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Research Article

Existence of Solution for Nonlinear Elliptic Equations with Unbounded Coefficients and L^1 Data

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An existence result of a renormalized solution for a class of nonlinear elliptic equations is established. The diffusion functions $a(x, u, \nabla u)$ may not be in $(L^1_{\text{loc}}(\Omega))^N$ for a finite value of the unknown and the data belong to $L^1(\Omega)$.

1. Introduction

In this paper we investigate the problem of existence of a renormalized solutions for elliptic equations of the type

$$\begin{gathered} -\operatorname{div}(a(x, u, \nabla u)) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with the data f in $L^1(\Omega)$. The operator $-\operatorname{div}(a(x, u, \nabla u))$ is a Leray-Lions operator defined on the weighted sobolev spaces $W_0^{1,p}(\Omega, w)$, but which is not restricted by any growth condition with respect to u (see assumptions (2.2), (2.4), and (2.5) of Section 3). The function $a(x, s, \xi)$ is controlled by a real function $b :]-\infty, m[\to \mathbb{R}$ which blows up for a finite value $m > 0$ (see (2.2), (2.3)).

There are mainly two types of difficulties that are studying Problem (1.1). One consists to give a sense to the flux $a(x, u, \nabla u)$ on the set $\{x \in \Omega; u(x) = m\}$. The second one is that the data f only belong to L^1, so that proving existence of a weak solution (i.e., in the distribution meaning) seems to be an arduous task. To overcome this difficulty we use in this paper the framework of renormalized solutions. This notion was introduced by DiPerna and Lions [1] for the study of Boltzmann equation (see also Lions [2] for a few applications to fluid mechanics models). This notion was then adapted to elliptic vesion of (1.1) in Boccardo et al. [3], and Murat [4, 5] (see also [6, 7] for nonlinear parabolic problems). At the same time the equivalent notion of entropy solutions has been developed independently by Bénilan et al. [8] for the study of nonlinear elliptic problems.

In the case where $a(x,u,\nabla u)$ is replaced by $(d(u)+A(u))\nabla u$ (problems with diffusion matrices that have at least one diagonal coefficient that blows up for a finite value of the unknown) and $f\in L^2(\Omega)$, existence and uniqueness has been established in Blanchard and Redwane [9, 10].

As far as we have the stationary and evolution equations case (1.1), the existence and a partial uniqueness of renormalized solutions have been proved in Blanchard et al. [11] in the case where $a(x,u,\nabla u)$ is replaced by $A(x,u)\nabla u$ (where $A(x,s)$ is a Carathéodory symmetric matrices, such that $A(x,s)$ blows up as $s\to m^-$ uniformly with respect to x). It has also been applied to the study of linear and nonlinear elliptic and parabolic equations when the diffusion coefficient has a singularity for a finite value of the unknown (see García Vázquez and Ortegón Gallego [12, 13] and Orsina [14]).

The paper is organized as follows. In Section 2 we will precise some basic properties of weighted Sobolev spaces. Section 3 is devoted to specify the assumptions on $a(x,s,\xi)$, $b(s)$, and f needed in the present study and gives the definition of a renormalized solution of (1.1). In Section 4 (Theorem 4.1) we establish the existence of such a solution.

2. Preliminaries

Throughout the paper, we assume that the following assumptions hold true. Ω is a bounded open subset on $\mathbb{R}^N$, $N\geq 1$. Let us suppose that $1<p<\infty$ is a real number, and $\omega(x)=\{\omega_i(x)\}_{\{0\leq i\leq N\}}$ is a vector of weight functions. Furthermore we suppose that every component $\omega_i(x)$ is a measurable function which is strictly positive and satisfies

$$\omega_i\in L^1_{\mathrm{loc}}(\Omega),\qquad \omega_i^{-1/(p-1)}\in L^1_{\mathrm{loc}}(\Omega). \tag{2.1}$$

We define the weighted Lebesgue space $L^p(\Omega,\omega_0)$ with weight ω_0, as the space of all real-valued measurable functions u for which

$$\|u\|_{p,\omega_0}=\left(\int_\Omega |u(x)|^p\omega_0(x)dx\right)^{1/p}<+\infty. \tag{2.2}$$

In order to define the weighted Sobolev space of $W^{1,p}(\Omega,\omega)$, as the space of all real-valued functions $u\in L^p(\Omega,\omega_0)$ such that the derivaties in the sense of distributions satisfy $\partial u/\partial x_i\in L^p(\Omega,\omega_i)$ for all $i=1,\dots,N$. This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,\omega}=\left(\int_\Omega |u(x)|^p\omega_0(x)dx+\sum_{i=1}^N\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^p\omega_i(x)dx\right)^{1/p}. \tag{2.3}$$

To deal with the Dirichlet problem, we use the space $X = W_0^{1,p}(\Omega,\omega)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p,\omega}$. Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega,\omega)$ and $(W_0^{1,p}(\Omega,\omega), \|\cdot\|_{1,p,\omega})$ is a reflexive Banach space. Note that the expression

$$\|u\|_X = \left(\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p \omega_i(x)dx\right)^{1/p} \tag{2.4}$$

is a norm defined on X and is equivalent to the norm (2.3). Moreover $(X, \|\cdot\|_X)$ is a reflexive Banach space, and there exist a weight function σ on Ω and a parameter $1 < q < \infty$ such that the Hardy inequality

$$\left(\int_\Omega |u|^q\sigma(x)dx\right)^{1/q} \leq C\left(\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p \omega_i(x)dx\right)^{1/p} \tag{2.5}$$

holds for every $u \in X$ with a constant $C > 0$ independent of u. Moreover, the imbedding $X \hookrightarrow L^q(\Omega,\sigma)$ is compact.

We recall that the dual of the weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega)$ is equivalent to $W^{-1,p'}(\Omega,\omega^*)$, where $\omega^* = \{\omega_i^* = \omega_i^{1-p'}; i = 1,\dots,N\}$ and $p' = p/(p-1)$ is the conjugate of p. For more details we refer the reader to [15] (see also [16]).

3. Assumptions on the Data and Definition of a Renormalized Solution

Throughout the paper, we assume that the following assumptions hold true. Ω is a bounded open set on $\mathbb{R}^N$, $N \geq 1$. Let $1 < p < \infty$, and let $\omega(x) = \{\omega_i(x)\}_{\{0\leq i\leq N\}}$ be a vector of weight functions.

Let now $-\operatorname{div}(a(x,u,\nabla u))$ be a Leray-Lions operator defined on $W_0^{1,p}(\Omega,\omega)$ into $W^{-1,p'}(\Omega,\omega^*)$ and where

$$a : \Omega\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N \text{ is a Carathéodory function, such that} \tag{3.1}$$

there exists a positive function $b \in C^0((-\infty,m))$ which satisfies

$$\begin{gathered}\lim_{r\to m^-} b(r) = +\infty; \quad \int_0^m b(s)ds < +\infty, \quad b(r) \geq \alpha > 0 \ \forall r \in]-\infty,m[, \\ a(x,s,\xi)\cdot\xi \geq b(s)^{p-1}\sum_{i=1}^{N}\omega_i(x)|\xi_i|^p, \quad a(x,s,0) = 0,\end{gathered} \tag{3.2}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

For any $i = 1,\dots,N$,

$$|a_i(x,s,\xi)| \leq \omega_i(x)^{1/p}\left[L(x) + \sigma(x)^{1/p'}|s|^{q/p'} + b(s)^{p-1}\sum_{j=1}^{N}\omega_j^{1/p'}(x)\left|\xi_j\right|^{p-1}\right], \tag{3.3}$$

for almost every $x \in \Omega$, for every s and ξ, and where $L(x)$ is a positive function in $L^{p'}(\Omega)$

$$[a(x,s,\xi) - a(x,s,\xi')][\xi - \xi'] \geq 0, \tag{3.4}$$

for any $\xi, \xi' \in \mathbb{R}^N$, for any $s \in \mathbb{R}$ and for almost every $x \in \Omega$

$$f \text{ is an element of } L^1(\Omega). \tag{3.5}$$

Remark 3.1. As already mentioned in the introduction, problem (1.1) does not admit a weak solution under assumptions (3.1)–(3.5) since the growth of $a(x,u,\nabla u)$ is not controlled with respect to u, the field $a(x,u,\nabla u)$ is not, in general, defined as a distribution because the difficulty is defining the field $a(x,u,\nabla u)$ on the subset $\{x \in \Omega; u(x) = m\}$ of Ω, (since on this set, $b(u) = +\infty$).

The following notations will be used throughout the paper. For any $K \geq 0$, the truncation at height K is defined by $T_K(r) = \max(-K, \min(r,K))$, for any positive numbers l and K, the functions T_l^K are defined by

$$T_l^K(r) = \begin{cases} -K, & \text{if } r \leq -K, \\ r, & \text{if } -K \leq r \leq l, \\ l, & \text{if } r \geq l. \end{cases} \tag{3.6}$$

We define for $n \geq 1$ fixed

$$\theta_n(r) = T_1(r - T_n(r)) = \begin{cases} 0, & \text{if } |r| \leq n, \\ r - n\ sg(s), & \text{if } n \leq |r| \leq n+1, \\ sg(s), & \text{if } |r| \geq n+1, \end{cases} \tag{3.7}$$

and $S_n(r) = 1 - |\theta_n(r)|$, for all $r \in \mathbb{R}$.

The definition of a renormalized solution for Problem (1.1) can be stated as follows.

Definition 3.2. A measurable function u defined on Ω is a renormalized solution of Problem (1.1) if

$$T_K(u) \in W_0^{1,p}(\Omega,\omega) \quad \forall K \geq 0, \tag{3.8}$$

$$u(x) \leq m \quad \text{for almost every } x \in \Omega, \tag{3.9}$$

$$a\left(x, T_m^K(u), \nabla T_m^K(u)\right)\chi_{\{u<m\}} \in \prod_{i=1}^{N} L^{p'}\left(\Omega, w_i^{1-p'}\right), \tag{3.10}$$

$$\int_{\{-n-1\le u(x)\le -n\}} a(x,u,\nabla u)\nabla u dx \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \tag{3.11}$$

$$\frac{1}{\delta}\int_{\{m-2\delta\le u(x)\le m-\delta\}} a(x,u,\nabla u)\nabla u dx \longrightarrow \int_{\{u=m\}} f dx \quad \text{as } \delta \longrightarrow 0, \tag{3.12}$$

and if, for every function S in $W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp}(S)$ is compact and $S(m)=0$, u satisfies

$$\int_\Omega a(x,u,\nabla u)\nabla(S(u)\varphi)dx = \int_\Omega fS(u)\varphi dx, \quad \forall\varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega). \tag{3.13}$$

The following remarks are concerned with Definition 3.2.

Remark 3.3. Notice that, thanks to our regularity assumptions (3.8), (3.9), (3.10) and the choice of S, all terms in (3.13) are well defined.

The following two identifications are made in (3.13):

(i) $a(x,u,\nabla u)\nabla(S(u)\varphi)$ identifies with $a(x,T_m^K(u),\nabla T_m^K(u))\nabla(S(u)\varphi)$ for almost every $x\in\Omega$, where $K>0$ and $\operatorname{supp}(S)\subset[-K,K]$. As a consequence of (3.8), (3.9), and (3.10), and of $S\in W^{1,\infty}(\mathbb{R})$, $\varphi\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$, it follows that

$$a\Big(x,T_m^K(u),\nabla T_m^K(u)\Big)\nabla(S'(u)\varphi)\in L^1(\Omega), \tag{3.14}$$

(ii) $fS(u)\varphi\in L^1(\Omega)$, because $f\in L^1(\Omega)$ and $S(u)\varphi\in L^\infty(\Omega)$.

4. Existence Result

This section is devoted to establish the following existence theorem.

Theorem 4.1. *Under assumptions* (3.1)–(3.5) *there exists a renormalized solution u of Problem* (1.1)*.*

Proof. The proof is divided into 7 steps. In Step 1, we introduce an approximate problem. Step 2 is devoted to establish a few a priori estimates, the limit u of the approximate solutions u^ε is introduced and it is shown that u satisfies (3.8) and (3.9). Step 3 is devoted to prove an energy estimate (Lemma 4.2) which is a key point for the monotonicity arguments that are developed in Step 4. Step 5 is devoted to prove that u satisfies (3.11). In Step 6 we prove that u satisfies (3.12). Finally, Step 7 is devoted to prove that u satisfies (3.13) of Definition 3.2. □

Step 1. Let us introduce the following regularization of the data:

$$b^\varepsilon(r) = b\Big(T_{m-\varepsilon}^{1/\varepsilon}(r)\Big) \quad \forall r\in\mathbb{R} \text{ for } \varepsilon>0, \tag{4.1}$$

$$a^\varepsilon(x,s,\xi) = a\Big(x,T_{m-\varepsilon}^{1/\varepsilon}(s),\xi\Big) \quad \text{a.e. in } \Omega,\ \forall s\in\mathbb{R},\ \forall\xi\in\mathbb{R}^N, \tag{4.2}$$

$$f^\varepsilon\in L^{p'}(\Omega); \quad \|f^\varepsilon\|_{L^1(\Omega)}\le\|f\|_{L^1(\Omega)} : f^\varepsilon\longrightarrow f \text{ strongly in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0. \tag{4.3}$$

Let us now consider the following regularized problem:

$$-\operatorname{div}(a^{\varepsilon}(x,u^{\varepsilon},\nabla u^{\varepsilon})) = f^{\varepsilon} \quad \text{in } \Omega, \tag{4.4}$$

$$u^{\varepsilon} = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

In view of (3.3), (4.1), and (4.2), a^{ε} satisfy. For $i = 1,\dots,N$

$$\left|a_i^{\varepsilon}(x,s,\xi)\right| \leq \omega_i(x)^{1/p}\left[L(x)+\sigma(x)^{1/p'}\left|T_{m-\varepsilon}^{1/\varepsilon}(s)\right|^{q/p'}+b^{\varepsilon}(s)^{p-1}\sum_{j=1}^{N}\omega_j^{1/p'}(x)\left|\xi_j\right|^{p-1}\right] \tag{4.6}$$

a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. And

$$\alpha \leq b^{\varepsilon}(r) \leq \max_{\{-1/\varepsilon\leq r\leq m-\varepsilon\}} b(r) = C_{\varepsilon} \quad \forall r \in \mathbb{R}. \tag{4.7}$$

As a consequence, proving existence of a weak solution $u^{\varepsilon} \in W_0^{1,p}(\Omega,\omega)$ of (4.4) and (4.5) is an easy task (see, e.g., Theorem 2.1 and Remark 2.1 in Chapter 2 of [17] and see also [18]).

Step 2. A priori estimates and pointwise convergence of u^{ε}.

Using $T_K(u^{\varepsilon})$ as a test function in (4.4) leads to

$$\int_{\Omega} a^{\varepsilon}(x,u^{\varepsilon},\nabla u^{\varepsilon})\nabla T_K(u^{\varepsilon})dx = \int_{\Omega} f^{\varepsilon}T_K(u^{\varepsilon})dx \leq K\|f\|_{L^1(\Omega)}. \tag{4.8}$$

Since a^{ε} satisfies (3.2), (4.2), and owing to (4.8) we have

$$\int_{\Omega} b^{\varepsilon}(u^{\varepsilon})^{p-1}\sum_{i=1}^{N}\left|\frac{\partial T_K(u^{\varepsilon})}{\partial x_i}\right|^p\omega_i(x)dx \leq K\|f\|_{L^1(\Omega)}, \tag{4.9}$$

$$\alpha^{p-1}\int_{\Omega}\sum_{i=1}^{N}\left|\frac{\partial T_K(u^{\varepsilon})}{\partial x_i}\right|^p\omega_i(x)dx \leq K\|f\|_{L^1(\Omega)}. \tag{4.10}$$

From (4.10) we deduce with a classical argument (see, e.g., [18]) that, for a subsequence still indexed by ε,

$$u^{\varepsilon} \longrightarrow u \quad \text{a.e. in } \Omega, \tag{4.11}$$

$$T_K(u^{\varepsilon}) \longrightarrow T_K(u) \text{ weakly in } W_0^{1,p}(\Omega,\omega) \text{ and strongly in } L^q(\Omega,\sigma), \tag{4.12}$$

as ε tends to 0, where u is a measurable function defined on Ω which is finite a.e. in Ω.

Taking now $Z^\varepsilon = \int_0^{T_m^K(u^\varepsilon)} b^\varepsilon(s)ds$ as a test function in (4.4) gives

$$\int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla Z^\varepsilon dx = \int_\Omega f^\varepsilon Z^\varepsilon dx. \tag{4.13}$$

Since a^ε satisfies (3.2) and b satisfies (3.1), permit to deduce from (4.13) that

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial Z^\varepsilon}{\partial x_i}\right|^p \omega_i(x)dx \le C_K \|f\|_{L^1(\Omega)}, \tag{4.14}$$

where $|Z^\varepsilon| \le \int_{-K}^m b(s)ds = C_K$ is a constant independent of ε.

Now for a fixed $K > 0$, assumption (3.3) gives for $i = 1, \dots, N$,

$$\begin{aligned}&\left|a_i^\varepsilon\left(x, T_m^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon)\right)\right| \\ &\quad \le \omega_i(x)^{1/p}\left[L(x) + \sigma(x)^{1/p'}\max(K,m)^{q/p'} + \sum_{j=1}^N \omega_j^{1/p'}(x)\left|\frac{\partial Z^\varepsilon}{\partial x_j}\right|^{p-1}\right].\end{aligned} \tag{4.15}$$

In view of (4.14) and (4.15), we deduce that

$$a^\varepsilon\left(x, T_m^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon)\right) \text{ is bounded in } \prod_{i=1}^N L^{p'}\left(\Omega, w_i^{1-p'}\right), \tag{4.16}$$

then there exists a function $X_K \in \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a^\varepsilon\left(x, T_m^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon)\right) \rightharpoonup X_K \text{ weakly in } \prod_{i=1}^N L^{p'}\left(\Omega, w_i^{1-p'}\right) \quad \text{as } \varepsilon \longrightarrow 0. \tag{4.17}$$

To prove that u is less or equal to m is an easy task which is performed exactly as in [10, 11]. Using $T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)$ as a test function in (4.4) leads to

$$\int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla\left(T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right)dx = \int_\Omega f^\varepsilon\left(T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right)dx, \tag{4.18}$$

which implies easily that

$$\int_\Omega a^\varepsilon\left(x, u^\varepsilon, \nabla\left(T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right)\right)\nabla\left(T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right)dx \le m\|f\|_{L^1(\Omega)}. \tag{4.19}$$

Then (3.2), (4.1), and (4.2) yield

$$b(m-\varepsilon)^{p-1}\int_\Omega \sum_{i=1}^N \left|\frac{\partial\left(T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right)}{\partial x_i}\right|^p \omega_i(x)dx \le m\|f\|_{L^1(\Omega)}. \tag{4.20}$$

With the help of Poincaré's inequality, we have

$$\int_{\Omega}\left|T_{2m}^{+}(u^{\varepsilon})-T_{m}^{+}(u^{\varepsilon})\right|^{p}\omega_0(x)dx\leq\frac{Cm}{b(m-\varepsilon)^{p-1}\|f\|_{L^1(\Omega)}},\tag{4.21}$$

where C does not depend on ε. Then in view of (3.1), (4.11), and $\omega_0>0$, we can pass to the limit in (4.21) as ε tends to 0, to deduce that

$$\begin{gathered}T_{2m}^{+}(u)-T_{m}^{+}(u)=0\quad\text{a.e. in }\Omega,\\ u\leq m\quad\text{a.e. in }\Omega.\end{gathered}\tag{4.22}$$

Let us now take $T_K(v^{\varepsilon})$ as a test function in (4.4), where $v^{\varepsilon}=\int_0^{u^{\varepsilon}}b^{\varepsilon}(s)ds$. We obtain

$$\int_{\Omega}a^{\varepsilon}(x,u^{\varepsilon},\nabla u^{\varepsilon})\nabla T_K(v^{\varepsilon})dx=\int_{\Omega}f^{\varepsilon}T_K(v^{\varepsilon})dx\leq K\|f\|_{L^1(\Omega)}.\tag{4.23}$$

Then (3.2) yields

$$\int_{\Omega}\sum_{i=1}^{N}\left|\frac{\partial T_K(v^{\varepsilon})}{\partial x_i}\right|^{p}\omega_i(x)dx\leq K\|f\|_{L^1(\Omega)}.\tag{4.24}$$

We deduce with a classical argument that, for a subsequence still indexed by ε,

$$v^{\varepsilon}\longrightarrow v\quad\text{a.e. in }\Omega,\tag{4.25}$$

$$T_K(v^{\varepsilon})\longrightarrow T_K(v)\ \text{weakly in }W_0^{1,p}(\Omega,\omega),\tag{4.26}$$

as ε tends to 0, where v is a measurable function defined on Ω which is finite a.e. in Ω.

Using the admissible test function $\theta_n(v^{\varepsilon})$ in (4.4) leads to

$$\int_{\Omega}a^{\varepsilon}(x,u^{\varepsilon},\nabla u^{\varepsilon})\nabla\theta_n(v^{\varepsilon})dx=\int_{\Omega}f^{\varepsilon}\theta_n(v^{\varepsilon})dx.\tag{4.27}$$

As a consequence of the previous convergence results, we are in a position to pass to the limit as ε tends to 0 in (4.27)

$$\lim_{\varepsilon\to 0}\int_{\Omega}a^{\varepsilon}(x,u^{\varepsilon},\nabla u^{\varepsilon})\nabla\theta_n(v^{\varepsilon})dx=\int_{\Omega}f\theta_n(v)dx.\tag{4.28}$$

Using the pointwise convergence of $\theta_n(u)$ to 0 as n tends to $+\infty$ and $|\theta_n(u)| \leq 1$ a.e. in Ω independently of n, since $f \in L^1(\Omega)$, Lebesgue's convergence theorem shows that $\int_\Omega f\theta_n(v)dx \to 0$, as n tends to $+\infty$. Passing to the limit in (4.28) we obtain

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_{\{n\leq|v^\varepsilon|\leq n+1\}} a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla v^\varepsilon dx = 0. \tag{4.29}$$

Step 3. In this step we prove the following monotonicity estimate.

Lemma 4.2. *The subsequence of u^ε defined in Step 1 satisfies for any $K \geq 0$*

$$\lim_{\varepsilon\to 0}\int_\Omega\left[\frac{a^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}} - \frac{a^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}}\right]\left[\nabla T_m^K(u^\varepsilon) - \nabla T_m^K(u)\right]dx = 0. \tag{4.30}$$

Proof. Let $K \geq 0$ be fixed. Equality (4.30) is split into

$$\begin{aligned}&\int_\Omega\left[\frac{a^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}} - \frac{a^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}}\right]\left[\nabla T_m^K(u^\varepsilon) - \nabla T_m^K(u)\right]dx\\ &\quad = A_1^\varepsilon + A_2^\varepsilon + A_3^\varepsilon,\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}A_1^\varepsilon &= \int_\Omega \frac{a^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^K(u^\varepsilon)dx\,ds\,dt,\\ A_2^\varepsilon &= -\int_\Omega \frac{a^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^K(u)dx\,ds\,dt,\\ A_3^\varepsilon &= -\int_\Omega \frac{a^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}}\left(\nabla T_m^K(u^\varepsilon) - \nabla T_m^K(u)\right)dx\,ds\,dt.\end{aligned} \tag{4.32}$$

In the sequel we pass to the limit in (4.31) when ε tends to 0.

Limit of A_1^ε

Using the admissible test function $S_n(v^\varepsilon)\int_0^{T_m^K(u)}(1/b(s)^{p-1})ds$ in (4.4) leads to

$$\begin{aligned}&\int_\Omega S_n(v^\varepsilon)a^\varepsilon(\mathrm{x},u^\varepsilon,\nabla u^\varepsilon)\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx + \int_\Omega a^\varepsilon(u^\varepsilon,\nabla u^\varepsilon)\nabla S_n(v^\varepsilon)\cdot\left(\int_0^{T_m^K(u)}\frac{1}{b(s)^{p-1}}ds\right)dx\\ &\quad = \int_\Omega f^\varepsilon S_n(v^\varepsilon)\int_0^{T_m^K(u)}\frac{1}{b(s)^{p-1}}ds dx,\end{aligned} \tag{4.33}$$

where $v^\varepsilon = \int_0^{u^\varepsilon} b^\varepsilon(s)ds$, pass to the limit as ε tends to 0 in (4.33).

Since $\operatorname{supp}(S_n) \subset [-(n+1), n+1]$ and $\{x \in \Omega; |v^\varepsilon| \le n+1\} \subset \{x \in \Omega; |u^\varepsilon| \le (n+1)/\alpha\}$, we have for $i = 1, \dots, N$ and $\varepsilon \le \alpha/(n+1)$

$$\begin{aligned}\left|a_i^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) S_n(v^\varepsilon)\right| &= \left|a_i^\varepsilon\left(x,\ T_{(n+1)/\alpha}\ (u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right) S_n(v^\varepsilon)\right| \\ &\le \|S_n\|_{L^\infty(\mathbb{R})} \omega_i(x)^{1/p} \Bigg[L(x) + \sigma(x)^{1/p'} \left|T_{(n+1)/\alpha}(u^\varepsilon)\right|^{q/p'} \\ &\qquad + \sum_{j=1}^N \omega_j^{1/p'}(x) \left|\frac{\partial T_{n+1}(v^\varepsilon)}{\partial x_j}\right|^{p-1} \Bigg].\end{aligned} \tag{4.34}$$

In view of (4.24), (4.34) we deduce that for fixed $n \ge 1$:

$$a^\varepsilon\left(x, T_{(n+1)/\alpha}(u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right) S_n(v^\varepsilon) \text{ is bounded in } \prod_{i=1}^N L^{p'}\left(\Omega, w_i^{1-p'}\right), \tag{4.35}$$

independently of $\varepsilon \le \alpha/(n+1)$. Then there exists a function $Y_n \in \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ such that for fixed $n \ge 1$:

$$S_n(v^\varepsilon) a^\varepsilon\left(x, T_{(n+1)/\alpha}(u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right) \rightharpoonup Y_n \text{ weakly in } \prod_{i=1}^N L^{p'}\left(\Omega, w_i^{1-p'}\right) \quad \text{as } \varepsilon \longrightarrow 0. \tag{4.36}$$

Now for $\max(K, m) \le n/\alpha$, we have

$$\begin{aligned}&S_n(v^\varepsilon) a^\varepsilon\left(x, T_{(n+1)/\alpha}(u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right) \chi_{\{-K<u^\varepsilon<m\}} \\ &\quad = S_n(v^\varepsilon) a^\varepsilon\left(x, T_K^m(u^\varepsilon), \nabla T_K^m(u^\varepsilon)\right) \chi_{\{-K<u^\varepsilon<m\}}\end{aligned} \tag{4.37}$$

a.e. in Ω, which implies that, through the use of (4.17), (4.25), and (4.36) and passing to the limit as ε tends to 0,

$$Y_n \chi_{\{-K<u<m\}} = S_n(v) X_K \chi_{\{-K<u<m\}} \tag{4.38}$$

a.e. in $\Omega - \{\{u = -K\} \cup \{u = m\}\}$ for $\max(K, m) \le n/\alpha$. As a consequence of (4.38) we have for $\max(K, m) \le n/\alpha$

$$Y_n \nabla T_m^K(u) = S_n(v) X_K \nabla_m^K(u) \quad \text{a.e. in } \Omega. \tag{4.39}$$

We are now in a position to exploit (4.33), which gives together with (4.36) and (4.39)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega S_n(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx \\
&\quad=\lim_{\varepsilon\to 0}\int_\Omega S_n(v^\varepsilon)a^\varepsilon\left(x,T_{(n+1)/\alpha}(u^\varepsilon),\nabla T_{(n+1)/\alpha}(u^\varepsilon)\right)\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx \\
&\quad=\int_\Omega Y_n\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx \\
&\quad=\int_\Omega S_n(v)X_K\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx.
\end{aligned}\tag{4.40}$$

Passing to the limit as n tends to $+\infty$ in (4.40) leads to

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega S_n(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx=\int_\Omega X_K\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx.\tag{4.41}$$

The second term of (4.33)

$$\begin{aligned}
&\left|\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla S_n(v^\varepsilon).\left(\int_0^{T_m^K(u)}\frac{1}{b(s)^{p-1}}ds\right)dx\right| \\
&\qquad\le\frac{\max(m,K)}{\alpha^{p-1}}\int_{\{n\le|v^\varepsilon|\le n+1\}}a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla v^\varepsilon dx.
\end{aligned}\tag{4.42}$$

Then (4.29) implies that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla S_n(v^\varepsilon).\left(\int_0^{T_m^K(u)}\frac{1}{b(s)^{p-1}}ds\right)dx=0.\tag{4.43}$$

In view (4.41) and (4.43), passing to the limit as ε tends to 0 and as n tends to $+\infty$ in (4.33) is an easy task and leads to

$$\int_\Omega X_K\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx=\int_\Omega f\int_0^{T_m^K(u)}\frac{1}{b(s)^{p-1}}ds\,dx.\tag{4.44}$$

We are now in a position to exploit (4.44).

The use of the test function $\int_0^{T_m^K(u^\varepsilon)}(1/b^\varepsilon(s)^{p-1})ds$ in (4.4), yields

$$\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\frac{\nabla T_m^K(u^\varepsilon)}{b^\varepsilon(u^\varepsilon)^{p-1}}dx=\int_\Omega f^\varepsilon\int_0^{T_m^K(u^\varepsilon)}\frac{1}{b^\varepsilon(s)^{p-1}}ds\,dx.\tag{4.45}$$

Passing to the limit as ε tends to 0 in (4.45), in view (4.44), we have

$$\lim_{\varepsilon\to 0} A_1^\varepsilon = \lim_{\varepsilon\to 0}\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\frac{\nabla T_m^K(u^\varepsilon)}{b^\varepsilon(u^\varepsilon)^{p-1}}dx = \int_\Omega X_K\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx. \tag{4.46}$$

Limit of A_2^ε

In view of (4.12), (4.17) and since $1/b^\varepsilon(u^\varepsilon)^{p-1}$ converges to $1/b(u)^{p-1}$ a.e. in Ω and due to the bound $1/b^\varepsilon(u^\varepsilon)^{p-1} \le 1/\alpha^{p-1}$ a.e. in Ω, we have

$$\lim_{\varepsilon\to 0} A_2^\varepsilon = -\int_\Omega X_K\frac{\nabla T_m^K(u)}{b(u)^{p-1}}dx. \tag{4.47}$$

Limit of A_3^ε

Let us remark that (3.1), (4.1), and (4.11) imply that

$$\frac{a^\varepsilon(x,T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}} \longrightarrow \frac{a(x,T_m^K(u),DT_m^K(u))}{b(u)^{p-1}} \quad \text{a.e. in } \Omega, \tag{4.48}$$

as ε tends to 0, and that for $i=1,\dots,N$

$$\begin{aligned}&\left|\frac{a_i^\varepsilon(T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}}\right| \\ &\quad\le w_i^{1/p}(x)\left[\frac{1}{\alpha^{p-1}}L(x)+\frac{\max(K,m)^{q/p'}}{\alpha^{p-1}}\sigma^{1/p'}(x)+\sum_{j=1}^N w_j^{1/p'}\left|\frac{\partial T_m^K(u)}{\partial x_j}\right|^{p-1}\right]\end{aligned} \tag{4.49}$$

a.e. in Ω, uniformly with respect to ε.

It follows that when ε tends to 0

$$\frac{a(x,T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}} \longrightarrow \frac{a(x,T_m^K(u),\nabla T_K(u))}{b(u)^{p-1}} \text{ strongly in } \prod_{i=1}^N L^{p'}\left(\Omega,w_i^{1-p'}\right). \tag{4.50}$$

In view of (4.12), we conclude that

$$\left(\nabla T_m^K(u^\varepsilon)-\nabla T_m^K(u)\right) \rightharpoonup 0 \text{ weakly in } \prod_{i=1}^N L^p(\Omega,w_i), \text{ as } \varepsilon \text{ goes to } 0. \tag{4.51}$$

As a consequence of (4.50) and (4.51) we have for all $K>0$

$$\lim_{\varepsilon\to 0} A_3^\varepsilon = 0. \tag{4.52}$$

Equations (4.46), (4.46), (4.47), and (4.46) allow to pass to the limit as ε tends to zero in (4.31) and to obtain (4.30) of Lemma 4.2. □

Step 4. In this step we identify the weak limit X_K and we prove the weak L^1 convergence of the "truncated" energy $(a^\varepsilon(x, T_m^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon))/b^\varepsilon(u^\varepsilon)^{p-1})\nabla T_m^K(u^\varepsilon)$ as ε tends to 0.

Lemma 4.3. *For fixed $K \geq 0$, one has*

$$X_K = a\left(x, T_m^K(u), \nabla T_m^K(u)\right) \quad \text{a.e. in } \{x \in \Omega; u(x) < m\}. \tag{4.53}$$

And as ε tends to 0

$$\frac{a^\varepsilon(x, T_m^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}} \nabla T_m^K(u^\varepsilon) \rightharpoonup \frac{a(x, T_m^K(u), \nabla T_m^K(u))}{b(u)^{p-1}} \nabla T_m^K(u) \text{ weakly in } L^1(\Omega). \tag{4.54}$$

Proof. Let $K \geq 0$ be fixed. From (4.11) and (4.50) together with (4.30) of Lemma 4.2, we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{a^\varepsilon(x, T_{\mathrm{m}}^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon))}{b\left(T_m^K(u^\varepsilon)\right)^{p-1}} \nabla T_m^K(u^\varepsilon) dx = \int_\Omega \frac{X_K}{b\left(T_m^K(u)\right)^{p-1}} \nabla T_K(u) dx. \tag{4.55}$$

We remark the monotone character a (with respect to ξ) and since $1/b^\varepsilon(u^\varepsilon)^{p-1}$ converges to $1/b(u)^{p-1}$ a.e. in Ω and due to the bound $1/b^\varepsilon(u^\varepsilon)^{p-1} \leq 1/\alpha^{p-1}$ a.e. in Ω, we conclude that for all $\varphi \in \prod_{i=1}^N L^p(\Omega, w_i)$ we have

$$\begin{aligned}
0 &\leq \lim_{\varepsilon \to 0} \int_\Omega \left[\frac{a^\varepsilon(x, T_m^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}} - \frac{a^\varepsilon(x, T_m^K(u^\varepsilon), \varphi)}{b^\varepsilon(u^\varepsilon)^{p-1}}\right]\left[\nabla T_m^K(u^\varepsilon) - \varphi\right] dx \\
&= \lim_{\varepsilon \to 0} \int_\Omega \frac{a^\varepsilon(x, T_m^K(u^\varepsilon), \nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}} \left[\nabla T_m^K(u^\varepsilon) - \varphi\right] dx \\
&\quad - \lim_{\varepsilon \to 0} \int_\Omega \frac{a^\varepsilon(x, T_m^K(u^\varepsilon), \varphi)}{b^\varepsilon(u^\varepsilon)^{p-1}} \left[\nabla T_m^K(u^\varepsilon) - \varphi\right] dx \\
&= \int_\Omega \frac{X_K}{b(u)^{p-1}} \left[\nabla T_m^K(u) - \varphi\right] dx - \int_\Omega \frac{a(x, T_m^K(u), \varphi)}{b(u)^{p-1}} \left[\nabla T_m^K(u) - \varphi\right] dx \\
&\quad \times \int_\Omega \left[\frac{X_K}{b(u)^{p-1}} - \frac{a(x, T_m^K(u), \varphi)}{b(u)^{p-1}}\right]\left[\nabla T_m^K(u) - \varphi\right] dx.
\end{aligned} \tag{4.56}$$

The usual Minty's argument applies in view of (4.56). It follows that

$$\frac{X_K}{b(u)^{p-1}} = \frac{a(x, T_m^K(u), \nabla T_m^K(u))}{b(u)^{p-1}} \quad \text{a.e. in } \Omega \tag{4.57}$$

which together with (4.20) yields (4.53) of Lemma 4.3.

In order to prove (4.54), we observe that the monotone character of a (with respect to ξ) and (4.30) give

$$\left[\frac{a^\varepsilon(x,T_m^K(u^\varepsilon),\nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}}-\frac{a^\varepsilon(x,T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}}\right]\left[\nabla T_m^K(u^\varepsilon)-\nabla T_m^K(u)\right]\longrightarrow 0 \tag{4.58}$$

strongly in $L^1(\Omega)$ as ε tends to 0. Moreover (4.12), (4.17), (4.50), and (4.53) imply that

$$\frac{a^\varepsilon(x,T_m^K(u^\varepsilon),\nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^K(u)\rightharpoonup\frac{a(x,T_m^K(u),\nabla T_m^K(u))}{b(u)^{p-1}}\nabla T_m^K(u) \tag{4.59}$$

weakly in $L^1(\Omega)$ as ε tends to 0

$$\frac{a^\varepsilon(x,T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^K(u^\varepsilon)\rightharpoonup\frac{a(x,T_m^K(u),\nabla T_m^K(u))}{b(u)^{p-1}}\nabla T_m^K(u) \tag{4.60}$$

weakly in $L^1(\Omega)$ as ε tends to 0, and

$$\frac{a^\varepsilon(x,T_m^K(u^\varepsilon),\nabla T_m^K(u))}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^K(u)\rightharpoonup\frac{a(x,T_m^K(u),\nabla T_m^K(u))}{b(u)^{p-1}}\nabla T_m^K(u) \tag{4.61}$$

strongly in $L^1(\Omega)$ as ε tends to 0.

Using the above convergence results (4.59), (4.60), and (4.61) in (4.58) we obtain that for any $K\geq 0$

$$\frac{a^\varepsilon(x,T_m^K(u^\varepsilon),\nabla T_m^K(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^K(u^\varepsilon)\rightharpoonup\frac{a(x,T_m^K(u),\nabla T_m^K(u))}{b(u)^{p-1}}\nabla T_m^K(u) \tag{4.62}$$

weakly in $L^1(\Omega)$ as ε tends to 0. □

Step 5. In this step we prove that u satisfies (3.11).

Using $(T_m^{n+1}(u^\varepsilon)-T_m^n(u^\varepsilon))$ as a test function in (4.4) leads to

$$\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\left(T_m^{n+1}(u^\varepsilon)-T_m^n(u^\varepsilon)\right)dx=\int_\Omega f^\varepsilon\left(T_m^{n+1}(u^\varepsilon)-T_m^n(u^\varepsilon)\right)dx. \tag{4.63}$$

Since $\operatorname{supp}(T_m^{n+1}(\cdot) - T_m^n(\cdot)) \subset [-(n+1), -n]$, we have

$$
\begin{aligned}
\int_{\{-n-1\le u^\varepsilon(x)\le -n\}} & a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon dx \\
&= \int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\Big(T_m^{n+1}(u^\varepsilon) - T_m^n(u^\varepsilon)\Big)dx \\
&= \int_\Omega \frac{a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)}{b^\varepsilon(u^\varepsilon)^{p-1}}\ \nabla\Big(T_m^{n+1}(u^\varepsilon) - T_m^n(u^\varepsilon)\Big)\ b\Big(T_{m-1}^{n+1}(u^\varepsilon)\Big)^{p-1}dx \\
&= \int_\Omega \frac{a^\varepsilon\big(x,T_m^{n+1}(u^\varepsilon),\nabla T_m^{n+1}(u^\varepsilon)\big)}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^{n+1}(u^\varepsilon) b\Big(T_{m-1}^{n+1}(u^\varepsilon)\Big)^{p-1}dx \\
&\quad - \int_\Omega \frac{a^\varepsilon(x,T_m^{n}(u^\varepsilon),\nabla T_m^{n}(u^\varepsilon))}{b^\varepsilon(u^\varepsilon)^{p-1}}\nabla T_m^{n}(u^\varepsilon) b\Big(T_{m-1}^{n+1}(u^\varepsilon)\Big)^{p-1}dx.
\end{aligned}
\tag{4.64}
$$

In view of (4.54) of Lemma 4.3 and since $b(T_{m-1}^{n+1}(u^\varepsilon))^{p-1}$ converges to $b(T_{m-1}^{n+1}(u))^{p-1}$ a.e. in Ω and due to the bound $b(T_{m-1}^{n+1}(u^\varepsilon))^{p-1} \le \max_{s\in[-n-1,m-1]} b(s)^{p-1}$ a.e. in Ω, we can pass to the limit as ε tends to 0 for fixed $n \ge 0$ to obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\{-n-1\le u^\varepsilon(x)\le -n\}} & a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon dx \\
&= \int_\Omega \frac{a\big(x,T_m^{n+1}(u),\nabla T_m^{n+1}(u)\big)}{b(u)^{p-1}}\nabla T_m^{n+1}(u) b\Big(T_{m-1}^{n+1}(u)\Big)^{p-1}dx \\
&\quad - \int_\Omega \frac{a(x,T_m^{n}(u),\nabla T_m^{n}(u))}{b(u)^{p-1}}\nabla T_m^{n}(u) b\Big(T_{m-1}^{n+1}(u)\Big)^{p-1}dx \\
&= \int_{\{-n-1\le u(x)\le -n\}} a(x,u,\nabla u)\nabla u dx.
\end{aligned}
\tag{4.65}
$$

Taking the limit as ε tends to 0 and n tends to $+\infty$ in (4.63) and using the estimate (4.64) and (4.65) show that

$$
\lim_{n\to+\infty}\int_{\{-n-1\le u(x)\le -n\}} a(x,u,\nabla u)\nabla u dx \le \lim_{n\to+\infty}\int_{\{u\le -n\}} |f| dx = 0. \tag{4.66}
$$

Step 6. In this step we prove that u satisfies (3.12).

Using $S_n(v^\varepsilon)(1/\delta)(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u))$ as a test function in (4.4) leads to

$$
\begin{aligned}
\frac{1}{\delta}\int_\Omega & S_n(v^\varepsilon) a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\big(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u)\big)dx \\
&= \int_\Omega S_n(v^\varepsilon) f^\varepsilon \frac{\big(T_{m-\delta}^+(u) - T_{m-2\delta}^+(u)\big)}{\delta}dx,
\end{aligned}
\tag{4.67}
$$

where $v^\varepsilon = \int_0^{u^\varepsilon} b^\varepsilon(s)ds$. Since $\operatorname{supp}(S_n) \subset [-(n+1), n+1]$ and $\{x \in \Omega; |v^\varepsilon| \le n+1\} \subset \{x \in \Omega; |u^\varepsilon| \le (n+1)/\alpha\}$ we have

$$\begin{aligned}
&\frac{1}{\delta}\int_\Omega S_n(v^\varepsilon) a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right)dx \\
&\quad = \frac{1}{\delta}\int_\Omega S_n(v^\varepsilon) a^\varepsilon\left(x, T_{(n+1)/\alpha}(u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right)\nabla\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right)dx.
\end{aligned} \tag{4.68}$$

In view of (4.22), (4.36), (4.39), and (4.53), passing to the limit as ε tends to 0 and n tends to $+\infty$

$$\begin{aligned}
&\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\frac{1}{\delta}\int_\Omega S_n(v^\varepsilon) a^\varepsilon\left(x, T_{(n+1)/\alpha}(u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right)\nabla\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right)dx \\
&\quad = \lim_{n\to+\infty}\frac{1}{\delta}\int_\Omega S_n(v) a\left(x, T_{(n+1)/\alpha}(u), \nabla T_{(n+1)/\alpha}(u)\right)\nabla\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right)dx \\
&\quad = \lim_{n\to+\infty}\frac{1}{\delta}\int_\Omega S_n(v) a(x, T_{m-\delta}(u), \nabla T_{m-\delta}(u))\nabla\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right)dx \\
&\quad = \frac{1}{\delta}\int_\Omega a(x, T_{m-\delta}(u), \nabla T_{m-\delta}(u))\nabla\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right)dx \\
&\quad = \frac{1}{\delta}\int_{\{m-2\delta\le u\le m-\delta\}} a(x, u, \nabla u)\nabla u dx.
\end{aligned} \tag{4.69}$$

Taking the limit as ε tends to 0, n tends to $+\infty$ and δ tends to 0 in (4.67) and using the estimate (4.68) and (4.69) show that

$$\begin{aligned}
\lim_{\delta\to 0}\frac{1}{\delta}\int_{\{m-2\delta\le u\le m-\delta\}} a(x, u, \nabla u)\nabla u dx &= \lim_{\delta\to 0}\int_\Omega f\frac{\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right)}{\delta}dx \\
&= \int_{\{u=m\}} f(x)dx.
\end{aligned} \tag{4.70}$$

Step 7. In this step, u is shown to satisfy (3.13). Let $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ and let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support and $S(m) = 0$. Let K be a positive real number such that $\operatorname{supp}(S) \subset [-K, K]$ and $v^\varepsilon = \int_0^{u^\varepsilon} b^\varepsilon(s)ds$. Using $S(u)S_n(v^\varepsilon)\varphi$ as a test function in (4.4) leads to

$$\begin{aligned}
&\int_\Omega S_n(v^\varepsilon) a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla(S(u)\varphi)dx + \int_\Omega S(u)\varphi a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla S_n(v^\varepsilon)dx \\
&\quad = \int_\Omega f^\varepsilon S_n(v^\varepsilon) S(u)\varphi dx.
\end{aligned} \tag{4.71}$$

In what follows we pass to the limit as ε tends to 0 and n tends to $+\infty$ in each term of (4.71).

Limit of First Term in (4.71)

Since supp $S_n \subset [-(n+1), n+1]$ and $\{x \in \Omega; |v^\varepsilon| \leq n+1\} \subset \{x \in \Omega; |u^\varepsilon| \leq (n+1)/\alpha\}$, we have

$$S_n(v^\varepsilon) a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) = S_n(v^\varepsilon) a^\varepsilon\left(x, T_{(n+1)/\alpha}(u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right) \quad \text{a.e. in } \Omega. \tag{4.72}$$

In view of (4.22), (4.36), (4.39), and (4.53), passing to the limit as ε tends to 0

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_\Omega S_n(v^\varepsilon) a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla\left(S(u)\varphi\right) dx \\
&\quad = \lim_{\varepsilon \to 0} \int_\Omega S_n(v^\varepsilon) a^\varepsilon\left(x, T_{(n+1)/\alpha}(u^\varepsilon), \nabla T_{(n+1)/\alpha}(u^\varepsilon)\right) \nabla\left(S(u)\varphi\right) dx \\
&\quad = \int_\Omega S_n(v) a\left(x, T_{(n+1)/\alpha}(u), \nabla T_{(n+1)/\alpha}(u)\right) \nabla\left(S(u)\varphi\right) dx \\
&\quad = \int_\Omega S_n(v) a\Big(x, T_m^K(u), \nabla T_m^K(u)\Big) \nabla\left(S(u)\varphi\right) dx, \\
&\lim_{n \to +\infty} \lim_{\varepsilon \to 0} \int_\Omega S_n(v^\varepsilon) a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla\left(S(u)\varphi\right) dx \\
&\quad = \lim_{n \to +\infty} \int_\Omega S_n(v) a\Big(x, T_m^K(u), \nabla T_m^K(u)\Big) \nabla\left(S(u)\varphi\right) dx \\
&\quad = \int_\Omega a\Big(x, T_m^K(u), \nabla T_m^K(u)\Big) \nabla\left(S(u)\varphi\right) dx = \int_\Omega a(x, u, \nabla u) \nabla\left(S(u)\varphi\right) dx.
\end{aligned} \tag{4.73}$$

Limit of Second Term in (4.71)

Since $\operatorname{supp}(S_n') \subset [-(n+1), -n] \cup [n+1, n]$ for any $n \geq 1$. As a consequence

$$\left| \int_\Omega S(u)\varphi a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla S_n(v^\varepsilon) dx \right| \leq \|S\|_{L^\infty(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \int_{\{n \leq |v^\varepsilon| \leq n+1\}} a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla v^\varepsilon dx. \tag{4.74}$$

Taking the limit as ε tends to 0 and n tends to $+\infty$ in (4.74) and using the estimate (4.29) show that

$$\lim_{n \to +\infty} \lim_{\varepsilon \to 0} \left| \int_\Omega S(u)\varphi a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla S_n(v^\varepsilon) dx \right| = 0. \tag{4.75}$$

Limit of the Right-Hand Side of (4.71)

Due to (4.3) and (4.25), we have

$$\lim_{n \to +\infty} \lim_{\varepsilon \to 0} \int_\Omega f^\varepsilon S_n(v^\varepsilon) S(u) \varphi dx = \int_\Omega f S(u) \varphi dx. \tag{4.76}$$

As a consequence of the previous convergence results, we are in a position to pass to the limit as ε tends to 0 in (4.71) and to conclude that u satisfies (3.13). The proof of Theorem 4.1 is achieved.

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