n = y_{\delta_n}$ such that $x_n \rightharpoonup x^* \in H, y_n \rightharpoonup y^* \in H$, and $T_1(x_n, y_n) \rightharpoonup u, T_2(x_n, y_n) \rightharpoonup v$ as $\varepsilon_n \to 0$ and $\delta_n \to 0$. Using the fact that $\|Px_n\| \to 0$ and $\|Py_n\| \to 0$ as $n \to \infty$ and taking the limit in the monotonicity relation, we get that

$$\begin{aligned}0 &\leq \lim_{n\to\infty} \langle Px - Px_n, x - x_n \rangle = \langle Px, x - x^* \rangle, \quad \forall x \in H, \\ 0 &\leq \lim_{n\to\infty} \langle Px - Py_n, x - y_n \rangle = \langle Px, x - y^* \rangle, \quad \forall x \in H.\end{aligned} \tag{3.13}$$

Set $x = x^* + tz$ with $t > 0$ and let z be arbitrarily chosen in H. Then $\langle P(x^* + tz), z \rangle \geq 0$ and by the hemicontinuity of P, we get that $\langle Px^*, z \rangle \geq 0$ for all $z \in H$. Hence $Px^* = 0$, that is, $x^* \in K$. Similarly, we have $Py^* = 0$, that is, $y^* \in K$.

By (3.1) and monotonicity of P, we have

$$\begin{aligned}\langle rT_1(x_n, y_n) - \omega_1, x^* - x_n \rangle &= \frac{1}{\varepsilon_n}\langle Px^* - Px_n, x^* - x_n \rangle \geq 0, \\ \langle sT_2(x_n, y_n) - \omega_2, y^* - y_n \rangle &= \frac{1}{\delta_n}\langle Py^* - Py_n, y^* - y_n \rangle \geq 0.\end{aligned} \tag{3.14}$$

Therefore

$$\begin{aligned} &\limsup_{n\to\infty}\langle rT_1(x_n,y_n),x_n-x^*\rangle \le \limsup_{n\to\infty}\langle \omega_1,x_n-x^*\rangle = 0,\\ &\limsup_{n\to\infty}\langle sT_2(x_n,y_n),y_n-y^*\rangle \le \limsup_{n\to\infty}\langle \omega_2,y_n-y^*\rangle = 0, \end{aligned} \tag{3.15}$$

that is,

$$\begin{aligned} &\limsup_{n\to\infty}\langle T_1(x_n,y_n),x_n-x^*\rangle \le 0,\\ &\limsup_{n\to\infty}\langle T_2(x_n,y_n),y_n-y^*\rangle \le 0. \end{aligned} \tag{3.16}$$

By Lemma 2.5(ii) and Definition 2.1, we deduce that

$$\begin{aligned} &u = T_1(x^*,y^*), \langle T_1(x_n,y_n),x_n\rangle \to \langle u,x^*\rangle \quad \text{as n} \to \infty,\\ &v = T_2(x^*,y^*), \langle T_2(x_n,y_n),y_n\rangle \to \langle v,y^*\rangle \quad \text{as n} \to \infty. \end{aligned} \tag{3.17}$$

By (3.1), (3.17), and monotonicity of P, we have

$$\begin{aligned} \langle rT_1(x^*,y^*),x^*-u\rangle &= \lim_{n\to\infty}\langle rT_1(x_n,y_n),x^*-u\rangle = \liminf_{n\to\infty}\langle rT_1(x_n,y_n),x^*-u\rangle\\ &= \liminf_{n\to\infty}\left\langle \omega_1-\frac{1}{\varepsilon_n}Px_n,x_n-u\right\rangle\\ &= \liminf_{n\to\infty}\langle \omega_1,x_n-u\rangle - \limsup_{n\to\infty}\frac{1}{\varepsilon_n}\langle Px_n-Px,x_n-u\rangle\\ &\le \langle \omega_1,x^*-u\rangle \quad \forall u\in K,\\ \langle sT_2(x^*,y^*),y^*-u\rangle &= \liminf_{n\to\infty}\langle sT_2(x_n,y_n),y^*-u\rangle\\ &= \liminf_{n\to\infty}\left\langle \omega_2-\frac{1}{\delta_n}Py_n,y_n-u\right\rangle\\ &= \liminf_{n\to\infty}\langle \omega_2,y_n-u\rangle - \limsup_{n\to\infty}\frac{1}{\delta_n}\langle Py_n-Px,y_n-u\rangle\\ &\le \langle \omega_2,y^*-u\rangle \quad \forall u\in K. \end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned} &\langle rT_1(x^*,y^*)-\omega_1,u-x^*\rangle \ge 0, \quad \forall \text{u} \in \text{K},\\ &\langle sT_2(x^*,y^*)-\omega_2,u-y^*\rangle \ge 0, \quad \forall \text{u} \in \text{K}. \end{aligned} \tag{3.19}$$

This completes the proof of Theorem 3.3. □

If $T_1(x, y) = Tx$, $r = 1$, $\omega_1 = \omega$, $\omega_2 = \theta$, and $s = 0$, then we have the following theorem.

Theorem 3.4. *Let H be a real Hilbert space and $K \subset H$ be a nonempty closed convex subset with $0 \in K$. Let $T : H \to H$ be α-strongly monotone, ξ-Lipschitz continuous, and coercive operator. If*

$$\sqrt{1 - 2r\alpha + r^2\xi^2} < 1, \tag{3.20}$$

then for each $\omega \in H$, there exists at least one solution x^* of problem (1.3).

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