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Research Article

Some Properties of (r, s)-T_0 and (r, s)-T_1 Spaces

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We define (r, s)-quasi-T_0, (r, s)-sub-T_0, (r, s)-T_0 and (r, s)-T_1 spaces in an intuitionistic fuzzy topological space and investigate some properties of these spaces and the relationships between them. Moreover, we study properties of subspaces and their products.

1. Introduction and preliminaries

Kubiak [1] and Šostak [2] introduced thefundamental concept of a fuzzytopological structure, as an extension of bothcrisp topology and fuzzy topology [3], in the sensethat not only the objectsare fuzzified, but also the axiomatics. In [4, 5], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al. [6] have redefined the same concept under the name gradation of openness. A general approach to the study of topological-type structures on fuzzy power sets was developed in [1, 7–10].

As a generalization of fuzzy sets, the notion of intuitionisticfuzzy sets was introduced by Atanassov [11]. By using intuitionistic fuzzy sets, Çoker [12], and Çoker and Dimirci [13] defined the topology of intuitionisticfuzzy sets. Recently, Mondal and Samanta [14] introduced the notion of intuitionistic gradation of openness of fuzzy sets, where to each fuzzy subset there is a definite grade of openness and there is a grade of nonopenness. Thus, the concept of intuitionistic gradation of openness is a generalization of the concept of gradation of openness and the topology of intuitionistic fuzzy sets.

In this paper, we define (r, s)-quasi-T_0, (r, s)-sub-T_0, (r, s)-T_0, and (r, s)-T_1 spaces in an intuitionistic fuzzy topological space and investigate some properties of these spaces and the relationships between them. Moreover, we study properties of subspaces and their products.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$, and $I_0 = (0, 1]$ and $I_1 = [0, 1)$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A *fuzzy point* x_t for $t \in I_0$ is an element of I^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases} \tag{1.1}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ if and only if $t < \lambda(x)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we denote it by $\lambda \overline{q} \mu$.

Definition 1.1 (see [14]). An intuitionistic gradation of openness (IGO, for short) on X is an ordered pair (τ, τ^*) of functions from I^X to I such that

(IGO1) $\tau(\lambda) + \tau^*(\lambda) \leq 1$, for all $\lambda \in I^X$,

(IGO2) $\tau(\underline{0}) = \tau(\underline{1}) = 1$, $\tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,

(IGO3) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in I^X$,

(IGO4) $\tau(\bigvee_{i\in\Delta}\lambda_i) \geq \bigwedge_{i\in\Delta}\tau(\lambda_i)$ and $\tau^*(\bigvee_{i\in\Delta}\lambda_i) \leq \bigvee_{i\in\Delta}\tau^*(\lambda_i)$, for each $\lambda_i \in I^X$, $i \in \Delta$.

The triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space (IFTS, for short). τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

An IFTS (X, τ, τ^*) is called stratified if
(S) $\tau(\underline{\alpha}) = 1$ and $\tau^*(\underline{\alpha}) = 0$ for each $\alpha \in I$.

Let $(\mathcal{U}, \mathcal{U}^*)$ and (τ, τ^*) be IGOs on X. We say $(\mathcal{U}, \mathcal{U}^*)$ is finer than (τ, τ^*) ((τ, τ^*) is coarser than $(\mathcal{U}, \mathcal{U}^*)$) if $\tau(\lambda) \leq \mathcal{U}(\lambda)$ and $\tau^*(\lambda) \geq \mathcal{U}^*(\lambda)$ for all $\lambda \in I^X$.

Theorem 1.2 (see [3, 15]). *Let (X, τ, τ^*) be an IFTS. For each $r \in I_0$, $s \in I_1$, $\lambda \in I^X$, an operator $\mathcal{C} : I^X \times I_0 \times I_1 \to I^X$ is defined as follows:*

$$\mathcal{C}(\lambda, r, s) = \bigwedge \{\mu \mid \mu \geq \lambda, \tau(\underline{1} - \mu) \geq r, \tau^*(\underline{1} - \mu) \leq s\}. \tag{1.2}$$

Then it satisfies the following properties:

(1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}$, $\mathcal{C}(\underline{1}, r, s) = \underline{1}$, *for all* $r \in I_0$, $s \in I_1$;

(2) $\mathcal{C}(\lambda, r, s) \geq \lambda$;

(3) $\mathcal{C}(\lambda_1, r, s) \leq \mathcal{C}(\lambda_2, r, s)$, *if* $\lambda_1 \leq \lambda_2$;

(4) $\mathcal{C}(\lambda \vee \mu, r, s) = \mathcal{C}(\lambda, r, s) \vee \mathcal{C}(\mu, r, s)$, *for all* $r \in I_0$, $s \in I_1$;

(5) $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\lambda, r', s')$, *if* $r \leq r'$, $s \geq s'$, *where* $r, r' \in I_0$, $s, s' \in I_1$;

(6) $\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) = \mathcal{C}(\lambda, r, s)$.

Definition 1.3 (see [14]). A function $f : (X, \tau, \tau^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is said to be as follows:

(1) IF continuous if $\tau(f^{-1}(\mu)) \geq \mathcal{U}(\mu)$ and $\tau^*(f^{-1}(\mu)) \leq \mathcal{U}^*(\mu)$, for each $\mu \in I^Y$;

(2) IF open if $\tau(\mu) \leq \mathcal{U}(f(\mu))$ and $\tau^*(\mu) \geq \mathcal{U}^*(f(\mu))$, for each $\mu \in I^X$;

(3) IF homeomorphism if and only if f is bijective and both f and f^{-1} are IF continuous.

Definition 1.4 (see [16]). Let $\underline{0} \notin \Theta_X$ be a subset of I^X. A pair (β, β^*) of functions $\beta, \beta^* : \Theta_X \to I$ is called an IF topological base on X if it satisfies the following conditions:

(B1) $\beta(\lambda) + \beta^*(\lambda) \leq 1, \forall \lambda \in \Theta_X$,

(B2) $\beta(\underline{1}) = 1$ and $\beta^*(\underline{1}) = 0$,

(B3) $\beta(\lambda_1 \wedge \lambda_2) \geq \beta(\lambda_1) \wedge \beta(\lambda_2)$ and $\beta^*(\lambda_1 \wedge \lambda_2) \leq \beta^*(\lambda_1) \vee \beta^*(\lambda_2)$, for each $\lambda_1, \lambda_2 \in \Theta_X$.

An IF topological base (β, β^*) always generates an IGO, $(\tau_\beta, \tau^*_{\beta^*})$ on X in the following sense.

Theorem 1.5 (see [16]). *Let (β, β^*) be an IF topological base for X. Define the functions $\tau_\beta, \tau^*_{\beta^*}$: $I^X \to I$ as follows: for each $\mu \in I^X$,*

$$\tau_\beta(\mu) = \begin{cases} \bigvee \Big\{ \bigwedge_{i \in J} \beta(\mu_i) \Big\}, & \text{if } \mu = \bigvee_{i \in J} \mu_i,\ \mu_i \in \Theta_X, \\ 1, & \text{if } \mu = \underline{0}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.3}$$

where $\bigvee$ is taken over all families $\{\mu_i \in \Theta_X \mid \mu = \bigvee_{i \in J} \mu_i\}$,

$$\tau^*_{\beta^*}(\mu) = \begin{cases} \bigwedge \Big\{ \bigvee_{i \in J} \beta^*(\mu_i) \Big\}, & \text{if } \mu = \bigvee_{i \in J} \mu_i,\ \mu_i \in \Theta_X, \\ 0, & \text{if } \mu = \underline{0}, \\ 1, & \text{otherwise}, \end{cases} \tag{1.4}$$

where $\bigwedge$ is taken over all families $\{\mu_i \in \Theta_X \mid \mu = \bigvee_{i \in J} \mu_i\}$. Then

(1) *$(X, \tau_\beta, \tau^*_{\beta^*})$ is an IFTS;*

(2) *A map $f : (Y, \tau, \tau^*) \to (X, \tau_\beta, \tau^*_{\beta^*})$ is IF continuous if and only if $\beta(\lambda) \leq \tau(f^{-1}(\lambda))$ and $\beta^*(\lambda) \geq \tau^*(f^{-1}(\lambda))$, for all $\lambda \in \Theta_X$.*

Lemma 1.6 (see [17]). *Let X be a product of the family $\{X_i \mid i \in \Gamma\}$ of sets, and for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection map. For each $\lambda \in I^X$, $i, j \in \Gamma$, and $\lambda_i \in I^{X_i}$, the following properties hold:*

(1) *$\pi_i(\pi_i^{-1}(\lambda_i) \wedge \lambda) = \lambda_i \wedge \pi_i(\lambda)$;*

(2) *if $\bigvee_{x^i \in X_i} \lambda_i(x^i) = \alpha_i$ for $i \in F$ with each finite index subset F of $\Gamma - \{j\}$ and put $\alpha = \bigwedge_{i \in F} \alpha_i$, then*

(a) *$\bigvee_{x \in X}(\bigwedge_{i \in F} \pi_i^{-1}(\lambda_i))(x) = \alpha$;*

(b) *$\pi_i(\bigwedge_{i \in F} \pi_i^{-1}(\lambda_i)) = \underline{\alpha}$.*

Definition 1.7. Let (X, τ, τ^*) be an IFTS, $\mu \in I^X$, $x_t \in P_t(X)$, $r \in I_0$, and $s \in I_1$. It holds that

$$Q(x_t, r, s) = \{\mu \in I^X \mid x_t q \mu,\ \tau(\mu) \geq r,\ \tau^*(\mu) \leq s\}. \tag{1.5}$$

A fuzzy set $\mu \in Q(x_t, r, s)$ is called (r, s)-Q open neighborhood of x_t.

2. Some properties of product intuitionistic fuzzy topological spaces

Theorem 2.1. *Let $\{(X_i, \tau_i, \tau_i^*)\}_{i\in\Gamma}$ be a family of IFTSs, let X be a set and for each $i \in \Gamma$, $f_i : X \to X_i$ a map. Let*

$$\Theta_X = \left\{ \underline{0} \neq \mu = \bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j}) \mid \tau_{k_j}(\nu_{k_j}) > 0, \forall k_j \in K \right\}, \tag{2.1}$$

for every finite set $K = \{k_1, \dots, k_n\} \subset \Gamma$. Define the functions $\beta, \beta^* : \Theta_X \to I$ on X by

$$\begin{aligned} \beta(\mu) &= \bigvee \left\{ \bigwedge_{j=1}^{n} \tau_{k_j}(\nu_{k_j}) \mid \mu = \bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j}) \right\}, \\ \beta^*(\mu) &= \bigwedge \left\{ \bigvee_{j=1}^{n} \tau_{k_j}^*(\nu_{k_j}) \mid \mu = \bigwedge_{j=1}^{n} f_{k_j}^{-1}(\nu_{k_j}) \right\}, \end{aligned} \tag{2.2}$$

where $\bigvee$ and $\bigwedge$ are taken over all finite subsets $K = \{k_1, k_2, \dots, k_n\} \subset \Gamma$. Then,

(1) *(β, β^*) is an IF topological base on X;*

(2) *the IGO, $(\tau_\beta, \tau_{\beta^*}^*)$ generated by (β, β^*) is the coarsest IGO on X for which each $i \in \Gamma$, f_i is IF continuous;*

(3) *a map $f : (Y, \tau_1, \tau_1^*) \to (X, \tau_\beta, \tau_{\beta^*}^*)$ is IF continuous if and only if for each $i \in \Gamma$, $f_i \circ f$ is IF continuous.*

Proof. (B1) It is trivial.

(B2) Since $\lambda = f_i^{-1}(\lambda)$ for each $\lambda \in \{\underline{0}, \underline{1}\}$, $\beta(\underline{1}) = \beta(\underline{0}) = 1$ and $\beta^*(\underline{1}) = \beta^*(\underline{0}) = 0$.

(B3) For all finite subsets $K = \{k_1, \dots, k_p\}$ and $J = \{j_1, \dots, j_q\}$ of Γ such that

$$\lambda = \bigwedge_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i}), \qquad \mu = \bigwedge_{i=1}^{q} f_{j_i}^{-1}(\mu_{j_i}), \tag{2.3}$$

we have

$$\lambda \wedge \mu = \left(\bigwedge_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i}) \right) \wedge \left(\bigwedge_{i=1}^{q} f_{j_i}^{-1}(\mu_{j_i}) \right). \tag{2.4}$$

Furthermore, we have for each $k \in K \cap J$,

$$f_k^{-1}(\lambda_k) \wedge f_k^{-1}(\mu_k) = f_k^{-1}(\lambda_k \wedge \mu_k). \tag{2.5}$$

Put $\lambda \wedge \mu = \bigwedge_{m_i \in K\cup J} f_{m_i}^{-1}(\mu_i)(\rho_{m_i})$, where

$$\rho_{m_i} = \begin{cases} \lambda_{m_i} & \text{if } m_i \in K - (K \cap J), \\ \mu_{m_i} & \text{if } m_i \in J - (K \cap J), \\ \lambda_{m_i} \wedge \mu_{m_i} & \text{if } m_i \in (K \cap J). \end{cases} \tag{2.6}$$

We have

$$\begin{aligned}\beta(\lambda\wedge\mu) &\geq \bigwedge_{j\in K\cup J}\tau_j(\rho_j)\\ &\geq \left(\bigwedge_{i=1}^{p}\tau_{k_i}(\lambda_{k_i})\right)\wedge\left(\bigwedge_{i=1}^{q}\tau_{j_i}(\mu_{j_i})\right),\\ \beta^*(\lambda\wedge\mu) &\leq \bigvee_{j\in K\cup J}\tau_j^*(\rho_j)\\ &\leq \left(\bigvee_{i=1}^{p}\tau_{k_i}^*(\lambda_{k_i})\right)\vee\left(\bigvee_{i=1}^{q}\tau_{j_i}^*(\mu_{j_i})\right).\end{aligned}\tag{2.7}$$

Then, $\beta(\lambda\wedge\mu)\geq\beta(\lambda)\wedge\beta(\mu)$ and $\beta^*(\lambda\wedge\mu)\leq\beta^*(\lambda)\vee\beta^*(\mu)$.

(2) For each $\lambda_i\in I^{X_i}$, one family $\{f_i^{-1}(\lambda_i)\}$, and $i\in\Gamma$, we have

$$\begin{aligned}\tau_\beta(f_i^{-1}(\lambda_i)) &\geq \beta(f_i^{-1}(\lambda_i))\geq\tau_i(\lambda_i),\\ \tau_{\beta^*}^*(f_i^{-1}(\lambda_i)) &\leq \beta^*(f_i^{-1}(\lambda_i))\leq\tau_i^*(\lambda_i).\end{aligned}\tag{2.8}$$

Thus, for each $i\in\Gamma$, $f_i:(X,\tau_\beta,\tau_{\beta^*}^*)\to(X_i,\tau_i,\tau_i^*)$ is IF continuous. Let $f_i:(X,\tau^\circ,\tau^{\circ*})\to(X_i,\tau_i,\tau_i^*)$ be IF continuous, that is, for each $i\in\Gamma$ and $\lambda_i\in I^{X_i}$, $\tau^\circ(f_i^{-1}(\lambda_i))\geq\tau_i(\lambda_i)$ and $\tau^{\circ*}(f_i^{-1}(\lambda_i))\leq\tau_i^*(\lambda_i)$. For all finite subsets $K=\{k_1,\dots,k_p\}$ of Γ such that $\lambda=\bigwedge_{i=1}^{p}f_{k_i}^{-1}(\lambda_{k_i})$, we have

$$\begin{aligned}\tau^\circ(\lambda) &\geq \bigwedge_{i=1}^{p}\tau^\circ(f_{k_i}^{-1}(\lambda_{k_i}))\geq\bigwedge_{i=1}^{p}\tau_{k_i}(\lambda_{k_i}),\\ \tau^{\circ*}(\lambda) &\leq \bigvee_{i=1}^{p}\tau^{\circ*}(f_{k_i}^{-1}(\lambda_{k_i}))\leq\bigvee_{i=1}^{p}\tau_{k_i}^*(\lambda_{k_i}).\end{aligned}\tag{2.9}$$

It implies $\tau^\circ(\lambda)\geq\beta(\lambda)$ and $\tau^{\circ*}(\lambda)\leq\beta^*(\lambda)$ for each $\lambda\in I^X$. By Theorem 1.5(2), $\tau^\circ\geq\tau_\beta$ and $\tau^{\circ*}\leq\tau_{\beta^*}^*$.

(3) ($\Rightarrow$) Let $f:(Y,\tau_1,\tau_1^*)\to(X,\tau_\beta,\tau_{\beta^*}^*)$ be an IF continuous. For each $i\in\Gamma$ and $\lambda_i\in I^{X_i}$, we have

$$\begin{aligned}\tau_1((f_i\circ f)^{-1}(\lambda_i)) &= \tau_1((f^{-1}(f_i^{-1}(\lambda_i)))\geq\tau_\beta(f_i^{-1}(\lambda_i)))\geq\tau_i(\lambda_i),\\ \tau_1^*((f_i\circ f)^{-1}(\lambda_i)) &= \tau_1^*((f^{-1}(f_i^{-1}(\lambda_i)))\leq\tau_{\beta^*}^*(f_i^{-1}(\lambda_i)))\leq\tau_i^*(\lambda_i).\end{aligned}\tag{2.10}$$

Hence, $f_i\circ f:(Y,\tau_1,\tau_1^*)\to(X_i,\tau_i,\tau_i^*)$ is IF continuous.

($\Leftarrow$) For all finite subsets $K=\{k_1,\dots,k_p\}$ of Γ such that $\lambda=\bigwedge_{i=1}^{p}f_{k_i}^{-1}(\lambda_{k_i})$, since $f_{k_i}\circ f:(Y,\tau_1,\tau_1^*)\to(X_{k_i},\tau_{k_i},\tau_{k_i}^*)$ is IF continuous,

$$\tau_1(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i})))\geq\tau_{k_i}(\lambda_{k_i}),\tag{A}$$

$$\tau_1^*(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i})))\leq\tau_{k_i}^*(\lambda_{k_i}).\tag{B}$$

Hence, we have

$$\begin{aligned}
\tau_1(f^{-1}(\lambda)) &= \tau_1\left(f^{-1}\left(\bigwedge_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i})\right)\right) \\
&= \tau_1\left(\bigwedge_{i=1}^{p} f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\right) \\
&\geq \bigwedge_{i=1}^{p} \tau_1(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))) \\
&\geq \bigwedge_{i=1}^{p} \tau_{k_i}(\lambda_{k_i}), \quad (\text{by (A)}), \\
\tau_1^*(f^{-1}(\lambda)) &= \tau_1^*\left(f^{-1}\left(\bigwedge_{i=1}^{p} f_{k_i}^{-1}(\lambda_{k_i})\right)\right) \\
&= \tau_1^*\left(\bigwedge_{i=1}^{p} f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))\right) \\
&\leq \bigvee_{i=1}^{p} \tau_1^*(f^{-1}(f_{k_i}^{-1}(\lambda_{k_i}))) \\
&\leq \bigvee_{i=1}^{p} \tau_{k_i}^*(\lambda_{k_i}), \quad (\text{by (B)}).
\end{aligned} \tag{2.11}$$

It implies $\tau_1(f^{-1}(\lambda)) \geq \beta(\lambda)$ and $\tau_1^*(f^{-1}(\lambda)) \leq \beta^*(\lambda)$ for all $\lambda \in I^X$. By Theorem 1.5(2), $f : (Y, \tau_1, \tau_1^*) \to (X, \tau_\beta, \tau_{\beta^*}^*)$ is IF continuous. □

Definition 2.2. Let (X, τ, τ^*) be an IFTS and $A \subset X$. The triple $(A, \tau|_A, \tau^*|_A)$ is said to be a subspace of (X, τ, τ^*) if $(\tau|_A, \tau^*|_A)$ is the coarsest IGO on A for which the inclusion map i is IF continuous.

Definition 2.3. Let X be the product $\prod_{i\in\Gamma} X_i$ of the family $\{(X_i, \tau_i, \tau_i^*) \mid i \in \Gamma\}$ of IFTSs. The coarsest IGO, $(\tau, \tau^*) = (\bigotimes \tau_i, \bigotimes \tau_i^*)$ on X for which each the projections $\pi_i : X \to X_i$ is IF continuous, is called the product IGO of $\{(\tau_i, \tau_i^*) \mid i \in \Gamma\}$ and (X, τ, τ^*) is called the product IFTS.

Lemma 2.4. *Let $(Y, \mathcal{U}, \mathcal{U}^*)$ be an IFTS and (β, β^*) an IF topological base on X. If $f : (X, \beta, \beta^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is a function such that $\beta(\lambda) \leq \mathcal{U}(f(\lambda))$ and $\beta^*(\lambda) \geq \mathcal{U}^*(f(\lambda))$ for all $\lambda \in \Theta_X$, then $f : (X, \tau_\beta, \tau_{\beta^*}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is IF open.*

Proof. Suppose there exists $\mu \in I^X$ such that

$$\tau_\beta(\mu) > \mathcal{U}(f(\mu)) \quad \text{or} \quad \tau_{\beta^*}^*(\mu) < \mathcal{U}^*(f(\mu)), \tag{2.12}$$

then there exists a family $\{\lambda_i \in \Theta_X \mid \mu = \bigvee_{i\in\Gamma} \lambda_i\}$ such that

$$\tau_\beta(\mu) \geq \bigwedge_{i\in\Gamma} \beta(\lambda_i) > \mathcal{U}(f(\mu)) \quad \text{or} \quad \tau_{\beta^*}^*(\mu) \leq \bigvee_{i\in\Gamma} \beta^*(\lambda_i) < \mathcal{U}^*(f(\mu)). \tag{2.13}$$

On the other hand, since $\beta(\lambda) \leq \mathcal{U}(f(\lambda))$ and $\beta^*(\lambda) \geq \mathcal{U}^*(f(\lambda))\ \forall \lambda \in \Theta_X$, then we have

$$\begin{aligned} \bigwedge_{i\in\Gamma}\beta(\lambda_i) &\leq \bigwedge_{i\in\Gamma}\mathcal{U}(f(\lambda_i)) \leq \mathcal{U}\left[\bigvee_{i\in\Gamma}(f(\lambda_i))\right] = \mathcal{U}\left[f\left(\bigvee_{i\in\Gamma}(\lambda_i)\right)\right] = \mathcal{U}(f(\mu)), \\ \bigvee_{i\in\Gamma}\beta^*(\lambda_i) &\geq \bigvee_{i\in\Gamma}\mathcal{U}^*(f(\lambda_i)) \geq \mathcal{U}^*\left[\bigvee_{i\in\Gamma}(f(\lambda_i))\right] = \mathcal{U}^*\left[f\left(\bigvee_{i\in\Gamma}(\lambda_i)\right)\right] = \mathcal{U}^*(f(\mu)). \end{aligned} \tag{2.14}$$

It is a contradiction. Hence f is IF open. □

Theorem 2.5. *Let $(X, \tau_\beta, \tau^*_{\beta^*})$ be a product space of a family $\{(X_i, \tau_i, \tau^*_i) \mid i \in \Gamma\}$ of IFTS's. Then the following statements are equivalent:*

(1) *a projection map $\pi_j : (X, \tau_\beta, \tau^*_{\beta^*}) \to (X_j, \tau_j, \tau^*_j)$ is IF open;*

(2) *for every $\mu = \bigwedge_{i\in\Gamma}\pi_i^{-1}(\lambda_i)$ such that $\bigvee_{x\in X_i}\lambda_i(x) = \alpha_i$ for each $\alpha_i \in I$ and $i \in \Gamma_\circ$ such that a finite index subset $\Gamma_\circ$ of $\Gamma - \{j\}$ and $\tau_i(\lambda_i) > 0$, then $\bigwedge_{i\in\Gamma_\circ}\tau_i(\lambda_i) \leq \tau_j(\underline{\alpha})$ and $\bigvee_{i\in\Gamma_\circ}\tau^*_i(\lambda_i) \geq \tau^*_j(\underline{\alpha})$, where $\alpha = \bigwedge_{i\in\Gamma_\circ}\alpha_i$.*

Proof. (1)⇒(2): For every $\mu = \bigwedge_{i\in\Gamma}\pi_i^{-1}(\lambda_i)$ such that $\bigvee_{x\in X_i}\lambda_i(x) = \alpha_i$ for each $\alpha_i \in I$ and $i \in \Gamma_\circ$ such that a finite index subset $\Gamma_\circ$ of $\Gamma - \{j\}$. By Lemma 1.6(2(b)), we have, for $\alpha = \bigwedge_{i\in\Gamma_\circ}\alpha_i$,

$$\pi_j(\mu) = \pi_j\left(\bigwedge_{i\in\Gamma_\circ}\pi_i^{-1}(\lambda_i)\right) = \underline{\alpha}. \tag{2.15}$$

Since $\mu \in \Theta_X$, by Theorem 2.1, we have

$$\bigwedge_{i\in\Gamma_\circ}\tau_i(\lambda_i) \leq \beta(\mu) \leq \tau_\beta(\mu), \qquad \bigvee_{i\in\Gamma_\circ}\tau^*_i(\lambda_i) \geq \beta^*(\mu) \geq \tau^*_{\beta^*}(\mu). \tag{2.16}$$

Furthermore, since π_j is IF open, we have

$$\tau_\beta(\mu) \leq \tau_j(\pi_j(\mu)) = \tau_j(\underline{\alpha}), \qquad \tau^*_{\beta^*}(\mu) \geq \tau^*_j(\pi_j(\mu)) = \tau^*_j(\underline{\alpha}). \tag{2.17}$$

Hence, $\bigwedge_{i\in\Gamma_\circ}\tau_i(\lambda_i) \leq \tau_j(\underline{\alpha})$ and $\bigvee_{i\in\Gamma_\circ}\tau^*_i(\lambda_i) \geq \tau^*_j(\underline{\alpha})$.

(2)⇒(1): From Lemma 2.4, we only show that $\beta(\mu) \leq \tau_j(\pi_j(\lambda))$ and $\beta^*(\mu) \geq \tau^*_j(\pi_j(\lambda))$ for all $\lambda \in \Theta_X$. Suppose that there exists $\nu \in \Theta_X$ such that $\beta(\nu) > \tau_j(\pi_j(\nu))$ or $\beta^*(\nu) < \tau^*_j(\pi_j(\nu))$. Then there exists a finite index subset $\Gamma_\circ$ of $\Gamma - \{j\}$ with $\nu = \pi_j^{-1}(\lambda_j) \wedge [\bigwedge_{i\in\Gamma_\circ}\pi_i^{-1}(\lambda_i)]$ (if necessary, we can take $\lambda_j = \underline{1}$) such that

$$\begin{aligned} \beta(\nu) &\geq \tau_j(\lambda_j) \wedge \left[\bigwedge_{i\in\Gamma_\circ}\tau_i(\lambda_i)\right] > \tau_j(\pi_j(\nu)), \\ \beta^*(\nu) &\leq \tau^*_j(\lambda_j) \vee \left[\bigvee_{i\in\Gamma_\circ}\tau^*_i(\lambda_i)\right] < \tau^*_j(\pi_j(\nu)). \end{aligned} \tag{2.18}$$

On the other hand, by Lemma 1.6(2), we have

$$\begin{aligned}\pi_j(\nu) &= \pi_j\left[\pi_j^{-1}(\lambda_j) \wedge \left(\bigwedge_{i\in\Gamma_\circ}\pi_i^{-1}(\lambda_i)\right)\right] \\ &= \lambda_j \wedge \pi_j\left[\bigwedge_{i\in\Gamma_\circ}\pi_i^{-1}(\lambda_i)\right] \\ &= \lambda_j \wedge \underline{\alpha},\end{aligned} \tag{2.19}$$

where $\bigvee_{x\in X_i}\lambda_i(x) = \alpha_i$ and $\alpha = \bigwedge_{i\in\Gamma_\circ}\alpha_i$. Since $\bigwedge_{i\in\Gamma_\circ}\tau_i(\lambda_i) \le \tau_j(\underline{\alpha})$ and $\bigvee_{i\in\Gamma_\circ}\tau_i^*(\lambda_i) \ge \tau_j^*(\underline{\alpha})$, we have

$$\begin{aligned}\tau_j(\pi_j(\nu)) &= \tau_j(\lambda_j \wedge \underline{\alpha}) \\ &\ge \tau_j(\lambda_j) \wedge \tau_j(\underline{\alpha}) \\ &\ge \tau_j(\lambda_j) \wedge \left(\bigwedge_{i\in\Gamma_\circ}\tau_i(\lambda_i)\right), \\ \tau_j^*(\pi_j(\nu)) &= \tau_j^*(\lambda_j \wedge \underline{\alpha}) \\ &\le \tau_j^*(\lambda_j) \wedge \tau_j^*(\underline{\alpha}) \\ &\le \tau_j^*(\lambda_j) \vee \left(\bigvee_{i\in\Gamma_\circ}\tau_i^*(\lambda_i)\right).\end{aligned} \tag{2.20}$$

It is a contradiction. □

Theorem 2.6. *Let (X,τ,τ^*) be a product space of a family $\{(X_i,\tau_i,\tau_i^*) \mid i \in \Gamma\}$ of IFTSs and (X_j,τ_j,τ_j^*) be stratified. Then, the following properties hold:*

(1) *(X,τ,τ^*) is stratified;*

(2) *a projection map $\pi_j : X \to X_j$ is IF open.*

Proof. (1) It is clear from the following: for all $\alpha \in I$,

$$\begin{aligned}\tau(\underline{\alpha}) \ge \beta(\underline{\alpha}) &= \bigvee\left\{\bigwedge_{i\in\Gamma_\circ}\tau_i(\lambda_i) \mid \underline{\alpha} = \bigwedge_{i\in\Gamma_\circ}\pi_i^{-1}(\lambda_i)\right\} \ge \tau_j(\underline{\alpha}) = 1, \\ \tau^*(\underline{\alpha}) \le \beta^*(\underline{\alpha}) &= \bigwedge\left\{\bigvee_{i\in\Gamma_\circ}\tau_i^*(\lambda_i) \mid \underline{\alpha} = \bigwedge_{i\in\Gamma_\circ}\pi_i^{-1}(\lambda_i)\right\} \le \tau_j^*(\underline{\alpha}) = 0.\end{aligned} \tag{2.21}$$

(2) Since $\tau_j(\underline{\alpha}) = 1$ and $\tau_j^*(\underline{\alpha}) = 0$ for all $\alpha \in I$, it satisfies the condition of Theorem 2.5(2). □

Theorem 2.7. *Let (X,τ,τ^*) be a product space of a family $\{(X_i,\tau_i,\tau_i^*) \mid i \in \Gamma\}$ of IFTSs and let (X_j,τ_j,τ_j^*) be stratified. Then for every $\widetilde{X}_j = X_j \times \prod\{y^i \mid i \neq j\}$ in X parallel to X_j, $\pi_j|_{\widetilde{X}_j} : \widetilde{X}_j \to X_j$ is an IF homeomorphism.*

Proof. Let $\widetilde{X}_j = X_j \times \prod\{y^i \mid i \neq j\}$. Since $i : \widetilde{X}_j \to \widetilde{X}_j$ and $\pi_j : \widetilde{X}_j \to X_j$ are IF continuous, $\pi_j \circ i = \pi_j|_{\widetilde{X}_j}$ is IF continuous. Moreover, $\pi_j|_{\widetilde{X}_j}$ is bijective.

Now we only show that $\pi_j|_{\widetilde{X}_j}$ is IF open. Suppose there exists $\mu \in I^{\widetilde{X}_j}$ such that

$$\tau|_{\widetilde{X}_j}(\mu) > \tau_j(\pi_j|_{\widetilde{X}_j}(\mu)) \quad \text{or} \quad \tau^*|_{\widetilde{X}_j}(\mu) < \tau_j^*(\pi_j|_{\widetilde{X}_j}(\mu)). \tag{2.22}$$

Then there exists $\nu \in I^X$ with $\mu = i^{-1}(\nu)$ such that

$$\tau|_{\widetilde{X}_j}(\mu) \geq \tau(\nu) > \tau_j(\pi_j|_{\widetilde{X}_j}(\mu)) \quad \text{or} \quad \tau^*|_{\widetilde{X}_j}(\mu) \leq \tau^*(\nu) < \tau_j^*(\pi_j|_{\widetilde{X}_j}(\mu)). \tag{2.23}$$

From the definition of (τ, τ^*), there exists a family $\{\nu_k \in \Theta_X \mid \nu = \bigvee_{k\in K} \nu_k\}$ such that

$$\tau(\nu) \geq \bigwedge_{k\in K} \beta(\nu_k) > \tau_j(\pi_j|_{\widetilde{X}_j}(\mu)) \quad \text{or} \quad \tau^*(\nu) \leq \bigvee_{k\in K} \beta^*(\nu_k) < \tau_j^*(\pi_j|_{\widetilde{X}_j}(\mu)). \tag{C}$$

On the other hand, since each $\nu_k \in \Theta_X$, there exists a finite index F_k of $\Gamma - \{j\}$ with $\nu_k = \pi_j^{-1}(\lambda_{k_j}) \wedge (\bigwedge_{i\in F_k} \pi_i^{-1}(\lambda_i))$. Since $\pi_i^{-1}(\lambda_i)(x) = y^i$ for $i \neq j$, then for each $x \in \widetilde{X}_j$, $\bigwedge_{i\in F_k} \pi_i^{-1}(\lambda_i)(x) = (\bigwedge_{i\in F_k} \lambda_i)(y^i)$. Put $\alpha_k = (\bigwedge_{i\in F_k} \lambda_i)(y^i)$. Let $\mu_k = i^{-1}(\nu_k)$ for each $k \in K$. Then,

$$\begin{aligned}
\pi_j|_{\widetilde{X}_j}(\mu_k)(x^j) &= \bigvee \{\mu_k(x) \mid x \in \widetilde{X}_j,\ \pi_j|_{\widetilde{X}_j}(x) = x^j\} \\
&= \bigvee \{i^{-1}(\nu_k)(x) \mid x \in \widetilde{X}_j,\ \pi_j(x) = x^j (\mu_k = i^{-1}(\nu_k))\} \\
&= \bigvee \{\nu_k(x) \mid x \in \widetilde{X}_j,\ \pi_j(x) = x^j\} \\
&= \bigvee \left\{\pi_j^{-1}(\lambda_{k_j})(x) \wedge \left(\bigwedge_{i\in F_k} \pi_i^{-1}(\lambda_i)(x) \mid x \in \widetilde{X}_j,\ \pi_j(x) = x^j\right)\right\} \\
&= \bigvee \left\{\lambda_{k_j}(\pi_j(x)) \wedge \left(\bigwedge_{i\in F_k} \lambda_i\right)(\pi_i(x)) \mid x \in \widetilde{X}_j,\ \pi_j(x) = x^j\right\} \\
&= \lambda_{k_j}(x^j) \wedge \left(\bigwedge_{i\in F_k} \lambda_i\right) y^i \\
&= \lambda_{k_j}(x^j) \wedge \alpha_k \\
&= (\lambda_{k_j} \wedge \underline{\alpha_k})(x^j).
\end{aligned} \tag{2.24}$$

Hence, $\pi_j|_{\widetilde{X}_j}(\mu_k) = \lambda_{k_j} \wedge \underline{\alpha_k}$. Thus,

$$\begin{aligned}
\tau_j(\pi_j|_{\widetilde{X}_j}(\mu_k)) &= \tau_j(\lambda_{k_j} \wedge \underline{\alpha_k}) \\
&\geq \tau_j(\lambda_{k_j}) \wedge \tau_j(\underline{\alpha_k}) \\
&= \tau_j(\lambda_{k_j}) \\
&\geq \tau_j(\lambda_{k_j}) \wedge \Big(\bigwedge_{i\in F_k} \lambda_i\Big), \\
\tau_j^*(\pi_j|_{\widetilde{X}_j}(\mu_k)) &= \tau_j^*(\lambda_{k_j} \wedge \underline{\alpha_k}) \\
&\leq \tau_j^*(\lambda_{k_j}) \vee \tau_j^*(\underline{\alpha_k}) \\
&= \tau_j^*(\lambda_{k_j}) \\
&\leq \tau_j^*(\lambda_{k_j}) \vee \Big(\bigwedge_{i\in F_k} \lambda_i\Big).
\end{aligned} \tag{2.25}$$

From the definition of (β, β^*), it implies

$$\tau_j(\pi_j|_{\widetilde{X}_j}(\mu_k)) \geq \beta(\nu_k), \qquad \tau_j^*(\pi_j|_{\widetilde{X}_j}(\mu_k)) \leq \beta^*(\nu_k). \tag{2.26}$$

Thus,

$$\begin{aligned}
\tau_j(\pi_j|_{\widetilde{X}_j}(\mu)) &\geq \bigwedge_{k\in K} \tau_j(\pi_j|_{\widetilde{X}_j}(\mu_k)) \geq \bigwedge_{k\in K} \beta(\nu_k), \\
\tau_j^*(\pi_j|_{\widetilde{X}_j}(\mu)) &\leq \bigvee_{k\in K} \tau_j^*(\pi_j|_{\widetilde{X}_j}(\mu_k)) \leq \bigvee_{k\in K} \beta^*(\nu_k).
\end{aligned} \tag{2.27}$$

It is a contradiction for (C). □

In an IFTS $\{(X_i, \tau_i, \tau_i^*) \mid i \in \Gamma\}$, $\widetilde{X}_j = X_j \times \prod\{y^i \mid i \neq j\}$ need not be homeomorphic to X from the following example.

Example 2.8. Let $X = \{x^1, x^2, x^3\}$, $Y = \{y^1, y^2\}$, and $Z = \{z^1, z^2\}$ be sets and $W = X \times Y \times Z$ a product set. Let $\pi_1 : W \to X$, $\pi_2 : W \to Y$, and $\pi_3 : W \to Z$be the projection maps. Define $\tau_1, \tau_1^* : I^X \to I$ by

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \dfrac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_1^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}, \\ \dfrac{1}{2} & \text{if } \lambda = \lambda_1, \\ 1 & \text{otherwise}, \end{cases} \tag{2.28}$$

where $\lambda_1(x^1) = 0.5$, $\lambda_1(x^2) = 0.2$, and $\lambda_1(x^3) = 0.3$. Also, $\widetilde{X}_j = \{(x, y^2, z^2) : x \in X\}$, define $\tau|_{\widetilde{X}_j}, \tau^*|_{\widetilde{X}_j} : I^{\widetilde{X}_j} \to I$ by

$$\tau|_{\widetilde{X}_j}(\mu) = \begin{cases} 1 & \text{if } \mu = \underline{0}, \underline{1}, \\ \dfrac{1}{2} & \text{if } \mu = \mu_1, \\ \dfrac{2}{3} & \text{if } \mu = \underline{0.1}, \\ \dfrac{1}{4} & \text{if } \mu = \underline{0.7}, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau^*|_{\widetilde{X}_j}(\mu) = \begin{cases} 0 & \text{if } \mu = \underline{0}, \underline{1}, \\ \dfrac{1}{2} & \text{if } \mu = \mu_1, \\ \dfrac{1}{3} & \text{if } \mu = \underline{0.1}, \\ \dfrac{3}{4} & \text{if } \mu = \underline{0.7}, \\ 1 & \text{otherwise}, \end{cases} \tag{2.29}$$

where $\mu_1(x^1, y^2, z^2) = 0.5$, $\mu_1(x^2, y^2, z^2) = 0.2$, and $\mu_1(x^3, y^2, z^2) = 0.3$. Then the projection map $\pi_j|_{\widetilde{X}_j} : \widetilde{X}_j \to X$ is bijective IF continuous, but $\pi_j|_{\widetilde{X}_j}$ is not IF open, because

$$\frac{2}{3} = \tau|_{\widetilde{X}_j}(\underline{0.1}) \not\leq \tau_1(\pi_1|_{\widetilde{X}_j}(\underline{0.1})) = 0. \tag{2.30}$$

Hence, $\widetilde{X}_j$ and X are not homeomorphic.

Theorem 2.9. *Let (X, τ, τ^*) be a product space of a family $\{(X_i, \tau_i, \tau_i^*) \mid i \in \Gamma\}$ of IFTSs. Then, the following properties hold:*

(1) $\mathcal{C}_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \leq \prod_{i\in\Gamma}\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s)$, $\forall \lambda_i \in I^{X_i}$, $r \in I_0$, $s \in I_1$;

(2) *if* $\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s) = \lambda_i$, $\forall \lambda_i \in I^{X_i}$, $r \in I_0$, $s \in I_1$, *then* $\mathcal{C}_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) = \prod_{i\in\Gamma}\lambda_i$.

Proof. (1) Suppose $\mathcal{C}_{\tau,\tau^*}(\prod_{i\in\Gamma}\lambda_i, r, s) \not\leq \prod_{i\in\Gamma}\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s)$. Then there exist $x \in X$ and $t \in (0, 1)$ such that

$$\mathcal{C}_{\tau,\tau^*}\left(\prod_{i\in\Gamma}\lambda_i, r, s\right)(x) \geq t > \prod_{i\in\Gamma}\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s)(x). \tag{D}$$

Since $\prod_{i\in\Gamma}\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s) < t$, there exists $j \in \Gamma$ such that $\prod_{i\in\Gamma}\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s) \leq \pi_j^{-1}(\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s)) < t$. Put $\pi_j(x) = x^j$. It implies $\mathcal{C}_{\tau_j,\tau_j^*}(\lambda_j, r, s)(x^j) < t$. From the definition of $\mathcal{C}_{\tau_j,\tau_j^*}$, there exists $\mu_j \in I^{X_j}$ with $\lambda_j \leq \mu_j$ and $\tau_j(\underline{1} - \mu_j) \geq r$, $\tau_j^*(\underline{1} - \mu_j) \leq s$ such that

$$\mathcal{C}_{\tau_j,\tau_j^*}(\lambda_j, r, s)(x^j) \leq \mu_j(x^j) < t. \tag{2.31}$$

On the other hand, we have

$$\begin{aligned} \lambda_j \leq \mu_j &\Longrightarrow \pi_j^{-1}(\lambda_j) \leq \pi_j^{-1}(\mu_j) \\ &\Longrightarrow \prod_{i\in\Gamma}\lambda_i = \bigwedge_{i\in\Gamma}\pi_j^{-1}(\lambda_i) \leq \pi_j^{-1}(\mu_j) \\ &\Longrightarrow \mathcal{C}_{\tau,\tau^*}\left(\prod_{i\in\Gamma}\lambda_i, r, s\right) \leq \pi_j^{-1}(\mu_j), \end{aligned} \tag{2.32}$$

because $\tau(\underline{1}-\pi_j^{-1}(\mu_j)) = \tau(\pi_j^{-1}(\underline{1}-\mu_j)) \geq \tau_j(\underline{1}-\mu_j) \geq r$ and $\tau^*(\underline{1}-\pi_j^{-1}(\mu_j)) = \tau^*(\pi_j^{-1}(\underline{1}-\mu_j)) \leq \tau_j^*(\underline{1}-\mu_j) \leq s$. Hence,

$$\mathcal{C}_{\tau,\tau^*}\left(\prod_{i\in\Gamma}\lambda_i, r, s\right)(x) \leq \pi_j^{-1}(\mu_j)(x) = \mu_j(x^j) < t. \tag{2.33}$$

It is a contradiction for (D). Hence,

$$\mathcal{C}_{\tau,\tau^*}\left(\prod_{i\in\Gamma}\lambda_i, r, s\right) \leq \prod_{i\in\Gamma}\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s). \tag{2.34}$$

(2) It is clear from the following:

$$\prod_{i\in\Gamma}\lambda_i \leq \mathcal{C}_{\tau,\tau^*}\left(\prod_{i\in\Gamma}\lambda_i, r, s\right) \leq \prod_{i\in\Gamma}\mathcal{C}_{\tau_i,\tau_i^*}(\lambda_i, r, s) = \prod_{i\in\Gamma}\lambda_i. \tag{2.35}$$

□

3. Some properties of (r,s)-T_0 and (r,s)-T_1 spaces

Definition 3.1. An IFTS (X, τ, τ^*) is said to be as follows.

(1) (r,s)-quasi-T_0 space if for each $x_t, x_m \in P_t(X)$ and $t < m$, there exists $\lambda \in Q(x_m, r, s)$ such that $x_t\overline{q}\lambda$.

(2) (r,s)-sub-T_0 space if for each $x \neq y \in X$, there exists $t \in I_0$ such that there exists $\lambda \in Q(x_t, r, s)$ such that $y_t\overline{q}\lambda$, or there exists $\mu \in Q(y_t, r, s)$ such that $x_t\overline{q}\mu$.

(3) (r,s)-T_0 space if for each $x_t, y_m \in P_t(X)$, there exists $\lambda \in Q(x_t, r, s)$ such that $y_m\overline{q}\lambda$, or there exists $\mu \in Q(y_m, r, s)$ such that $x_t\overline{q}\mu$.

(4) (r,s)-T_1 space if for each $x_t, y_m \in P_t(X)$ such that $x_t \not\leq y_m$, there exists $\lambda \in Q(x_t, r, s)$ such that $y_m\overline{q}\lambda$.

Theorem 3.2. *Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent:*

(1) *(X, τ, τ^*) is (r,s)-T_0 space;*

(2) *for each $x_t, y_m \in P_t(X)$, $Q(x_t, r, s) \neq Q(y_m, r, s)$;*

(3) *for each $x_t, y_m \in P_t(X)$, then $x_t \notin \mathcal{C}_{\tau,\tau^*}(y_m, r, s)$ or $y_m \notin \mathcal{C}_{\tau,\tau^*}(x_t, r, s)$.*

Proof. (1)⇒(2): It is trivial.

(2)⇒(3): Let $\lambda \in Q(x_t, r, s)$ and $\lambda \notin Q(y_m, r, s)$. Since $\lambda \notin Q(y_m, r, s)$, we have

$$y_m \leq \underline{1} - \lambda, \qquad \tau(\lambda) \geq r, \qquad \tau^*(\lambda) \leq s. \tag{3.1}$$

By Theorem 1.2, we have $\mathcal{C}_{\tau,\tau^*}(y_m, r, s) \leq \underline{1} - \lambda$. Since $x_t q \lambda$ and $\lambda \leq \underline{1} - \mathcal{C}_{\tau,\tau^*}(y_m, r, s)$, then $x_t q[\underline{1} - \mathcal{C}_{\tau,\tau^*}(y_m, r, s)]$. Hence, $x_t \notin \mathcal{C}_{\tau,\tau^*}(y_m, r, s)$.

(3)⇒(1): Let $x_t, y_m \in P_t(X)$ and $x_t \notin C_{\tau,\tau^*}(y_m, r, s)$. Then $t > C_{\tau,\tau^*}(y_m, r, s)(x)$ implies $x_t q[\underline{1} - C_{\tau,\tau^*}(y_m, r, s)]$. Since $C_{\tau,\tau^*}(y_m, r, s) = \bigwedge\{\mu \mid \mu \geq y_m,\ \tau(\underline{1} - \mu) \geq r,\ \tau^*(\underline{1} - \mu) \leq s\}$. Since $\tau(\bigvee(\underline{1} - \mu)) \geq \bigwedge \tau(\underline{1} - \mu)$ and $\tau^*(\bigvee(\underline{1} - \mu)) \leq \bigvee \tau^*(\underline{1} - \mu)$, we have $\tau(\underline{1} - C_{\tau,\tau^*}(y_m, r, s)) \geq r$ and $\tau^*(\underline{1} - C_{\tau,\tau^*}(y_m, r, s)) \leq s$. Hence, $[\underline{1} - C_{\tau,\tau^*}(y_m, r, s)] \in Q(x_t, r, s)$. Since $y_m \in C_{\tau,\tau^*}(y_m, r, s)$ and $y_m \overline{q}[\underline{1} - C_{\tau,\tau^*}(y_m, r, s))]$. Thus, (X, τ, τ^*) is (r, s)-T_0 space. □

We can prove the following corollaries as a similar method as Theorem 3.2.

Corollary 3.3. *Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent:*

(1) *(X, τ, τ^*) is (r, s)-quasi-T_0 space;*

(2) *for each $x_t, x_m \in P_t(X)$, $Q(x_t, r, s) \neq Q(x_m, r, s)$;*

(3) *for each $x_t, x_m \in P_t(X)$, then $x_t \notin C_{\tau,\tau^*}(x_m, r, s)$ or $x_m \notin C_{\tau,\tau^*}(x_t, r, s)$.*

Corollary 3.4. *Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent:*

(1) *(X, τ, τ^*) is (r, s)-sub-T_0 space;*

(2) *for each $x \neq y \in X$, there exists $t \in I_0$ such that $Q(x_t, r, s) \neq Q(y_t, r, s)$;*

(3) *for each $x \neq y \in X$, there exists $t \in I_0$ such that $x_t \notin C_{\tau,\tau^*}(y_t, r, s)$ or $y_t \notin C_{\tau,\tau^*}(x_t, r, s)$.*

Example 3.5. Let $X = \{x, y\}$ be a set. We define an IGO (τ, τ^*) on X as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{1} \text{ or } \underline{0}, \\ \dfrac{1}{2} & \text{if } \lambda = x_{0.7}, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{1} \text{ or } \underline{0}, \\ \dfrac{1}{2} & \text{if } \lambda = x_{0.7}, \\ 1 & \text{otherwise.} \end{cases} \tag{3.2}$$

For each $x \neq y \in X$, there exists $0.4 \in I_0$ such that $x_{0.7} \in Q(x_{0.4}, 1/2, 1/2)$ and $y_{0.4} \overline{q} x_{0.7}$. Hence, (X, τ, τ^*) is $(1/2, 1/2)$-sub-T_0 space. On the other hand, since $Q(y_{0.5}, 1/2, 1/2) = Q(y_{0.6}, 1/2, 1/2) = \{\underline{1}\}$, by Corollary 3.3(2), (X, τ, τ^*) is not $(1/2, 1/2)$-quasi-T_0 space.

Theorem 3.6. *Let (X, τ, τ^*) be an IFTS. Then the following statements are equivalent:*

(1) *(X, τ, τ^*) is (r, s)-T_1 space;*

(2) *for each $x_t \in P_t(X)$, $x_t = C_{\tau,\tau^*}(x_t, r, s)$;*

(3) *for each $\lambda \in I^X$, $\lambda = \bigwedge\{\mu \mid \lambda \leq \mu,\ \tau(\mu) \geq r,\ \tau^*(\mu) \leq s\}$.*

Proof. (1)⇒(2): We only show that $C_{\tau,\tau^*}(x_t, r, s) \leq x_t$. Let $y_m \in C_{\tau,\tau^*}(x_t, r, s)$. Suppose that $y_m \not\leq x_t$. Since (X, τ, τ^*) is (r, s)-T_1 space, there exists $\lambda \in Q(y_m, r, s)$ such that $x_t \overline{q} \lambda$. It implies $x_t \leq \underline{1} - \lambda$ with $\tau(\lambda) \geq r$ and $\tau^*(\lambda) \leq s$. Hence, $C_{\tau,\tau^*}(x_t, r, s) \leq \underline{1} - \lambda$. Since $y_m \in C_{\tau,\tau^*}(x_t, r, s) \leq \underline{1} - \lambda$, we have $\lambda \notin Q(y_m, r, s)$. It is a contradiction. Hence, $y_m \leq x_t$. Since $y_m \in C_{\tau,\tau^*}(x_t, r, s)$ implies $y_m \leq x_t$, then $C_{\tau,\tau^*}(x_t, r, s) \leq x_t$.

(2)⇒(3): Let $\rho = \bigwedge\{\mu \mid \lambda \leq \mu,\ \tau(\mu) \geq r,\ \tau^*(\mu) \leq s\}$. We only show that $\rho \leq \lambda$. Suppose there exist $x \in X$ and $t \in (0, 1)$ such that

$$\rho(x) > 1 - t \geq \lambda(x). \tag{3.3}$$

Then, $\lambda \leq \underline{1} - x_t$. Since $x_t = \mathcal{C}_{\tau,\tau^*}(x_t, r, s)$,

$$\tau(\underline{1} - x_t) = \tau(\underline{1} - \mathcal{C}_{\tau,\tau^*}(x_t, r, s)) \geq r, \qquad \tau^*(\underline{1} - \mathcal{C}_{\tau,\tau^*}(x_t, r, s)) \leq s. \tag{3.4}$$

Hence, $\rho \leq \underline{1} - x_t$. It is a contradiction.

(3)$\Rightarrow$(1): For each $x_t, y_m \in P_t(X)$ such that $x_t \not\leq y_m$, $\underline{1} - x_t \not\geq \underline{1} - y_m$. From (3), since $\underline{1} - y_m = \bigwedge\{\mu \mid \underline{1} - y_m \leq \mu,\ \tau(\mu) \geq r,\ \tau^*(\mu) \leq s\}$, there exists $\mu = \underline{1} - y_m \in I^X$ such that

$$\tau(\underline{1} - y_m) \geq r, \qquad \tau^*(\underline{1} - y_m) \leq s. \tag{3.5}$$

Moreover, since $\underline{1} - x_t \not\geq \underline{1} - y_m$, we have $x_t q[\underline{1} - y_m]$. Thus $\underline{1} - y_m \in Q(x_t, r, s)$ such that $y_m \overline{q}[\underline{1} - y_m]$. Hence, (X, τ, τ^*) is (r, s)-T_1 space. □

Theorem 3.7. *Let (X, τ, τ^*) be a stratified IFTS. Then (X, τ, τ^*) is (r, s)-quasi-T_0 space for all $r \in I_0$, $s \in I_1$.*

Proof. Let $x_t, x_m \in P_t(X)$ such that $t < m$. Then there exists $\alpha \in I_0$ such that

$$t \leq 1 - \alpha < m. \tag{3.6}$$

Since (X, τ, τ^*) is stratified IFTS, we have $\tau(\underline{\alpha}) = 1$ and $\tau^*(\underline{\alpha}) = 0$. Hence, $\underline{\alpha} \in Q(x_m, r, s)$ such that $x_t \overline{q} \underline{\alpha}$. □

Theorem 3.8. (1) *Every (r, s)-T_0 space is both (r, s)-quasi-T_0 and (r, s)-sub-T_0.*

(2) *Every (r, s)-T_1 space is (r, s)-T_0.*

Proof. (1) For each $x_t, x_m \in P_t(X)$ such that $t < m$. By (r, s)-T_0 space, there exists $\lambda \in Q(x_t, r, s)$ such that $x_m \overline{q} \lambda$. Since $t < m$, we have $x_t \overline{q} \lambda$. So, X is (r, s)-quasi-T_0.

(2) For each $x_t, y_m \in P_t(X)$, if $x_t \not\leq y_m$, by (r, s)-T_1 space, there exists $\lambda \in Q(x_t, r, s)$ such that $y_m \overline{q} \lambda$. Also, if $y_m \not\leq x_t$, by (r, s)-T_1 space, there exists $\mu \in Q(y_m, r, s)$ such that $x_t \overline{q} \mu$. Hence, X is (r, s)-T_0. □

The converse of Theorem 3.8 is not true from the following examples.

Example 3.9. Let $X = \{x, y\}$ be a set. We define an IGO (τ, τ^*) on X as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \dfrac{1}{3} & \text{if } \lambda = \mu_{pq}, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \dfrac{2}{3} & \text{if } \lambda = \mu_{pq}, \\ 1 & \text{otherwise}, \end{cases} \tag{3.7}$$

where for each $0 < p \leq 0.4$, $\mu_{pq}(x) = p$ and $\mu_{pq}(y) = q$, $0 \leq q < p$. Since (X, τ, τ^*) is a stratified IFTS, by Theorem 3.7, (X, τ, τ^*) is (r, s)-quasi-T_0 space for each $r \in I_0$ and $s \in I_1$.

If $r > 1/3$, $s < 2/3$, and $t \in I_0$, then for each $x_t, y_t \in P_t(X)$, we have

$$Q(x_t, r, s) = Q(y_t, r, s) = \{\underline{\alpha} \mid 1 - t < \alpha \leq 1\}. \tag{3.8}$$

By Theorem 3.2(2) and Corollary 3.4(2), (X, τ, τ^*) is neither (r,s)-sub-T_0 nor (r,s)-T_0 for $r > 1/3$ and $s < 2/3$.

If $0 < r \leq 1/3$, $2/3 \leq s < 1$, and $x \neq y \in X$, there exists $0.7 \in I_0$ such that there exists $\mu_{(2/5)0} \in Q(x_{0.7}, r, s)$ with $y_{0.7}\overline{q}\mu_{(2/5)0}$. Hence, (X, τ, τ^*) is (r,s)-sub-T_0 for $0 < r \leq 1/3$ and $2/3 \leq s < 1$.

For $x_{0.3}, y_{0.3} \in P_t(X)$, $0 < r \leq 1/3$, and $2/3 \leq s < 1$, we have $Q(x_{0.3}, r, s) = Q(y_{0.3}, r, s) = \{\underline{\alpha} \mid 0.7 < \alpha\}$. Hence, it is not (r,s)-T_0 for $0 < r \leq 1/3$ and $2/3 \leq s < 1$. For $0 < r \leq 1/3$ and $2/3 \leq s < 1$, (X, τ, τ^*) is both (r,s)-quasi-T_0 and (r,s)-sub-T_0, but not (r,s)-T_0.

Example 3.10. Let $X = \{x, y\}$ be a set. We define an IGO (τ, τ^*) on X as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \dfrac{1}{2} & \text{if } \lambda = \mu_{pq}, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ \dfrac{1}{2} & \text{if } \lambda = \mu_{pq}, \\ 1 & \text{otherwise}, \end{cases} \tag{3.9}$$

where for each $0 < p < 1$, $\mu_{pq}(x) = p$ and $\mu_{pq}(y) = q$, $0 \leq q < p$. Let $z_t, z_m \in P_t(X)$ with $t \neq m$ for $z = x$ or y. We have

$$Q\left(z_t, \frac{1}{2}, \frac{1}{2}\right) \neq Q\left(z_m, \frac{1}{2}, \frac{1}{2}\right). \tag{3.10}$$

For $x_t, y_m \in P_t(X)$, for $p > 1 - t$, we have $\mu_{p0} \in Q(x_t, 1/2, 1/2)$ with $y_m\overline{q}\mu_{p0}$. Hence, (X, τ, τ^*) is $(1/2, 1/2)$-T_0 space. On the other hand, let $y_{0.5} \nleq x_{0.5}$. For each $\mu_{pq} \in Q(y_{0.5}, 1/2, 1/2)$, since $q + 0.5 > 1$ and $p > q$, we have $x_{0.5}q\mu_{pq}$, that is, $Q(y_{0.5}, 1/2, 1/2) \subset Q(x_{0.5}, 1/2, 1/2)$. Thus (X, τ, τ^*) is not (r,s)-T_1 space.

Theorem 3.11. *Every subspace of (r,s)-quasi-T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1) space is (r,s)-quasi-T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1) space.*

Proof. Let (X, τ, τ^*) be (r,s)-T_1 space. Let $a_t, b_m \in P_t(A)$ such that $a_t \nleq b_m$. Then, $a_t, b_m \in P_t(X)$ such that $a_t \nleq b_m$. Since (X, τ, τ^*) is (r,s)-T_1, there exists $\lambda \in Q(a_t, r, s)$ such that $b_m\overline{q}\lambda$. Since $\tau_A(i^{-1}(\lambda)) \geq \tau(\lambda) \geq r$ and $\tau_A^*(i^{-1}(\lambda)) \leq \tau^*(\lambda) \leq s$, we have $i^{-1}(\lambda) \in Q_{\tau|_A, \tau^*|_A}(a_t, r, s)$ such that $b_m\overline{q}i^{-1}(\lambda)$. The others are similarly proved. □

We can prove the following theorem as a similar method as Theorem 3.11.

Theorem 3.12. *Every IF homeomorphic space of (r,s)-quasi-T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1) space is (r,s)-quasi-T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1) space.*

Theorem 3.13. *Let $\{(X_i, \tau_i, \tau_i^*) \mid i \in \Gamma\}$ be a family of (r,s)-quasi T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1) space. Let (τ, τ^*) be the product IGO on $X = \prod_{i \in \Gamma} X_i$. Then (X, τ, τ^*) is (r,s)-quasi-T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1) space.*

Proof. Let $x_t, y_m \in P_t(X)$ such that $x_t \nleq y_m$. Then there exists $i \in \Gamma$ such that $(\pi_i(x))_t \nleq (\pi_i(y))_m$. Since (X_i, τ_i, τ_i^*) is (r,s)-T_1 space, there exists $\lambda \in I^{X_i}$ such that

$$\lambda \in Q_{\tau_i, \tau_i^*}\left((\pi_i(x))_t, r, s\right), \quad (\pi_i(y))_m\overline{q}\lambda. \tag{3.11}$$

Since $\pi_i(x_t) = (\pi_i(x))_t q\lambda$ if and only if $x_t q \pi_i^{-1}(\lambda)$, we have

$$\pi_i^{-1}(\lambda) \in Q(x_t, r, s), \quad y_m \overline{q} \pi_i^{-1}(\lambda). \tag{3.12}$$

Therefore, (X, τ, τ^*) is (r, s)-T_1 space. The others are similarly proved. □

Theorem 3.14. *Let $\{(X_i, \tau_i, \tau_i^*) \mid i \in \Gamma\}$ be a family of IFTSs. Let (τ, τ^*) be the product IGO on $X = \prod_{i\in\Gamma} X_i$. If (X, τ, τ^*) is (r, s)-sub-T_0 space, then (X_i, τ_i, τ_i^*) is $(r - \epsilon,\ s + \epsilon)$-sub-$T_0$ space for each $\epsilon > 0$ and for each $i \in \Gamma$.*

Proof. Let $x^j, y^j \in X_j$ such that $x^j \neq y^j$. Then there exists $x^i \in X_i$ for all $i \in \Gamma - \{j\}$ such that $x \neq y \in X$ and

$$\pi_i(x) = \begin{cases} x^i & \text{if } i \in \Gamma - \{j\}, \\ x_j & \text{if } i = j, \end{cases} \qquad \pi_i(y) = \begin{cases} x^i & \text{if } i \in \Gamma - \{j\}, \\ y_j & \text{if } i = j. \end{cases} \tag{3.13}$$

Since (X, τ, τ^*) is (r, s)-sub-T_0 space, there exists $t \in (0, 1)$ such that

$$\rho \in Q(x_t, r, s), \quad y_t \overline{q} \rho. \tag{3.14}$$

Let (β, β^*) be a base for (τ, τ^*). Since $\tau(\rho) \geq r$ and $\tau^*(\rho) \leq s$, by Theorem 1.5, for $\epsilon > 0$, there exists a family $\{\rho_k \mid \rho = \bigvee_{k\in\Delta} \rho_k\}$ such that

$$\tau(\rho) \geq \bigwedge_{k\in\Delta} \beta(\rho_k) > r - \epsilon, \qquad \tau^*(\rho) \leq \bigvee_{k\in\Delta} \beta^*(\rho_k) < s + \epsilon. \tag{3.15}$$

Since $x_t q[\rho = \bigvee_{k\in\Delta} \rho_k]$, there exists $k \in \Gamma$ such that $x_t q \rho_k$, $\beta(\rho_k) > r - \epsilon$ and $\beta^*(\rho_k) < s + \epsilon$. Then, there exists a family $\{\lambda_i \mid \rho_k = \bigwedge_{i\in F} \pi_i^{-1}(\lambda_i)\}$ which F is a finite subset of Γ such that

$$\beta(\rho_k) \geq \bigwedge_{i\in F} \tau_i(\lambda_i) > r - \epsilon, \qquad \beta^*(\rho_k) \leq \bigvee_{i\in F} \tau_i^*(\lambda_i) < s + \epsilon. \tag{E}$$

Without loss of generality, we may assume $j \in F$ because we can take $F_1 = F \cup \{j\}$ such that $\lambda_j = \underline{1}$, $\tau_j(\underline{1}) = 1$, and $\tau_j^*(\underline{1}) = 0$, if necessary.

Since $x_t q \rho_k$ and $y_t \overline{q} \rho_k$,

$$t > \left[\bigvee_{i\in F-\{j\}} (\underline{1} - \lambda_i)(\pi_i(x))\right] \vee (\underline{1} - \lambda_j)(x^j), \tag{F}$$

$$t \leq \left[\bigvee_{i\in F-\{j\}} (\underline{1} - \lambda_i)(\pi_i(x))\right] \vee (\underline{1} - \lambda_j)(y^j). \tag{G}$$

If $(\bigvee_{i\in F-\{j\}} (\underline{1} - \lambda_i)(\pi_i(x)) \geq t$, it is a contradiction for (F) and (G). Thus

$$\bigvee_{i\in F-\{j\}} (\underline{1} - \lambda_i)(\pi_i(x)) < t. \tag{3.16}$$

It implies

$$t > (\underline{1} - \lambda_j)(x^j), \qquad t \leq (\underline{1} - \lambda_j)(y^j). \tag{3.17}$$

Furthermore, by (E), we have $\tau_j(\lambda_j) > r - \epsilon$ and $\tau_j^*(\lambda_j) < s + \epsilon$. Hence,

$$\lambda_j \in Q_{\tau_j,\tau_j^*}\big((x^j)_t, r - \epsilon, s + \epsilon\big), \quad (y^j)_t \overline{q} \lambda_j. \tag{3.18}$$

Thus, (X_j, τ_j, τ_j^*) is $(r - \epsilon, s + \epsilon)$-sub-$T_0$ space. □

In the above theorem, if (X, τ, τ^*) is (r, s)-T_1 (resp., (r, s)-quasi-T_0 and (r, s)-T_0), it is not true from the following example.

Example 3.15. Let $X = \{x\}$ and $Y = \{y\}$ be sets. Define IGO (τ_1, τ_1^*) on X as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_1^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 1 & \text{otherwise,} \end{cases} \tag{3.19}$$

and IGO (τ_2, τ_2^*) on Y as follows:

$$\tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0} \text{ or } \underline{1}, \\ \dfrac{1}{2} & \text{if } \lambda = y_{0.2}, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_2^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0} \text{ or } \underline{1}, \\ \dfrac{1}{2} & \text{if } \lambda = y_{0.2}, \\ 1 & \text{otherwise.} \end{cases} \tag{3.20}$$

Let $X \times Y = \{(x, y)\}$ be a productset and $(\tau_1 \bigotimes \tau_2,\ \tau_1^* \bigotimes \tau_2^*)$ the product IGO on $X \times Y$. Since $(x, y)_{0.2} = \pi_1^{-1}(\underline{0.2}) = \pi_2^{-1}(y_{0.2})$, by Theorem 1.5, we have

$$(\tau_1 \bigotimes \tau_2)(\underline{0.2}) = \tau_1(\underline{0.2}) \vee \tau_2(y_{0.2}) = 1, \qquad (\tau_1^* \bigotimes \tau_2^*)(\underline{0.2}) = \tau_2(\underline{0.2}) \wedge \tau_2^*(y_{0.2}) = 0. \tag{3.21}$$

We can obtain the product IGO, $(\tau_1 \bigotimes \tau_2,\ \tau_1^* \bigotimes \tau_2^*)$ as follows:

$$\begin{gathered} \tau_1 \bigotimes \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 0 & \text{otherwise,} \end{cases} \\ \tau_1^* \bigotimes \tau_2^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{\alpha} \text{ for } \alpha \in I, \\ 1 & \text{otherwise.} \end{cases} \end{gathered} \tag{3.22}$$

Then, $(X \times Y, \tau_1 \bigotimes \tau_2, \tau_1^* \bigotimes \tau_2^*)$ are (r, s)-T_1, (r, s)-T_0, and (r, s)-quasi-T_0 for all $r \in I_0$, $s \in I_1$. But (Y, τ_2, τ_2^*) is not (r, s)-quasi-T_0 for all $r \in I_0$, $s \in I_1$. Hence, it is neither (r, s)-T_0 nor (r, s)-T_1 for all $r \in I_0$, $s \in I_1$.

Theorem 3.16. *Let $\{(X_i, \tau_i, \tau_i^*) \mid i \in \Gamma\}$ be a family of IFTSs and (τ, τ^*) be the product IGO on $X = \prod_{i \in \Gamma} X_i$. If (X, τ, τ^*) is (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0, and (r, s)-T_1) and (X_j, τ_j, τ_j^*) is stratified for $j \in \Gamma$, then (X_j, τ_j, τ_j^*) is (r, s)-quasi-T_0 (resp., (r, s)-sub-T_0, (r, s)-T_0, and (r, s)-T_1).*

Proof. Let (X, τ, τ^*) and $\widetilde{X}_j = X_j \times \prod\{y^i \mid i \neq j\}$ of X parallel to X_j. Since $(\widetilde{X}_j, \tau|_{\widetilde{X}_j}, \tau^*|_{\widetilde{X}_j})$ is a subspace of (X, τ, τ^*), by Theorem 3.11, $(\widetilde{X}_j, \tau|_{\widetilde{X}_j}, \tau^*|_{\widetilde{X}_j})$ is (r,s)-quasi-T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1). Since (X_j, τ_j, τ_j^*) is stratified, by Theorem 2.7, $\pi_j|_{\widetilde{X}_j} : (\widetilde{X}_j, \tau|_{\widetilde{X}_j}, \tau^*|_{\widetilde{X}_j}) \to (X_j, \tau_j, \tau_j^*)$ is IF homeomorphism. From Theorem 3.12, (X_j, τ_j, τ_j^*) is (r,s)-quasi-T_0 (resp., (r,s)-sub-T_0, (r,s)-T_0, and (r,s)-T_1). □

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