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Research Article

Fixed Points for Multivalued Mappings in Uniformly Convex Metric Spaces

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The purpose of this paper is to ensure the existence of fixed points for multivalued nonexpansive weakly inward nonself-mappings in uniformly convex metric spaces. This extends a result of Lim (1980) in Banach spaces. All results of Dhompongsa et al. (2005) and Chaoha and Phon-on (2006) are also extended.

1. Introduction

In 1974, Lim [1] developed a result concerning the existence of fixed points for multivalued nonexpansive self-mappings in uniformly convex Banach spaces. This result was extended to nonself-mappings satisfying the inwardness condition independently by Downing and Kirk [2] and Reich [3]. This result was extended to weak inward mappings independently by Lim [4] and Xu [5]. Recently, Dhompongsa et al. [6] presented an analog of Lim-Xu's result in CAT(0) spaces. In this note, we extend the result to uniformly convex metric spaces which improve results of both Lim-Xu and Dhompongsa et al. In addition, we also give a new proof of a result of Lim [7] by using Caristi's theorem [8]. Finally, we give some basic properties of fixed point sets for quasi-nonexpansive mappings for these spaces.

2. Preliminaries

A concept of convexity in metric spaces was introduced by Takahashi [9].

Definition 2.1. Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times I$ and $z \in X$,

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda) d(z, y). \tag{2.1}$$

A metric space (X, d) together with a convex structure W is called a *convex metric* space which will be denoted by (X, d, W).

Definition 2.2. A convex metric space (X, d, W) is said to be *uniformly convex* [10] if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$,

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r(1 - \delta). \tag{2.2}$$

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces. By using the (CN) inequality [11], it is easy to see that CAT(0) spaces are also uniformly convex.

For $x, y \in X$, $C \subset X$, and $\lambda \in I$, we denote $W(x, y, \lambda) := \lambda x \oplus (1 - \lambda) y$, $[x, y] := \{\lambda x \oplus (1 - \lambda) y : \lambda \in I\}$, $(x, y] := [x, y] \setminus \{x\}$, and $\lambda x \oplus (1 - \lambda) C := \{\lambda x \oplus (1 - \lambda) z : z \in C\}$. So we can define the *inward set* $I_C(x)$ of x as follows:

$$I_C(x) := \{x\} \cup \{z : (x, z] \cap C \neq \varnothing\}. \tag{2.3}$$

Let C be a nonempty subset of a metric space X. Then C is called convex if for $x, y \in C$, $[x, y] \subset C$. We will denote by $\mathcal{F}(C)$ the family of nonempty closed subsets of C, by $\mathcal{K}(C)$ the family of nonempty compact subsets of C, and by $\mathcal{KC}(C)$ the family of nonempty compact convex subsets of C. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $\mathcal{F}(X)$. That is,

$$H(A, B) = \max\left\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\right\}, \quad A, B \in \mathcal{F}(X). \tag{2.4}$$

Definition 2.3. A multivalued mapping $T : C \to \mathcal{F}(X)$ is said to be *inward* on C if for some $p \in C$,

$$\lambda p \oplus (1 - \lambda) T x \subset I_C(x) \quad \forall x \in C,\ \forall \lambda \in [0, 1], \tag{2.5}$$

and *weakly inward* on C if for some $p \in C$,

$$\lambda p \oplus (1 - \lambda) T x \subset \overline{I_C(x)} \quad \forall x \in C,\ \forall \lambda \in [0, 1], \tag{2.6}$$

where $\overline{A}$ denotes the closure of a subset A of X. In a Banach space setting, if C is convex, then so is $I_C(x)$. Therefore, the conditions above can be replaced by $Tx \subset I_C(x)$ and $Tx \subset \overline{I_C(x)}$, respectively.

Definition 2.4. A multivalued mapping $T : C \to \mathcal{F}(X)$ satisfying

$$H(Tx, Ty) \leq k d(x, y), \quad x, y \in C, \tag{2.7}$$

is called a *contraction* if $k \in [0, 1)$ and *nonexpansive* if $k = 1$. A point x is a fixed point of T if $x \in Tx$.

Given a metric space X, one way to describe a metric space ultrapower $\widetilde{X}$ of X is to first embed X as a closed subset of a Banach space E (see, e.g., [12, page 129]). Let $\widetilde{E}$ denote a Banach space ultrapower of E relative to some nontrivial ultrafilter $\mathcal{U}$ (see, e.g., [13]). Then take

$$\widetilde{X} := \{\tilde{x} = [\{x_n\}] \in \widetilde{E} : x_n \in X \ \forall n\}. \tag{2.8}$$

One can then let $\tilde{d}$ denote the metric on $\widetilde{X}$ inherited from the ultrapower norm $\|\cdot\|_{\mathcal{U}}$ in $\widetilde{E}$. If X is complete, then so is $\widetilde{X}$ since $\widetilde{X}$ is a closed subset of the Banach space $\widetilde{E}$. In particular, the metric $\tilde{d}$ on $\widetilde{X}$ is given by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{\mathcal{U}} \|x_n - y_n\| = \lim_{\mathcal{U}} d(x_n, y_n), \tag{2.9}$$

with $\{u_n\} \in [\{x_n\}]$ if and only if $\lim_{\mathcal{U}} \|x_n - u_n\| = 0$.

If (X, d, W) is a convex metric space, we consider a metric space ultrapower $(\widetilde{X}, \tilde{d})$ of (X, d) and define a function $\widetilde{W} : \widetilde{X} \times \widetilde{X} \times I \to \widetilde{X}$ by

$$\widetilde{W}(\tilde{x}, \tilde{y}, \lambda) = [W(x_n, y_n, \lambda)]. \tag{2.10}$$

In order to show that the function $\widetilde{W}$ is well defined, we need the following condition. For each $p, x, y \in X$ and $\lambda \in [0, 1]$,

$$d((1-\lambda)p \oplus \lambda x, (1-\lambda)p \oplus \lambda y) \leq \lambda d(x, y), \tag{2.11}$$

which is equivalent to

$$d((1-\lambda)p \oplus \lambda x, (1-\lambda)q \oplus \lambda y) \leq \lambda d(x, y) + (1-\lambda) d(p, q), \tag{2.12}$$

for all $p, q, x, y \in X$ and $\lambda \in [0, 1]$.

By using condition (2.12), it is easy to see that $\widetilde{W}$ is a convex structure on $\widetilde{X}$. This implies that $(\widetilde{X}, \tilde{d}, \widetilde{W})$ is a convex metric space.

Example 2.5. Every Banach space satisfies condition (2.11).

Example 2.6. Condition (2.11) is satisfied for spaces of hyperbolic type (for more details of these spaces see [14]). This is also true for CAT(0) spaces and $\mathbb{R}$-trees.

Example 2.7. Let H be a hyperconvex metric space. Then there exists a nonexpansive retract $R : l_\infty(H) \to H$ (see, e.g., [15] for more on this). For any $x, y \in H$ and $t \in [0, 1]$, we let

$$tx \oplus (1-t)y = R(tx + (1-t)y). \tag{2.13}$$

Since $l_\infty(H)$ is a Banach space, H also satisfies condition (2.11).

Let $\mathcal{U}$ be a nontrivial ultrafilter on the natural number $\mathbb{N}$. If (X, d, W) is a uniformly convex metric space satisfying condition (2.11), then the metric space ultrapower

$(\widetilde{X}, \widetilde{d}, \widetilde{W})$ relative to $\mathcal{U}$ is also uniformly convex. Indeed, let any $\varepsilon > 0$ and let δ be a positive number corresponding to the uniform convexity of X. Let $r > 0$ and $\tilde{x}, \tilde{y}, \tilde{z} \in \widetilde{X}$ be such that

$$\tilde{d}(\tilde{z}, \tilde{x}) \le r, \qquad \tilde{d}(\tilde{z}, \tilde{y}) \le r, \qquad \tilde{d}(\tilde{x}, \tilde{y}) \ge r\varepsilon. \tag{2.14}$$

Then there are some representatives (x_n) and (y_n) of $\tilde{x}$ and $\tilde{y}$ and a set $I \in \mathcal{U}$ such that

$$d(z_n, x_n) \le r, \quad d(z_n, y_n) \le r, \quad d(x_n, y_n) \ge r\varepsilon \quad \forall n \in I. \tag{2.15}$$

For such $n, d(z_n, W(x_n, y_n, 1/2)) \le r(1-\delta)$. This implies $\tilde{d}(\tilde{z}, \widetilde{W}(\tilde{x}, \tilde{y}, 1/2)) \le r(1-\delta)$.

Recall that a subset C of a metric space X is said to be (uniquely) *proximinal* if each point $x \in X$ has a (unique) nearest point in C. A convex metric space X is said to have *property (C)* if every decreasing sequence of nonempty bounded closed convex subsets of X has nonempty intersection. In [10], Shimizu and Takahashi proved that property (C) holds in complete uniformly convex metric spaces. This implies that every nonempty closed convex subset of a complete uniformly convex metric space is uniquely proximinal. Indeed, let C be a nonempty closed convex subset of a complete uniformly convex metric space X, and $x_0 \in X$.

Let $N = \{x \in C : d(x_0, x) = \operatorname{dist}(x_0, C)\}$. For each n, we define

$$C_n := \left\{ y \in C : d(x_0, y) \le \operatorname{dist}(x_0, C) + \frac{1}{n} \right\}. \tag{2.16}$$

Then (C_n) is a decreasing sequence of nonempty bounded closed convex subsets of C. Moreover,

$$N = \bigcap_{n=1}^{\infty} C_n,$$

which is nonempty by the above observation. The uniqueness follows from the uniform convexity of X.

3. Main results

We first establish the following lemma.

Lemma 3.1. *Let X be a complete uniformly convex metric space satisfying condition* (2.11)*, C a nonempty closed convex subset of X, $x \in X$, and $p(x)$ the unique nearest point of x in C. Then*

$$d(x, p(x)) < d(x, y) \quad \forall y \in \overline{I_C(p(x))} \setminus \{p(x)\}. \tag{3.1}$$

Proof. Let $y \in \overline{I_C(p(x))} \setminus \{p(x)\}$. Then there is a sequence (y_n) in $I_C(p(x))$ and $y_n \to y$. Choose $n_0 \in \mathbb{N}$ such that $(p(x), y_n] \cap C \neq \varnothing$ for all $n \ge n_0$. For such n, let $z_n \in (p(x), y_n] \cap C$ and write $z_n = (1-\alpha_n)p(x) \oplus \alpha_n y_n$, $\alpha_n \in (0,1]$. Then

$$d(x, p(x)) \le d(x, z_n) \le (1-\alpha_n) d(x, p(x)) + \alpha_n d(x, y_n). \tag{3.2}$$

This implies

$$d(x, p(x)) \le d(x, y). \tag{3.3}$$

If $d(x, p(x)) = d(x, y)$, we let $u = (1/2)p(x) \oplus (1/2)y$. By the uniform convexity of X, we have $d(x, u) < d(x, p(x))$. On the other hand, for each $n \ge n_0$, let $u_n = (1/2)p(x) \oplus (1/2)y_n$. We will show that $u_n \in I_C(p(x))$.

Case 1. $\alpha_n = 1/2$. We are done.
Case 2. $1/2 < \alpha_n$. Let $v_n = (1 - 1/2\alpha_n)p(x) \oplus (1/2\alpha_n)z_n$. This implies

$$d(p(x), v_n) = \frac{1}{2\alpha_n} d(p(x), z_n) = \frac{1}{2} d(p(x), y_n) = d(p(x), u_n), \tag{3.4}$$

$$\begin{aligned} d(v_n, y_n) \le d(v_n, z_n) + d(z_n, y_n) &= \left(1 - \frac{1}{2\alpha_n}\right) d(p(x), z_n) + (1 - \alpha_n) d(p(x), y_n) \\ &= \left(\alpha_n - \frac{1}{2}\right) d(p(x), y_n) + (1 - \alpha_n) d(p(x), y_n) = \frac{1}{2} d(p(x), y_n) = d(u_n, y_n). \end{aligned} \tag{3.5}$$

Therefore

$$d(v_n, y_n) \le d(u_n, y_n). \tag{3.6}$$

We claim that $u_n = v_n$. If not, let $w_n = (1/2)u_n \oplus (1/2)v_n$.

From (3.4), (3.6), and the uniform convexity of X, we have

$$\begin{aligned} d(p(x), w_n) &< d(p(x), u_n), \\ d(w_n, y_n) &< d(u_n, y_n). \end{aligned} \tag{3.7}$$

This implies

$$d(p(x), y_n) \le d(p(x), w_n) + d(w_n, y_n) < d(p(x), u_n) + d(u_n, y_n) = d(p(x), y_n), \tag{3.8}$$

which is a contradiction. Hence $u_n = v_n \in [p(x), z_n]$ and so $u_n \in I_C(p(x))$ by the convexity of C.
Case 3. $\alpha_n < 1/2$. let $v_n = (1 - 2\alpha_n)p(x) \oplus 2\alpha_n u_n$. By the same arguments in the proof of Case 2, we can show that $z_n \in (p(x), u_n]$. This means $u_n \in I_C(p(x))$.

By condition (2.11), $\lim_n u_n = u$, which implies $u \in \overline{I_E(p(x))} \setminus \{p(x)\}$. By the same arguments in the first part of the proof, $d(x, p(x)) \le d(x, u)$ which is a contradiction. Hence $d(x, p(x)) < d(x, y)$ as desired. □

From [6, Lemma 3.4], we observe that the space is not necessarity assumed to be a CAT(0) space since the proof is only involved with condition (2.11) which is weaker than the (CN) inequality (see [11, Lemma 3]). Therefore, we can obtain the following lemma.

Lemma 3.2. *Let X be a complete convex metric space satisfying condition* (2.11)*, C a nonempty closed subset of X, and $T : C \to \mathcal{F}(X)$ a contraction mapping satisfying, for all $x \in C$,*

$$Tx \subset \overline{I_C(x)}. \tag{3.9}$$

Then T has a fixed point.

This lemma was first proved in Banach spaces by Lim [7], using transfinite induction, while we apply directly Caristi's theorem. For completeness, we include the details.

Proof of Lemma 3.2. Let $0 \leq k < 1$ be the contraction constant of T and let $\varepsilon > 0$ be such that $\varepsilon + (k + 2\varepsilon)(1 + \varepsilon) < 1$. Let $M = \{(x,z) : z \in Tx,\ x \in C\}$ and define a metric ρ on M by $\rho((x,z),(u,v)) = \max\{d(x,u), d(z,v)\}$. It is easy to see that (M,ρ) is a complete metric space.

Now define $\psi : M \to [0,\infty)$ by $\psi(x,z) = d(x,z)/\varepsilon$. Then ψ is continuous on M. Suppose that T has no fixed points, that is, $\operatorname{dist}(x,Tx) > 0$ for all $x \in C$. Let $(x,z) \in M$. By (3.9), we can find $z' \in I_C(x)$ satisfying $d(z,z') < \varepsilon \operatorname{dist}(x,Tx)$. Now choose $u \in (x,z'] \cap C$ and write $u = (1-\delta)x \oplus \delta z'$ for some $0 < \delta \leq 1$. For such δ, we have

$$\delta\varepsilon + (1-\delta) + (k + 2\varepsilon)\delta(1+\varepsilon) < 1. \tag{3.10}$$

Since $d(x,u) > 0$, we can find $v \in Tu$ satisfying

$$d(z,v) \leq H(Tx,Tu) + \varepsilon d(x,u) \leq (k+\varepsilon)d(x,u). \tag{3.11}$$

Now we define a mapping $g : M \to M$ by $g(x,z) = (u,v)$ for all $(x,z) \in M$. We claim that g satisfies

$$\rho\big((x,z), g(x,z)\big) < \psi(x,z) - \psi\big(g(x,z)\big) \quad \forall (x,z) \in M. \tag{3.12}$$

Caristi's theorem [8] then implies that g has a fixed point, which contradicts to the strict inequality (3.12) and the proof is complete. So it remains to prove (3.12). In fact, it is enough to show that

$$\rho\big((x,z),(u,v)\big) < \frac{1}{\varepsilon}\big(d(x,z) - d(u,v)\big). \tag{3.13}$$

But $d(z,v) \leq d(x,u)$, it only needs to prove that $d(x,u) < (1/\varepsilon)\big(d(x,z) - d(u,v)\big)$.

Now,

$$d(x,u) = \delta d(x,z') \leq \delta\big(d(x,z) + \varepsilon \operatorname{dist}(x,Tx)\big) \leq \delta\big(d(x,z) + \varepsilon d(x,z)\big) \leq \delta(1+\varepsilon)d(x,z). \tag{3.14}$$

Therefore

$$d(x,u) \leq \delta(1+\varepsilon)d(x,z). \tag{3.15}$$

It follows that

$$d(z,v) \leq (k+\varepsilon)d(x,u) \leq (k+\varepsilon)\delta(1+\varepsilon)d(x,z). \tag{3.16}$$

We now let $y = (1-\delta)x \oplus \delta z$, then by the condition (2.11),

$$\begin{aligned} d(u,v) &\leq d(u,y) + d(y,z) + d(z,v) \leq \delta d(z,z') + (1-\delta)d(x,z) + (k+\varepsilon)\delta(1+\varepsilon)d(x,z) \\ &\leq \delta\varepsilon d(x,z) + \big((1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z). \end{aligned} \tag{3.17}$$

Thus

$$d(u,v) \leq \big(\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z). \tag{3.18}$$

Inequalities (3.15), (3.18), and (3.10) imply that

$$\begin{aligned} \varepsilon d(x,u) + d(u,v) &\leq \varepsilon\delta(1+\varepsilon)d(x,z) + \big(\delta\varepsilon + (1-\delta) + (k+\varepsilon)\delta(1+\varepsilon)\big)d(x,z) \\ &= \big(\delta\varepsilon + (1-\delta) + (k+2\varepsilon)\delta(1+\varepsilon)\big)d(x,z) < d(x,z). \end{aligned} \tag{3.19}$$

Therefore $d(x,u) < (1/\varepsilon)(d(x,z) - d(u,v))$ as desired. □

By Lemmas 3.1 and 3.2 with the same arguments in the proof of Theorem 3.3 of [6], we can obtain the following theorem which extends [4, Theorem 8] by Lim and [6, Theorem 3.3] by Dhompongsa et al.

Theorem 3.3. *Let X be a complete uniformly convex metric space satisfying condition* (2.11)*, C a nonempty bounded closed convex subset of X, and $T : C \to \mathcal{K}(X)$ a nonexpansive weakly inward mapping. Then T has a fixed point.*

As an immediate consequence of Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let X be a complete uniformly convex metric space satisfying condition* (2.11)*, C a nonempty bounded closed convex subset of X, and $T : C \to \mathcal{K}(C)$ a nonexpansive mapping. Then T has a fixed point.*

In fact, this corollary is a special case of [10, Theorem 2] in which condition (2.11) was not assumed. An interesting question is whether condition (2.11) in Theorem 3.3 can be dropped.

Let C be a nonempty subset of a metric space X. Recall that a single-valued mapping $t : C \to C$ and a multivalued mapping $T : C \to 2^C \setminus \varnothing$ are said to be *commuting* if $ty \in Ttx$ for all $y \in Tx$ and $x \in C$. If $t : C \to C$ is nonexpansive with C being bounded closed convex and X complete uniformly convex satisfying condition (2.11), then $\mathrm{Fix}(t)$ is nonempty by the above corollary. Moreover, by a standard argument, we can show that it is also closed and convex. So we can obtain a common fixed point theorem in uniformly convex metric spaces as [6, Theorem 4.1] (see also [16, Theorem 4.2] for a related result in Banach spaces).

Theorem 3.5. *Let X be a complete uniformly convex metric space satisfying condition* (2.11)*, let C be a nonempty bounded closed convex subset of X, and let $t : C \to C$ and $T : C \to \mathcal{KC}(C)$ be nonexpansive. Assume that for some $p \in \mathrm{Fix}(t)$,*

$$\alpha p \oplus (1-\alpha)Tx \text{ is convex} \quad \forall x \in C,\ \forall \alpha \in [0,1]. \tag{3.20}$$

If t and T are commuting, then there exists a point $z \in C$ such that $tz = z \in Tz$.

4. Fixed point sets of quasi-nonexpansive mappings

Let X be a metric space. Recall that a mapping $f : X \to X$ is said to be *quasi-nonexpansive* if $d(f(x),p) \le d(x,p)$ for all $x \in X$ and $p \in \mathrm{Fix}(f)$. In this case, we will assume that $\mathrm{Fix}(f) \neq \varnothing$. In [17], Chaoha and Phon-on showed that if X is a CAT(0) space, then $\mathrm{Fix}(f)$ is closed convex. Furthermore, they gave an explicit construction of a continuous function defined on X whose fixed point set is any prescribed closed subset of X. In this section, we extend these results to uniformly convex metric spaces.

We begin by proving the following lemma.

Lemma 4.1. *Let X be a uniformly convex metric space, and let $x, y, z \in X$ for which*

$$d(x,z) + d(z,y) = d(x,y). \tag{4.1}$$

Then $z \in [x,y]$.

Proof. Let $u \in [x, y]$ be such that $d(x, u) = d(x, z)$. Then $d(x, y) = d(x, u) + d(u, y)$ and also $d(z, y) = d(u, y)$ by (4.1). We will show that $z = u$. Suppose not, we let $v = (1/2)z \oplus (1/2)u$ and $r = d(x, u) = d(x, z)$. Since $d(z, u) > 0$, choose $\varepsilon > 0$ so that $d(z, u) > r\varepsilon$. By the uniform convexity of X, there exists $\delta > 0$ such that

$$d(x, v) \leq r(1 - \delta) < r = d(x, z). \tag{4.2}$$

By using the same arguments, we can show that $d(y, v) < d(y, z)$. Therefore

$$d(x, y) \leq d(x, v) + d(y, v) < d(x, z) + d(y, z) = d(x, y), \tag{4.3}$$

which is a contradiction. □

By using the above lemma with the proof of Theorem 1.3 of [17], we obtain the following result.

Theorem 4.2. *Let X be a convex subset of a uniformly convex metric space and $f : X \to X$ a quasi-nonexpansive mapping whose fixed point set is nonempty. Then* $\mathrm{Fix}(f)$ *is closed convex.*

In [17], the authors constructed a continuous function defined on a CAT(0) space X whose fixed point set is any prescribed closed subset of X by using the following two implications of the (CN) inequality:

$$d\big((1-t)x \oplus ty, (1-s)x \oplus sy\big) = |t - s| d(x, y) \quad \forall x, y \in X,\ t, s \in [0, 1], \tag{4.4}$$

$$d\big((1-t)x \oplus ty, (1-t)x \oplus tz\big) \leq d(y, z) \quad \forall x, y, z \in X,\ t \in [0, 1]. \tag{4.5}$$

In fact, condition (4.4) holds in uniformly convex metric spaces as the following lemma shows.

Lemma 4.3. *Condition* (4.4) *holds in uniformly convex metric spaces.*

Proof. We first note that the conclusion holds if $s = 0$ or $t = 0$. We now let X be a uniformly convex metric space, $x, y \in X$, and $t, s \in (0, 1]$. Let $u = (1-t)x \oplus ty$ and $z = (1-s)x \oplus sy$. Without loss of generality, we can assume that $t < s$. Let $v = (1 - t/s)x \oplus (t/s)z$, then

$$\begin{gathered} d(x, v) = \frac{t}{s} d(x, z) = t d(x, y), \\ d(v, y) \leq \left(1 - \frac{t}{s}\right) d(x, y) + \frac{t}{s} d(z, y) = (1 - t) d(x, y). \end{gathered} \tag{4.6}$$

If $u \neq v$, we let $w = (1/2)u \oplus (1/2)v$. Then by the uniform convexity of X, we can show that $d(x, w) < d(x, u)$ and $d(y, w) < d(y, u)$. This implies

$$d(x, y) < d(x, u) + d(u, y) = d(x, y), \tag{4.7}$$

which is a contradiction, hence $u = v$. Therefore

$$d(z, u) = d(z, v) = \left(1 - \frac{t}{s}\right) d(x, z) = |s - t| d(x, y). \tag{4.8}$$

□

It is unclear that condition (4.5) holds for uniformly convex metric spaces. However, the following theorem is a generalization of [17, Theorem 2.1], we omit the proof because it is similar to the one given in [17].

Theorem 4.4. *Let A be a nonempty subset of a uniformly convex metric space X satisfying condition* (4.5)*. Then there exists a continuous function $f : X \to X$ such that* $\mathrm{Fix}(f) = \overline{A}$.

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